Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1065–1069

AUTOMORPHISM GROUPS OF ALGEBRAS OF FINITE TYPE

MATTHEW GOULD

By "algebra" we shall mean a *finitary* universal algebra, that is, a pair $\langle A; F\rangle$ where A and F are nonvoid sets and every element of F is a function, defined on A, of some finite number of variables. Armbrust and Schmidt showed in [**1**] that for any finite nonvoid set A, every group G of permutations of A is the automorphism group of an algebra defined on A and having only one operation, whose rank is the cardinality of A. In [**6**], Jónsson gave a necessary and sufficient condition for a given permutation group to be the automorphism group of an algebra, whereupon Plonka [**8**] modified Jónsson's condition to characterize the automorphism groups of algebras whose operations have ranks not exceeding a prescribed bound.

The goal of the present paper is to characterize those permutation groups which are automorphism groups of algebras having finitely many operations (such algebras are said to be of *finite similarity type*, or simply of *finite type*). A problem posed by Jónsson in [**7**, p. 41] asks for such a characterization. It will follow from our characterization that the automorphism group of any algebra whose operations are of bounded rank, is the automorphism group of an algebra of finite type. Moreover, it will be shown that the automorphism group of any algebra of finite type is the automorphism group of an algebra having precisely one operation. As a byproduct of our characterization we shall obtain a new proof of Jónsson's result cited above; this new proof will use only a countable number of operations, whereas in the algebras constructed by Jónsson the number of operations was, in the infinite case, equal to the cardinality of the underlying set.

1. Terminology.

For the most part our notation will be taken from [**5**]. One exception is that we will use the symbol $\mathrm{Aut}(\mathfrak{A})$ to denote the automorphism group of an algebra $\mathfrak{A}$. Moreover, we shall need the following terminology: If G is a group of permutations of a nonvoid set A, and n is a natural number, we denote by $T_n(G)$ the statement "A permutation ϕ of A is a member of G if for every n-element subset X of A there is a member of G that agrees with ϕ on X".

Let $T(G)$ denote the statement obtained from $T_n(G)$ by substituting "finite" for "n-element". $T_n(G)$ is a modification of the condition denoted

Received October 4, 1971 and in revised form, February 1, 1972. This research was partially supported by NSF Grant GP-21059.

https://doi.org/10.4153/CJM-1972-109-0 Published online by Cambridge University Press

$\beta_n(G)$ in [**6**], in which paper Jónsson showed that $T(G)$ (therein denoted $\beta_\omega(G)$) is necessary and sufficient for G to be the automorphism group of an algebra.

A final word on notation: if ϕ is a mapping of a set A into itself, and $x = \langle x_0, x_1, \ldots, x_{n-1}\rangle$ is a sequence of elements of A, the symbol $x\phi$ will denote the sequence $\langle x_0\phi, x_1\phi, \ldots, x_{n-1}\phi\rangle$.

2. Results. The proof of our main result requires the following proposition, due to Vopěnka, Hedrlín, and Pultr [**9**].

PROPOSITION. *There exists on any set S a binary relation ρ such that the only endomorphism of the system $\langle S; \rho\rangle$ is the identity.*

THEOREM 1. *Given a group G of permutations of a nonvoid set A, the following statements are equivalent.*

(i) $G = \mathrm{Aut}(\mathfrak{A})$ *for an algebra $\mathfrak{A}$ whose operations are of bounded rank.*

(ii) $T_n(G)$ *holds for some n.*

(iii) $G = \mathrm{Aut}(\mathfrak{A})$ *for an algebra $\mathfrak{A}$ of finite type.*

(iv) $G = \mathrm{Aut}(\mathfrak{A})$ *for an algebra $\mathfrak{A}$ having precisely one operation.*

Proof. (i) $\Rightarrow$ (ii): If A is finite, $T_{|A|}(G)$ holds, so suppose A is infinite. Let n be any integer exceeding the rank of every operation of $\mathfrak{A} = \langle A; F\rangle$, and let ϕ be a permutation of A having the property that for every n-element subset of A there is a member of G agreeing with ϕ on that set. If $f \in F$ has positive rank k, and $x_0, x_1, \ldots, x_{k-1} \in A$, let X be any n-element subset of A containing $\{x_0, x_1, \ldots, x_{k-1}, f(x_0, x_1, \ldots, x_{k-1})\}$ and choose $\alpha \in G$ such that α agrees with ϕ on X. Then $f(x_0, x_1, \ldots, x_{k-1})\phi = f(x_0, x_1, \ldots, x_{k-1})\alpha = f(x_0\alpha, x_1\alpha, \ldots, x_{k-1}\alpha) = f(x_0\phi, x_1\phi, \ldots, x_{k-1}\phi)$, and so ϕ is compatible with f. Similarly, by expanding every singleton subset of A to an n-element subset, we see that ϕ is also compatible with every nullary operation in F. Thus $\phi \in \mathrm{Aut}(\mathfrak{A}) = G$, and so $T_n(G)$ holds. (This proof of (i) $\Rightarrow$ (ii) differs only slightly from Jónsson's proof of a similar statement in [**6**].)

(ii) $\Rightarrow$ (iii): Since the theorem is trivial if A has only one element, we assume that A has more than one element. Moreover, since $T_0(G)$ implies $T_1(G)$, which in turn implies $T_2(G)$, we assume that $T_n(G)$ holds for some $n > 1$. If $|A| < n$, then $T_n(G)$ implies that G consists of all permutations of A, and so $G = \mathrm{Aut}(\mathfrak{A})$ for the algebra $\mathfrak{A}$ whose only operation is the identity function on A. Thus, we further assume that $|A| \geqq n$.

Let $A^{(n)}$ denote the set of all sequences $x = \langle x_0, x_1, \ldots, x_{n-1}\rangle \in A^n$ consisting of n distinct elements of A. Define on $A^{(n)}$ an equivalence relation $\sim$ by: $x \sim y$ if and only if there is some $\alpha \in G$ such that $x = y\alpha$. Set $S = A^{(n)}/\sim$, the set of equivalence classes induced by $\sim$. If S has only one element then G consists of all permutations of A, so we assume that S has more than one element. For $x \in A^{(n)}$, the equivalence class of x under $\sim$ will be denoted $[x]$.

Using the proposition quoted above, let ρ be a binary relation on S having the property that the only endomorphism of $\langle S; \rho\rangle$ is the identity. Since S has more than one element, no constant mapping is an endomorphism of $\langle S; \rho\rangle$, whence it follows that there is no $[x] \in S$ such that $[x] \rho [x]$.

Define $2n$-ary operations f and g_i $(i < n)$ as follows: For all $x, y \in A^n$,

$$f(x, y) = \begin{cases} x_0 \text{ if } x, y \in A^{(n)} \text{ and } x \sim y \\ x_1 \text{ otherwise} \end{cases}$$

and

$$g_i(x, y) = \begin{cases} x_i \text{ if } x, y \in A^{(n)} \text{ and } [x] \rho [y] \\ y_i \text{ otherwise.} \end{cases}$$

Let $\mathfrak{A}$ denote the algebra $\langle A; \{f\} \cup \{g_i | i < n\}\rangle$. It is straight-forward to verify that $G \subseteq \mathrm{Aut}(\mathfrak{A})$: for $\phi \in G$ the key point in showing that ϕ is compatible with f is the observation that for all $x, y \in A^{(n)}$, $x \sim y$ if and only if $x\phi \sim y\phi$; the key point in showing compatibility with g_i is the trivial observation that $[x] = [x\phi]$ for all $x \in A^{(n)}$.

We now show that $\mathrm{Aut}(\mathfrak{A}) \subseteq G$. Let $\phi \in \mathrm{Aut}(\mathfrak{A})$. In view of $T_n(G)$, to show that $\phi \in G$ it will suffice to show that $[x\phi] = [x]$ for all $x \in A^{(n)}$. To achieve this, we define a map $\phi^*: S \to S$ by $[x]\phi^* = [x\phi]$ for all $x \in A^{(n)}$, and we show that ϕ^* is the identity map on S.

First it must be shown that ϕ^* is well-defined. To this end, suppose there exist $x, y \in A^{(n)}$ such that $[x] = [y]$ but $[x\phi] \neq [y\phi]$. Then $x_0\phi = f(x, y)\phi = f(x\phi, y\phi) = x_1\phi$, which implies $x_0 = x_1$, a contradiction. Thus, ϕ^* is well-defined.

To show that ϕ^* is the identity map on S it suffices, by the choice of ρ, to show that ϕ^* is an endomorphism of $\langle S; \rho\rangle$. Suppose this is not the case. Then there exist $[x], [y] \in S$ such that $[x] \rho [y]$ but it is false that $[x\phi] \rho [y\phi]$. It follows that for each $i < n$, $x_i\phi = g_i(x, y)\phi = g_i(x\phi, y\phi) = (y\phi)_i = y_i\phi$, whence $x_i = y_i$. Since this holds for all i, we have $x = y$, hence $[x] \rho [x]$, a contradiction. Thus ϕ^* is the identity map on S, so $[x\phi] = [x]$ for all $x \in A^{(n)}$, which by $T_n(G)$ implies $\phi \in G$, and (ii) $\Rightarrow$ (iii) is established.

(iii) $\Rightarrow$ (iv): Since any operation may be replaced by one of higher rank without altering the automorphism structure (e.g., a nullary operation may be replaced by the corresponding unary operation, and a unary operation f may be replaced by the binary operation g defined by $g(x, y) = f(x)$), we may assume that the operations in F all have the same positive rank n; and clearly we may also assume that A contains more than one element.

Partition F into disjoint pairs of distinct operations, possibly with one operation left over. If there is a left-over operation replace it by an operation of rank $n + 1$ inducing the same automorphisms. For each pair $\{f_1, f_2\}$ in the partition define an $(n + 1)$-ary operation f_3 by:

$$f_3(x_0, \ldots, x_n) = \begin{cases} f_1(x_0, \ldots, x_{n-1}) \text{ if } x_n = x_0 \\ f_2(x_0, \ldots, x_{n-1}) \text{ if } x_n \neq x_0. \end{cases}$$

It is straight-forward to verify that f_3 induces the same automorphisms as $\{f_1, f_2\}$. In this way F has been replaced by a set F' of $(n + 1)$-ary operations such that $\text{Aut}(\mathfrak{A}) = \text{Aut}(A; F')$ and the cardinality of F' is the first integer not less than $|F|/2$. Now partition F' in the same way and continue until a single operation f is obtained such that $\text{Aut}(\mathfrak{A}) = \text{Aut}(A; f)$; the rank of f will be n plus the first integer not less than $\log_2|F|$.

The implication (iv) $\Rightarrow$ (i) being trivial, the theorem is proved.

As mentioned earlier, the equivalence (i) $\Leftrightarrow$ (ii) in the following theorem was first proved by Jónsson in [**6**].

THEOREM 2. *Given a group G of permutations of a nonvoid set A, the following statements are equivalent.*

(i) $G = \text{Aut}(\mathfrak{A})$ *for some algebra* $\mathfrak{A}$.

(ii) $T(G)$ *holds.*

(iii) $G = \text{Aut}(\mathfrak{A})$ *for an algebra* $\mathfrak{A}$ *having countably many operations.*

Proof. Since the proof of (i) $\Rightarrow$ (ii) is essentially the same as in Theorem 1, and since (iii) $\Rightarrow$ (i) is trivial, we need only prove (ii) $\Rightarrow$ (iii). If A is finite, $T_{|A|}(G)$ holds, whence (iii) follows from Theorem 1.

We now assume that A is infinite and $T(G)$ holds. For each $n < \omega$ let G_n denote the set of all permutations ϕ of A having the property that for every n-element subset X of A there is a member of G that agrees with ϕ on X. It is easily verifiable that G_n is a group and that $T_n(G_n)$ holds. By Theorem 1 choose for each n a finite set F_n of operations such that $G_n = \text{Aut}(A; F_n)$. Now, $T(G)$ implies that G is the intersection of all the groups G_n, whence $G = \text{Aut}(A; F)$ where F is the union of the sets F_n.

3. Examples and questions. Although it seems intuitively clear, an example is needed to show that the conditions of Theorem 1 are not equivalent with those of Theorem 2; i.e., an example is needed of a permutation group G for which $T(G)$ holds but $T_n(G)$ fails for every n. For each $n < \omega$ Jónsson gives in [**6**, Example 2] an example of a group G_n of permutations of a set A_n, such that $T(G_n)$ holds but $T_n(G_n)$ fails. Taking the sets A_n to be disjoint, set $A = \bigcup (A_n | n < \omega)$ and $G = \{\alpha | \alpha$ is a permutation of A and for all n, $\alpha|_{A_n} \in G_n\}$. It readily follows that G is a group and that $T(G)$ holds, but $T_n(G)$ fails for every n.

The equivalence of (i), (iii), and (iv) in Theorem 1 provides an interesting contrast with the situation that obtains when subalgebra structures are considered instead of automorphism groups. In [**3**] it is shown that in the subalgebra context (iii) and (iv) remain equivalent, but (i) no longer implies (iii). However, the subalgebra structure of any algebra whose operations are of bounded rank and countable in number, can be realized as the subalgebra structure of an algebra having precisely one operation [**3**, Lemma 2.1].

One can ask in reference to automorphism structure all the questions studied in [**3**] concerning the relationship between subalgebra structure and the number of operations that an algebra has of each rank. For example, it is shown in [**3**] that for every $n < \omega$ and every non-zero cardinal m, there is an algebra whose operations are n-ary and m in number, and whose subalgebra structure is not the subalgebra structure of any algebra whose operations are n-ary but fewer than m in number. We pose the question of whether the same result holds with "automorphism group" in place of "subalgebra structure".

Finally, we mention one special case of Theorem 1. In [**2**], Birkhoff showed that for any abstract group A, the set $R(A)$ of all right-translations of A is the automorphism group of an algebra defined on A whose operations are all unary and form a set whose cardinality is that of a generating set for the group. It follows from Theorem 1 that $R(A)$ is the automorphism group of an algebra having a single operation. This operation, if constructed as indicated in the proof of the theorem, will have rank 6. In this special case the proof can be modified to show that a single operation of rank 5 will work. The author has been unable to decrease the rank any further, except under the assumption that G is finite or countably infinite, in which case a single binary operation will suffice (see [**4**]).

References

1. M. Armbrust and J. Schmidt, *Zum Cayleyschen Darstellungssatz*, Math. Ann. *154* (1964), 70–73.

2. G. Birkhoff, *Sobre los grupos de automorfismos*, Rev. Un. Mat. Argentina *11* (1946), 155–157.

3. M. Gould, *Multiplicity type and subalgebra structure in universal algebras*, Pacific J. Math. *26* (1968), 469–485.

4. ——— *A note on automorphisms of groupoids*, Algebra Universalis *2* (1972), 36–38.

5. G. Grätzer, *Universal algebra* (D. Van Nostrand Co., Princeton, N.J., 1968).

6. B. Jónsson, *Algebraic structures with prescribed automorphism groups*, Colloq. Math. *XIX* (1968), 1–4.

7. ——— *Topics in universal algebra* (lecture notes, Vanderbilt University, 1969–70).

8. E. Plonka, *On a problem of B. Jónsson concerning automorphisms of a general algebra*, Colloq. Math. *XIX* (1968), 5–8.

9. P. Vopěnka, A. Pultr, and Z. Hedrlin, *A rigid relation exists on any set*, C.M.U.C. *6* (1965), 149–155.

Vanderbilt University,
Nashville, Tennessee