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On Asymptotically Orthonormal Sequences

Emmanuel Fricain and Rishika Rupam

Abstract. An asymptotically orthonormal sequence is a sequence that is nearly orthonormal in the sense that it satisfies the Parseval equality up to two constants close to one. In this paper, we explore such sequences formed by normalized reproducing kernels for model spaces and de Branges–Rovnyak spaces.

1 Introduction

When working in Hilbert spaces, it is very natural and useful to deal with orthonormal bases. However, in many situations, the system we are interested in does not form an orthonormal basis but is close to one. The investigation of such bases has a long history. It began with the works of Paley–Wiener [16] and Levinson [13], mainly for exponential systems. In this context, functional models have been used in [12] together with some other tools from operator theory. The model spaces K_Θ of the unit disc are subspaces of the Hardy space $H^2(\mathbb{D})$ invariant under the adjoints of multiplications. Their theory is connected to dilation theory for contractions on Hilbert spaces. The paper [12] has inspired a fruitful line of research on geometric properties of systems formed by reproducing kernels for K_Θ. Not only did it enable the recapture of all classical results on exponential systems, but it also provided many new results in a more general context. In [4], following the line of research in [12], the authors studied the case when the system of normalized reproducing kernels $(\kappa_{\lambda_n}^\Theta)_n$ for K_Θ is asymptotically close to an orthonormal basis (see definition below). This is a particular case of unconditional basis where more rigidity is required. It should be noted that in [12] and [4], the additional assumption

$$\sup_{n\geq 1} |\Theta(\lambda_n)| < 1 \tag{1.1}$$

is required. Under that assumption, the projection method developed in [12] and used in [4] linked the properties of $(\kappa_{\lambda_n}^\Theta)_n$ with those of normalized reproducing kernels $(\kappa_{\lambda_n})_n$ for $H^2(\mathbb{D})$. Volberg proved in [19] that $(\kappa_{\lambda_n})_n$ is an asymptotically orthonormal basis for its closed span if and only if $(\lambda_n)_n$ is a thin sequence (a stronger condition than Carleson's condition). This beautiful result was recently reproved by

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Gorkin, McCarthy, Pott, and Wick [11] by a direct and easier method using ideas from interpolation theory.

Following the work of Baranov [2] for Riesz bases, we are interested here in investigating asymptotically orthonormal bases of reproducing kernels for K_Θ without requiring assumption (1.1). In this situation, the projection method no longer applies, and the main tool here will be Bernstein's type inequalities. We also work in the more general context where model spaces K_Θ are replaced by de Branges–Rovnyak spaces $\mathcal{H}(b)$. We should mention that we work in the upper-half plane, but most results transfer easily to the unit disc.

The plan of the paper is the following. The next section contains preliminary material; in particular, an analogue of Bari's theorem is given, which completes a result given in [4]. In Section 3, we study the stability of asymptotically orthonormal sequences with respect to perturbation of frequencies. The main results of the paper are Theorem 3.6, Corollary 3.13, Theorem 3.24, and Corollary 3.27. In Section 4, we study the case of exponential systems. Finally, in the last section, we examine what happens when one projects an AOB $(\kappa^{b_1}_{\lambda_n})_{n\geq1}$ on a subspace $\mathcal{H}(b_2)$ of $\mathcal{H}(b_1)$.

2 Preliminaries

2.1 Asymptotically Orthonormal Sequences

Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{X} = (x_n)_{n\geq1}$ be a sequence of vectors in $\mathcal{H}$. We recall that $\mathcal{X}$ is said to be:

(a) *minimal* if for every $n \geq 1$, $x_n \notin \operatorname{span}(x_\ell : \ell \neq n)$, where $\operatorname{span}(\cdots)$ denotes the closure of the finite linear combination of $(\cdots)$;

(b) a *Riesz sequence* (RS) if there exists two constants $c, C > 0$ such that

$$c \sum_{n\geq1} |a_n|^2 \leq \| \sum_{n\geq1} a_n x_n \|^2_{\mathcal{H}} \leq C \sum_{n\geq1} |a_n|^2$$

for every finitely supported sequence of complex numbers $(a_n)_n$;

(c) an *asymptotically orthonormal sequence* (AOS) if there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ there are constants $c_N, C_N > 0$ verifying

$$c_N \sum_{n\geq N} |a_n|^2 \leq \| \sum_{n\geq N} a_n x_n \|^2_{\mathcal{H}} \leq C_N \sum_{n\geq N} |a_n|^2 \tag{2.1}$$

for every finitely supported sequence of complex numbers $(a_n)_n$ and

$$\lim_{N\to\infty} c_N = 1 = \lim_{N\to\infty} C_N;$$

(d) an *asymptotically orthonormal basic sequence* (AOB) if it is an AOS with $N_0 = 1$;

(e) a *Riesz basis* (RB) for $\mathcal{H}$ if it is a complete Riesz sequence, that is, a Riesz sequence satisfying $\operatorname{span}(x_n : n \geq 1) = \mathcal{H}$.

It is easy to see that $(x_n)_{n\geq1}$ is an AOB if and only if it is an AOS as well as an RS. Also, $(x_n)_{n\geq1}$ is an AOB if and only if it is minimal and an AOS. The well-known result of Köthe–Toeplitz ([14, p. 136]) says that if $\mathcal{X} = (x_n)_{n\geq1}$ is a complete and minimal sequence of normalized vectors in $\mathcal{H}$, then $\mathcal{X}$ is a Riesz basis for $\mathcal{H}$ if and only if $\mathcal{X}$ is an unconditional basis for $\mathcal{H}$. The reader should pay attention to the fact that AOB does not imply completeness; an AOB is a basis for its span but not necessarily for the whole space.

We recall also that for a sequence $\mathcal{X} = (x_n)_{n\geq1}$, the *Gram matrix* $\Gamma_{\mathcal{X}} = (\Gamma_{n,p})_{n,p\geq1}$ is defined by

$$\Gamma_{n,p} = \langle x_n, x_p\rangle_{\mathcal{H}}, \qquad (n, p \geq 1).$$

If $\mathcal{X} = (x_n)_{n\geq1}$ is a complete and minimal sequence and $\mathcal{X}^* = (x_n^*)_{n\geq1}$ is its biorthogonal sequence, that is, the unique sequence $(x_n^*)_{n\geq1}$ in $\mathcal{H}$ satisfying

$$\langle x_\ell, x_n^*\rangle_{\mathcal{H}} = \delta_{n,\ell} = \begin{cases} 1 & \text{if } n = \ell, \\ 0 & \text{if } n \neq \ell, \end{cases}$$

then the *interpolation operator* $J_{\mathcal{X}}$ is defined as

$$J_{\mathcal{X}} x = (\langle f, x_n^*\rangle_{\mathcal{H}})_{n\geq1}, \qquad (x \in \mathcal{H}).$$

We finally recall that $\mathcal{X} = (x_n)_{n\geq1}$ is a Riesz basis for $\mathcal{H}$ if and only if there exists a (unique) invertible operator $U_{\mathcal{X}}: \mathcal{H} \to \ell^2$ such that $U_{\mathcal{X}}(x_n) = e_n$, $n \geq 1$, where $(e_n)_{n\geq1}$ is the canonical orthonormal basis for ℓ^2. The operator $U_{\mathcal{X}}$ is called the *orthonormalizer* of $\mathcal{X}$. We refer the reader to [9, 14, 20] for details on general geometric properties of sequences in an Hilbert space.

Bari's theorem (see [14, p. 132]) gives several characterizations for a sequence to be a RB in terms of its Gram matrix and the interpolation operator. An analogue of Bari's result for complete AOB is also available. A part of this can be found in [4]. To complete the picture, we need two preliminaries results. First, we introduce a notation. Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. We say that $T \in \mathcal{UK}(\mathcal{H}_1, \mathcal{H}_2)$ if T is invertible from $\mathcal{H}_1$ onto $\mathcal{H}_2$ and can be written as $T = U + K$, where $U, K \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, U is unitary, and K is compact.

Lemma 2.2 *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be Hilbert spaces.*

(i) *If $T_1 \in \mathcal{UK}(\mathcal{H}_1, \mathcal{H}_2)$ and $T_2 \in \mathcal{UK}(\mathcal{H}_2, \mathcal{H}_3)$, then $T_2T_1 \in \mathcal{UK}(\mathcal{H}_1, \mathcal{H}_3)$.*
(ii) *If $T \in \mathcal{UK}(\mathcal{H}_1, \mathcal{H}_2)$, then $T^{-1} \in \mathcal{UK}(\mathcal{H}_2, \mathcal{H}_1)$.*
(iii) *If $T \in \mathcal{UK}(\mathcal{H}_1, \mathcal{H}_2)$, then $T^* \in \mathcal{UK}(\mathcal{H}_2, \mathcal{H}_1)$.*

Proof The proofs of (i) and (iii) are straightforward and are left to the reader. Let us prove (ii). Assume that $T = U + K$ is invertible with U unitary and K compact. Then write $T = U(I + U^*K) = UV$ with $V = I + U^*K$. It is clear that V is invertible and $I = V^{-1} + V^{-1}U^*K$. Hence, $V^{-1} = I - V^{-1}U^*K$, and we get

$$T^{-1} = V^{-1}U^* = U^* - V^{-1}U^*KU^*,$$

which implies that $T^{-1} \in \mathcal{UK}(\mathcal{H}_2, \mathcal{H}_1)$. ∎

Lemma 2.3 *Let $\mathcal{X} = (x_n)_{n\geq1}$ be a complete AOB for $\mathcal{H}$ and let C_N be the constant appearing in the right inequality of (2.1). Then for every $N \geq 1$ and $f \in \mathcal{H}$, we have*

$$\sum_{n\geq N} |\langle f, x_n\rangle_{\mathcal{H}}|^2 \leq C_N \|f\|_{\mathcal{H}}^2.$$

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Proof Let us denote by $P_N: \ell^2 \to \ell^2$ the orthogonal projection defined by

$$P_N\Big(\sum_{n\geq 1} a_n e_n\Big) = \sum_{n\geq N} a_n e_n,$$

where $(e_n)_{n\geq 1}$ is the canonical orthonormal basis of ℓ^2. For every $a = (a_n)_{n\geq 1} \in \ell^2$, define

$$V_{\mathcal{X}} a = \sum_{n\geq 1} a_n x_n.$$

Since $\mathcal{X}$ is a Riesz basis, this map $V_{\mathcal{X}}$ defines a continuous and invertible operator from ℓ^2 onto $\mathcal{H}$. Moreover, for $a \in \ell^2$, we have

$$\|V_{\mathcal{X}} P_N a\|_{\mathcal{H}}^2 = \|\sum_{n\geq N} a_n x_n\|_{\mathcal{H}}^2 \leq C_N \sum_{n\geq N} |a_n|^2 \leq C_N \|a\|_{\ell^2}^2,$$

which gives $\|P_N V_{\mathcal{X}}^*\| = \|V_{\mathcal{X}} P_N\| \leq \sqrt{C_N}$. But is is easy to see that $V_{\mathcal{X}}^* = J_{\mathcal{X}^*}$, whence $\|P_N J_{\mathcal{X}^*}\| \leq \sqrt{C_N}$, and we get the desired inequality. ∎

Theorem 2.4 *Let $\mathcal{X} = (x_n)_{n\geq 1}$ be a complete and minimal sequence of vectors in $\mathcal{H}$, $\mathcal{X}^* = (x_n^*)_{n\geq 1}$ its biorthogonal sequence. The following assertions are equivalent:*

(i) *The sequence $\mathcal{X}$ is a complete AOB for $\mathcal{H}$.*

(ii) *There exists an operator $U_{\mathcal{X}} \in \mathcal{UK}(\mathcal{H}, \ell^2)$ such that $U_{\mathcal{X}}(x_n) = e_n$, $n \geq 1$.*

(iii) *The Gram matrix defines a bounded and invertible operator on ℓ^2 of the form $I+K$ with K compact.*

(iv) $J_{\mathcal{X}^*} \in \mathcal{UK}(\mathcal{H}, \ell^2)$.

(v) *The sequence $\mathcal{X}^*$ is a complete AOB for $\mathcal{H}$.*

(vi) *There exists an invertible operator $U_{\mathcal{X}}: \mathcal{H} \to \ell^2$ such that $U_{\mathcal{X}}(x_n) = e_n$, $n \geq 1$, and if $U_{\mathcal{X},N}: \operatorname{span}(x_n : n \geq N) \to \operatorname{span}(e_n : n \geq N)$ is the restriction of $U_{\mathcal{X}}$ to $\operatorname{span}(x_n : n \geq N)$, then*

$$\lim_{N\to\infty} \|U_{\mathcal{X},N}\| = 1 = \lim_{N\to\infty} \|U_{\mathcal{X},N}^{-1}\|.$$

(vii) *For every $N \geq 1$, there are two constants $C_N, C_N^* > 0$ such that*

$$C_N^{*\,-1} \|f\|_{\mathcal{H}}^2 \leq \sum_{n\geq N} |\langle f, x_n\rangle_{\mathcal{H}}|^2 \leq C_N \|f\|_{\mathcal{H}}^2 \tag{2.5}$$

for every $f \in \mathcal{H} \ominus \operatorname{span}(x_1, x_2, \dots, x_{N-1})$ and $\lim_{N\to\infty} C_N = 1 = \lim_{N\to\infty} C_N^*$.

(viii) *The sequence $\mathcal{X}^*$ is complete in $\mathcal{H}$ and for every $N \geq 1$, there are two constants $C_N, C_N^* > 0$ such that*

$$\sum_{n\geq N} |\langle f, x_n\rangle_{\mathcal{H}}|^2 \leq C_N \|f\|_{\mathcal{H}}^2 \quad \text{and} \quad \sum_{n\geq N} |\langle f, x_n^*\rangle_{\mathcal{H}}|^2 \leq C_N^* \|f\|_{\mathcal{H}}^2, \tag{2.6}$$

for every $f \in \mathcal{H}$ and $\lim_{N\to\infty} C_N = 1 = \lim_{N\to\infty} C_N^*$.

Proof The equivalences between (i), (ii), and (iii) are contained in [4, Proposition 3.2]. The equivalence with (iv) follows from Bari's theorem, the fact that $J_{\mathcal{X}^*} = V_{\mathcal{X}}^* = (U_{\mathcal{X}}^{-1})^*$, and Lemma 2.2. Let us now prove the others implications.

(ii) ⇒ (v): Since

$$\delta_{n,\ell} = \langle U_{\mathcal{X}} x_n, U_{\mathcal{X}} x_\ell\rangle_{\ell^2} = \langle x_n, U_{\mathcal{X}}^* U_{\mathcal{X}} x_\ell\rangle_{\mathcal{H}},$$

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we have $x_\ell^* = U_{\mathcal{X}}^* U_{\mathcal{X}} x_\ell = U_{\mathcal{X}}^* e_\ell$, $\ell \geq 1$. Thus, $U_{\mathcal{X}^*} = (U_{\mathcal{X}}^*)^{-1}$ and $\mathcal{X}^*$ is a complete and minimal sequence. Now (v) follows from Lemma 2.2 and the implication (ii) ⇒ (i) applied to $\mathcal{X}^*$.

(v) ⇒ (i): Use the implication (i) ⇒ (v) applied to $\mathcal{X}^*$.

(i) ⇒ (vi): By Bari's theorem, $U_{\mathcal{X}}$ is a bounded and invertible operator from $\mathcal{H}$ onto ℓ^2. Moreover, for every $x = \sum_{n\geq N} a_n x_n$, we have

$$\|U_{\mathcal{X},N} x\|_{\ell^2}^2 = \Big\| \sum_{n\geq N} a_n e_n \Big\|_{\ell^2}^2 = \sum_{n\geq N} |a_n|^2,$$

and using (2.1), we get

$$c_N \|U_{\mathcal{X},N} x\|_{\ell^2}^2 \leq \|x\|_{\mathcal{H}}^2 \leq C_N \|U_{\mathcal{X},N} x\|_{\ell^2}^2.$$

Thus, $C_N^{-1/2} \leq \|U_{\mathcal{X},N}\| \leq c_N^{-1/2}$ and $\|U_{\mathcal{X},N}\| \to 1$ as N goes to ∞. Similarly, we prove that $\|U_{\mathcal{X},N}^{-1}\| \to 1$ as N goes to ∞.

(vi)⇒ (i): By Bari's theorem, $\mathcal{X}$ is a Riesz basis. Moreover, we have

$$\Big\| \sum_{n\geq N} a_n x_n \Big\|_{\mathcal{H}}^2 = \Big\| U_{\mathcal{X},N}^{-1}\Big(\sum_{n\geq N} a_n e_n \Big) \Big\|_{\mathcal{H}}^2 \leq \|U_{\mathcal{X},N}^{-1}\|^2 \sum_{n\geq N} |a_n|^2,$$

$$\sum_{n\geq N} |a_n|^2 = \Big\| U_{\mathcal{X},N}\Big(\sum_{n\geq N} a_n x_n \Big) \Big\|_{\ell^2}^2 \leq \|U_{\mathcal{X},N}\|^2 \Big\| \sum_{n\geq N} a_n x_n \Big\|_{\mathcal{H}}^2.$$

Then we obtain

$$\|U_{\mathcal{X},N}\|^{-2} \sum_{n\geq N} |a_n|^2 \leq \Big\| \sum_{n\geq N} a_n x_n \Big\|_{\mathcal{H}}^2 \leq \|U_{\mathcal{X},N}^{-1}\|^2 \sum_{n\geq N} |a_n|^2.$$

Since $\|U_{\mathcal{X},N}\|$ and $\|U_{\mathcal{X},N}^{-1}\|$ go to 1 as N goes to ∞, we get that $(x_n)_{n\geq 1}$ is a complete AOB for $\mathcal{H}$.

(i)⇒ (vii): The right inequality in (2.5) follows from Lemma 2.3. Since $(x_n^*)_{n\geq 1}$ is also a complete AOB for $\mathcal{H}$, for every $N \geq 1$, there are two positive constants c_N^*, C_N^* satisfying

$$c_N^* \sum_{n\geq N} |a_n|^2 \leq \Big\| \sum_{n\geq N} a_n x_n^* \Big\|_{\mathcal{H}}^2 \leq C_N^* \sum_{n\geq N} |a_n|^2, \tag{2.7}$$

and c_N^*, C_N^* go to 1 as N goes to ∞. Moreover, for every $f \in \mathcal{H} \ominus \operatorname{span}(x_1, x_2, \dots, x_{N-1})$, we have

$$f = \sum_{n\geq N} \langle f, x_n \rangle_{\mathcal{H}} x_n^*,$$

and (2.7) gives

$$\|f\|_{\mathcal{H}}^2 \leq C_N^* \sum_{n\geq N} |\langle f, x_n \rangle_{\mathcal{H}}|^2.$$

This proves the left inequality in (2.5).

(vii) ⇒ (v): Since

$$C_1^{*-1} \|f\|_{\mathcal{H}}^2 \leq \sum_{n\geq 1} |\langle f, x_n \rangle_{\mathcal{H}}|^2 \leq C_1 \|f\|_{\mathcal{H}}^2,$$

for every $f \in \mathcal{H}$, the operator $J_{\mathcal{X}^*}$ is invertible from $\mathcal{H}$ onto ℓ^2. Hence, according to Bari's theorem, the sequences $\mathcal{X}$ and $\mathcal{X}^*$ are Riesz basis for $\mathcal{H}$. Moreover, every $f = \sum_{n\geq N} a_n x_n^*$ with $(a_n)_{n\geq N} \in \ell^2$ satisfies $f \in \mathcal{H} \ominus \operatorname{span}(x_1, \dots, x_{N-1})$ and $\langle f, x_k \rangle_{\mathcal{H}} = a_k$, $k \geq N$. Hence by (2.5), we have

$$C_N^{*\,-1} \Big\| \sum_{n\geq N} a_n x_n^* \Big\|_{\mathcal{H}}^2 \leq \sum_{n\geq N} |a_n|^2 \leq C_N \Big\| \sum_{n\geq N} a_n x_n^* \Big\|_{\mathcal{H}}^2,$$

and $(x_n^*)_{n\geq 1}$ is an AOB.

(i) ⇒ (viii): Follows immediately from Lemma 2.3 and the fact that (i)⇒(v).

(viii) ⇒(i): Let $f = \sum_{n\geq N} a_n x_n$, where $(a_n)_{n\geq N}$ is a finitely supported sequence of complex numbers. Then applying the second inequality in (2.6) gives us

$$\sum_{n\geq N} |a_n|^2 \leq C_N^* \Big\| \sum_{n\geq N} a_n x_n \Big\|_{\mathcal{H}}^2.$$

On the other hand, by duality and Cauchy–Schwarz inequality, we have

$$\begin{aligned} \Big\| \sum_{n\geq N} a_n x_n \Big\|_{\mathcal{H}}^2 &= \sup_{\substack{g\in\mathcal{H} \\ \|g\|_{\mathcal{H}}\leq 1}} \Big| \Big\langle \sum_{n\geq N} a_n x_n, g \Big\rangle_{\mathcal{H}} \Big|^2 = \sup_{\substack{g\in\mathcal{H} \\ \|g\|_{\mathcal{H}}\leq 1}} \Big| \sum_{n\geq N} a_n \langle x_n, g \rangle_{\mathcal{H}} \Big|^2 \\ &\leq \sum_{n\geq N} |a_n|^2 \sup_{\substack{g\in\mathcal{H} \\ \|g\|_{\mathcal{H}}\leq 1}} \sum_{n\geq N} |\langle x_n, g \rangle_{\mathcal{H}}|^2 \leq C_N \sum_{n\geq N} |a_n|^2. \end{aligned}$$ ■

We now give two simple conditions on the Gram matrix, one necessary and the other one sufficient for a sequence to be an AOB.

Proposition 2.8 *Let $\mathcal{X} = (x_n)_{n\geq 1}$ be a sequence of normalized vectors in $\mathcal{H}$ and let $\Gamma_{\mathcal{X}} = (\Gamma_{n,p})_{n,p\geq 1}$ be its Gram matrix. The following hold:*

(i) *If*

$$\lim_{N\to\infty} \Big(\sup_{n\geq N} \sum_{\substack{p\geq N \\ p\neq n}} |\Gamma_{n,p}| \Big) = 0,$$

then $(x_n)_{n\geq 1}$ is an AOS.

(ii) *If $(x_n)_{n\geq 1}$ is an AOB, then*

$$\lim_{n\to\infty} \Big(\sum_{\substack{p\geq 1 \\ p\neq n}} |\Gamma_{n,p}|^2 \Big) = 0.$$

Proof (i) Let $(a_n)_{n\geq 1}$ be a finitely supported sequence of complex numbers and denote by

$$\varepsilon_N = \sup_{n\geq N} \sum_{\substack{p\geq N \\ p\neq n}} |\Gamma_{n,p}|.$$

Write

$$\Big\| \sum_{n\geq N} a_n x_n \Big\|_{\mathcal{H}}^2 = \sum_{n,p\geq N} a_n \overline{a_p} \langle x_n, x_p \rangle_{\mathcal{H}} = \sum_{n\geq N} |a_n|^2 + \sum_{\substack{n,p\geq N \\ n\neq p}} a_n \overline{a_p}\, \Gamma_{n,p}.$$

We will prove the inequality

$$\Big|\sum_{\substack{n,p\geq N\\ n\neq p}} a_n\overline{a_p}\,\Gamma_{n,p}\Big| \leq \varepsilon_N \sum_{n\geq N}|a_n|^2. \tag{2.9}$$

Using that $ab \leq (a^2+b^2)/2$, for every real numbers a and b, and $|\Gamma_{n,p}| = |\Gamma_{p,n}|$, we have

$$\begin{aligned}\Big|\sum_{\substack{n,p\geq N\\ n\neq p}} a_n\overline{a_p}\,\Gamma_{n,p}\Big| &\leq \frac{1}{2}\sum_{\substack{n,p\geq N\\ n\neq p}}(|a_n|^2+|a_p|^2)|\Gamma_{n,p}|\\ &= \sum_{\substack{n,p\geq N\\ n\neq p}}|a_n|^2|\Gamma_{n,p}| = \sum_{n\geq N}|a_n|^2\sum_{\substack{p\geq N\\ p\neq n}}|\Gamma_{n,p}|,\end{aligned}$$

which gives (2.9). Therefore,

$$(1-\varepsilon_N)\sum_{n\geq N}|a_n|^2 \leq \Big\|\sum_{n\geq N}a_n x_n\Big\|_{\mathcal{H}}^2 \leq (1+\varepsilon_N)\sum_{n\geq N}|a_n|^2.$$

Since $\varepsilon_N \to 0$ as $N\to\infty$, the sequence $(x_n)_{n\geq1}$ is an AOS.

(ii) Since $\mathcal{X} = (x_n)_{n\geq1}$ is an AOB, we know from Theorem 2.4 that $\Gamma_{\mathcal{X}} = I + K$, with K compact. In particular, we have

$$\|(\Gamma_{\mathcal{X}} - I)e_n\|_{\ell^2}^2 = \|Ke_n\|_{\ell^2}^2 \to 0, \quad \text{as } n\to\infty.$$

It remains to note that

$$\|(\Gamma_{\mathcal{X}} - I)e_n\|_{\ell^2}^2 = \sum_{\substack{p\geq1\\ p\neq n}}|\Gamma_{n,p}|^2. \qquad \blacksquare$$

We end this subsection with two stability results. The first one is inspired by an analogue result of Baranov for the Riesz basis property [2]. The second one is a generalization of a result appearing in [4, Proposition 3.3].

Proposition 2.10 *Let $(x_n)_{n\geq1}$ and $(x'_n)_{n\geq1}$ be two sequences in $\mathcal{H}$. Assume that there exists $N_0\in\mathbb{N}$ such that for all $N\geq N_0$ there is $\varepsilon_N > 0$ verifying*

$$\sum_{n\geq N}|\langle x, x_n - x'_n\rangle|^2 \leq \varepsilon_N\|x\|_{\mathcal{H}}^2, \tag{2.11}$$

for every $x\in\mathcal{H}$ and $\lim_{N\to\infty}\varepsilon_N = 0$. Then $(x_n)_{n\geq1}$ is an AOS if and only if $(x'_n)_{n\geq1}$ is an AOS. Furthermore, if $N_0 = 1$ and ε_1 is sufficiently small, then $(x_n)_{n\geq1}$ is a complete AOB for $\mathcal{H}$ if and only if $(x'_n)_{n\geq1}$ is a complete AOB for $\mathcal{H}$.

Proof Let $(a_n)_n$ be a finitely supported sequence of complex numbers. For the first part, since (2.11) is symmetric with respect to x_n and x'_n, it is sufficient to show that if $(x_n)_{n\geq1}$ is an AOS and if c_N and C_N are the constants appearing in (2.1), then we have

$$(c_N + \varepsilon_N - 2\sqrt{c_N\varepsilon_N})\sum_{n\geq N}|a_n|^2 \leq \Big\|\sum_{n\geq N}a_n x'_n\Big\|_{\mathcal{H}}^2 \leq (C_n + \varepsilon_N + 2\sqrt{C_N\varepsilon_N})\sum_{n\geq N}|a_n|^2.$$

For simplicity, define $g_N := \sum_{n\geq N}a_n x_n$ and $g'_N = \sum_{n\geq N}a_n x'_n$ and write

$$\|g_N - g'_N\|_{\mathcal{H}}^2 = \Big\langle g_N - g'_N, \sum_{n\geq N}a_n(x_n - x'_n)\Big\rangle_{\mathcal{H}} = \sum_{n\geq N}\overline{a_n}\langle g_N - g'_N, x_n - x'_n\rangle.$$

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Then using the Cauchy–Schwarz inequality and (2.11), we get

$$\begin{aligned}\|g_N - g'_N\|^2_{\mathcal{H}} &\le \Big(\sum_{n\ge N} |a_n|^2\Big)^{1/2}\Big(\sum_{n\ge N} |\langle g_N - g'_N, x_n - x'_n\rangle|^2\Big)^{1/2}\\ &\le \sqrt{\varepsilon_N}\Big(\sum_{n\ge N}|a_n|^2\Big)^{1/2}\|g_N - g'_N\|_{\mathcal{H}}.\end{aligned}$$

We thus have $\|g_N - g'_N\|_{\mathcal{H}} \le \sqrt{\varepsilon_N}\|(a_n)_{n\ge N}\|_{\ell^2}$. We now obtain the desired inequalities as follows:

$$\begin{aligned}\Big\|\sum_{n\ge N} a_n x'_n\Big\|_{\mathcal{H}} &\ge \Big\|\sum_{n\ge N} a_n x_n\Big\|_{\mathcal{H}} - \Big\|\sum_{n\ge N} a_n(x_n - x'_n)\Big\|_{\mathcal{H}}\\ &\ge \sqrt{c_N}\Big(\sum_{n\ge N}|a_n|^2\Big)^{1/2} - \sqrt{\varepsilon_N}\Big(\sum_{n\ge N}|a_n|^2\Big)^{1/2}\\ &= (\sqrt{c_N} - \sqrt{\varepsilon_N})\Big(\sum_{n\ge N}|a_n|^2\Big)^{1/2}.\end{aligned}$$

And similarly,

$$\Big\|\sum_{n\ge N} a_n x'_n\Big\|_{\mathcal{H}} \le (\sqrt{C_N} + \sqrt{\varepsilon_n})\Big(\sum_{n\ge N}|a_n|^2\Big)^{1/2}.$$

Assume now that $(x_n)_{n\ge1}$ is a complete AOB for $\mathcal{H}$. Then we know that the operator $J_{\mathcal{X}^*}$, defined by $J_{\mathcal{X}^*}x = (\langle x, x_n\rangle)_{n\ge1}$, is an isomorphism from $\mathcal{H}$ onto ℓ^2. The inequality (2.11) for $N = 1$ implies that $\|J_{\mathcal{X}^*} - J_{\mathcal{X}'^*}\| \le \sqrt{\varepsilon_1}$. Therefore for ε_1 sufficiently small, the operator $J_{\mathcal{X}'^*}$ is also an isomorphism from $\mathcal{H}$ onto ℓ^2. It follows from Bari's theorem that $(x'_n)_{n\ge1}$ is a Riesz basis for $\mathcal{H}$ and thus a complete AOB for $\mathcal{H}$. ■

Proposition 2.12 *Let $\mathcal{X} = (x_n)_{n\ge1}$ be a complete AOB for $\mathcal{H}$ and let $(x'_n)_{n\ge1}$ be a sequence in $\mathcal{H}$ satisfying*

$$\sum_{n\ge1}\|x_n - x'_n\|^2_{\mathcal{H}} < \|U_{\mathcal{X}}\|^{-2}.$$

Then $(x'_n)_{n\ge1}$ is a complete AOB for $\mathcal{H}$.

Proof Let $x \in \mathcal{H}$. Then we have

$$\sum_{n\ge N}|\langle x, x_n - x'_n\rangle|^2 \le \|x\|^2_{\mathcal{H}}\sum_{n\ge N}\|x_n - x'_n\|^2_{\mathcal{H}} = \varepsilon_N\|f\|^2_{\mathcal{H}},$$

where $\varepsilon_N = \sum_{n\ge N}\|x_n - x'_n\|^2_{\mathcal{H}}$. It follows by hypothesis that $\varepsilon_N \to 0$ as N goes to ∞. Hence, by Proposition 2.10, the sequence $(x'_n)_{n\ge1}$ is an AOS. It remains to prove that $(x'_n)_{n\ge1}$ is minimal and complete. For that purpose, define $T\colon\mathcal{H}\to\mathcal{H}$ by $T(x_n) = x'_n$, $n\ge1$, and let $\delta > 0$ such that

$$\sum_{n\ge1}\|x_n - x'_n\|^2_{\mathcal{H}} \le \delta < \|U_{\mathcal{X}}\|^{-2}.$$

Then for every finitely supported sequence of complex numbers $(a_n)_{n\geq1}$, we have

$$\begin{aligned}\Big\|(I-T)\sum_{n\geq1}a_nx_n\Big\| &= \Big\|\sum_{n\geq1}a_n(x_n-x_n')\Big\|\\ &\leq\Big(\sum_{n\geq1}|a_n|^2\Big)^{1/2}\Big(\sum_{n\geq1}\|x_n-x_n'\|^2\Big)^{1/2}\\ &\leq\sqrt{\delta}\Big(\sum_{n\geq1}|a_n|^2\Big)^{1/2}\leq\sqrt{\delta}\|U_{\mathcal{X}}\|\Big\|\sum_{n\geq1}a_nx_n\Big\|.\end{aligned}$$

Since $(x_n)_{n\geq1}$ is a Riesz basis for $\mathcal{H}$, the operator $I-T$ is bounded and $\|I-T\|\leq\sqrt{\delta}\|U_{\mathcal{X}}\|<1$. Thus, $T=I-(I-T)$ is bounded and invertible. In particular, we deduce that $(x_n')_{n\geq1}$ is complete and minimal. ∎

2.2 De Branges–Rovnyak Spaces

Let H^∞ denote the space of bounded analytic functions on the upper half-plane $\mathbb{C}_+=\{z\in\mathbb{C}:\Im(z)>0\}$ normed by $\|f\|_\infty=\sup_{z\in\mathbb{C}_+}|f(z)|$ and let $H_1^\infty=\{g\in H^\infty:\|g\|_\infty\leq1\}$ be the closed unit ball of H^∞. For $b\in H_1^\infty$, the de Branges–Rovnyak space $\mathcal{H}(b)$ is the reproducing kernel Hilbert space of analytic functions on $\mathbb{C}_+$ whose kernel is given by

$$k_\lambda^b(z)=\frac{i}{2\pi}\frac{1-\overline{b(\lambda)}b(z)}{z-\overline{\lambda}},\qquad\lambda,z\in\mathbb{C}_+.$$

By definition, $f(\lambda)=\langle f,k_\lambda^b\rangle_b$ for all $f\in\mathcal{H}(b)$ and $\lambda\in\mathbb{C}_+$, where $\langle\cdot,\cdot\rangle_b$ represents the inner product in $\mathcal{H}(b)$. The space $\mathcal{H}(b)$ can also be defined as the range space $(I-T_bT_b^*)^{1/2}H^2$ equipped with the norm that makes $(I-T_bT_b^*)^{1/2}$ a partial isometry. Here H^2 is the Hardy space of $\mathbb{C}_+$, that is, the space of analytic functions f on $\mathbb{C}_+$ verifying

$$\|f\|_2^2=\sup_{y>0}\Big(\int_{-\infty}^{\infty}|f(x+iy)|^2\,dx\Big)<\infty,$$

and T_φ is the Toeplitz operator on H^2 with symbol $\varphi\in L^\infty(\mathbb{R})$ defined by $T_\varphi(f)=P_+(\varphi f)$, $f\in H^2$, where P_+ denotes the orthogonal projection of $L^2(\mathbb{R})$ onto H^2.

These spaces (and, more precisely, their general vector-valued version) were introduced by de Branges and Rovnyak [6,7] as universal model spaces for Hilbert space contractions. Thanks to the pioneer works of Sarason, we know that de Branges–Rovnyak spaces play an important role in numerous questions of complex analysis and operator theory. The book [17] is the classical reference for $\mathcal{H}(b)$ spaces. See also the recent monograph [9].

In the special case where $b=\Theta$ is an inner function (that is, $|\Theta|=1$ a.e. on $\mathbb{R}$), the operator $(\mathrm{Id}-T_\Theta T_\Theta^*)^{1/2}$ is an orthogonal projection and $\mathcal{H}(\Theta)$ becomes a closed (ordinary) subspace of H^2 that coincides with the so-called model subspace

$$K_\Theta=H^2\ominus\Theta H^2=H^2\cap\Theta\overline{H^2}.$$

For the model space theory, see [10,15].

It turns out that the boundary behavior of functions in $\mathcal{H}(b)$ is controlled by the boundary behavior of the function b itself. More precisely, let $b=BI_\mu O_b$ be the canonical factorization of b, where

$$B(z) = \prod_n e^{i\alpha_n} \frac{z - z_n}{z - \overline{z_n}}$$

is a Blaschke product, the singular inner function I_μ is given by

$$I_\mu(z) = \exp\Big(iaz - \frac{i}{\pi} \int_{\mathbb{R}} \Big(\frac{1}{z-t} + \frac{t}{t^2+1} \Big) \, d\mu(t) \Big)$$

with a positive measure μ on $\mathbb{R}$ singular with respect to Lebesgue measure dt such that $\int_{\mathbb{R}} (1+t^2)^{-1} \, d\mu(t) < \infty$ and $a \geq 0$, and O_b is the outer function

$$O_b(z) = \exp\Big(\frac{i}{\pi} \int_{\mathbb{R}} \Big(\frac{1}{z-t} + \frac{t}{t^2+1} \Big) \log|b(t)| \, dt \Big).$$

For $x_0 \in \mathbb{R}$ and $\ell \geq 1$, let

$$S_\ell(x_0) := \sum_{n=1}^{\infty} \frac{\Im(z_n)}{|x_0 - z_n|^\ell} + \int_{\mathbb{R}} \frac{d\mu(t)}{|x_0 - t|^\ell} + \int_{\mathbb{R}} \frac{|\log|b(t)||}{|x_0 - t|^\ell} \, dt$$

and $E_\ell(b) = \{x_0 \in \mathbb{R} : S_\ell(x_0) < \infty\}$. The set $E_\ell(b)$ is related to nontangential boundary limits of functions (and their derivatives) in $\mathcal{H}(b)$. More precisely, if $S_2(x_0) < \infty$, then it was proved in [8] that for each $f \in \mathcal{H}(b)$, the nontangential limit

$$f(x_0) = \lim_{\substack{z \longrightarrow x_0 \\ \sphericalangle}} f(z)$$

exists, the function

$$k_{x_0}^b(z) = \frac{i}{2\pi} \frac{1 - \overline{b(x_0)} b(z)}{z - x_0}, \qquad z \in \mathbb{C}_+,$$

belongs to $\mathcal{H}(b)$, and $\langle f, k_{x_0}^b \rangle_b = f(x_0)$, $f \in \mathcal{H}(b)$. In that case, we also have $\|k_{x_0}^b\|_b^2 = S_2(x_0) = |b'(x_0)|$. Moreover, if $S_4(x_0) < \infty$, for every function $f \in \mathcal{H}(b)$, $f(z)$, and $f'(z)$ have finite limits as z tends nontangentially to x_0. In [3], a Bernstein's type inequality is proved in the context of $\mathcal{H}(b)$ spaces. To state this inequality, we need to introduce the following kernel. For $z_0 \in \mathbb{C}_+ \cup E_4(b)$, we define

$$\mathfrak{K}_{z_0}^b(t) = \overline{b(z_0)} \frac{2 - \overline{b(z_0)} b(t)}{(t - \overline{z_0})^2}.$$

It is not difficult to see that $\rho^{1/q} \mathfrak{K}_{z_0}^b \in L^q(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} \frac{|\log|b(t)||}{|t - z_0|^{2q}} \, dt < \infty,$$

where $\rho(t) = 1 - |b(t)|^2$, $t \in \mathbb{R}$. Now, for $1 < p \leq 2$ and q its conjugate exponent, we define

$$w_p(z) := \min\Big\{ \|(k_z^b)^2\|_q^{-p/(p+1)}, \|\rho^{1/q} \mathfrak{K}_z^b\|_q^{-p/(p+1)} \Big\}, \qquad z \in \overline{\mathbb{C}_+},$$

where $\|\cdot\|_q$ denotes the $L^q(\mathbb{R})$-norm with respect to Lebesgue measure dt on $\mathbb{R}$.

We assume that $w_p(x) = 0$, whenever $x \in \mathbb{R}$ and at least one of the functions $(k_x^b)^2$ or $\rho^{1/q} \mathfrak{K}_x^b$ is not in $L^q(\mathbb{R})$. Note that if $f \in \mathcal{H}(b)$ and $1 < p \leq 2$, then $f' w_p$ is well defined on $\mathbb{R}$. Indeed, if $S_4(x) < \infty$, then $f'(x)$ and $w_p(x)$ are finite. If $S_4(x) = \infty$, then as shown in [3, 8], $\|(k_x^b)^2\|_q = \infty$, which, by definition, implies that $w_p(x) = 0$, and thus we can assume that $(f'w_p)(x) = 0$. Moreover, note that in the inner case, we have $\rho(t) = 0$ for a.e. $t \in \mathbb{R}$, and the second term in the definition of the weight w_p disappears. We will need two useful estimates for the weight w_p. The first one, proved in [3, Lemma 3.5], is valid for every function $b \in H_1^\infty$: there is a constant $C = C(p) > 0$ such that

$$w_p(z) \geq C \frac{\Im z}{(1-|b(z)|)^{\frac{p}{q(p+1)}}}, \qquad (z \in \mathbb{C}_+). \tag{2.13}$$

The second one, proved in [1] and valid when $b = \Theta$ is an inner function, says that there is two constants $C_1, C_2 > 0$ such that

$$C_1 v_0(x) \leq w_p(x) \leq C_2 |\Theta'(x)|^{-1}, \qquad (x \in \mathbb{R}),$$

where $v_0(x) = \min(d_0(x), |\Theta'(x)|^{-1})$, $d_0(x) = \operatorname{dist}(x, \sigma(\Theta))$ and $\sigma(\Theta)$ is the spectrum of the inner function Θ defined as the set of all $\zeta \in \overline{\mathbb{C}_+} \cup \{\infty\}$ such that

$$\liminf_{z \to \zeta \, z \in \mathbb{C}_+} |\Theta(z)| = 0.$$

It is known that every function $f \in K_\Theta$ has an analytic continuation through $\mathbb{R} \setminus (\sigma(\Theta) \cap \mathbb{R})$. Moreover, the quantity v_0 has a simple geometrical meaning related to the sublevel sets $\Omega(\Theta, \delta) = \{z \in \mathbb{C}_+ : |\Theta(z)| \leq \delta\}$. Namely, $v_0(x) \asymp \operatorname{dist}(x, \Omega(\Theta, \delta))$ with the constants depending only on $\delta \in (0,1)$.

We also recall that a Borel measure μ on the closed upper half-plane $\overline{\mathbb{C}_+}$ is said to be a *Carleson measure* if there is a constant $C > 0$ such that

$$\mu(S(x,h)) \leq C\,h, \tag{2.14}$$

for all squares $S(x,h) = [x, x+h] \times [0,h]$, $x \in \mathbb{R}$, $h > 0$, with the lower side on the real axis. We denote the class of Carleson measures by $\mathcal{C}$, and the best constant satisfying (2.14) is called the *Carleson constant* of μ and is denoted by C_μ. Recall that, according to a classical theorem of Carleson, $\mu \in \mathcal{C}$ if and only if $H^p \subset L^p(\mu)$ for some (all) $p > 0$. In [3], it is proved that if $\mu \in \mathcal{C}$, $1 < p < 2$, then there exists a constant $K = K(\mu, p) > 0$ such that

$$\|f' w_p\|_{L^2(\mu)} \leq K \|f\|_b, \qquad f \in \mathcal{H}(b). \tag{2.15}$$

In other words, the map $f \mapsto f' w_p$ is a bounded operator from $\mathcal{H}(b)$ into $L^2(\mu)$. If $p = 2$, then this map is of weak type $(2,2)$ as an operator from $\mathcal{H}(b)$ to $L^2(\mu)$.

3 Some Stability Results

This section contains results about the stability of AOBs under certain perturbations. We will often use techniques developed by Baranov [2] concerning the stability problem for the Riesz bases for K_Θ.

For $\lambda \in \mathbb{C}_+ \cup E_2(b)$, we denote by κ_λ^b the normalized reproducing kernel at the point λ, that is, $\kappa_\lambda^b = k_\lambda^b / \|k_\lambda^b\|_b$. Let $(\lambda_n)_{n\geq1} \subset \mathbb{C}_+ \cup E_2(b)$ and $G = \bigcup_n G_n \subset \mathbb{C}_+ \cup E_2(b)$. We say that G is an *admissible set* for $(\lambda_n)_{n\geq1}$ if it satisfies the following properties:

(i) $\lambda_n \in G_n$.

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(ii) For every $z_n \in G_n$, we have

$$\lim_{n\to\infty} \frac{\|k^b_{z_n}\|_b}{\|k^b_{\lambda_n}\|_b} = 1.$$

(iii) For every $z_n \in G_n$, the measure $\nu = \sum_n \delta_{[\lambda_n, z_n]}$ is a Carleson measure. Moreover, the Carleson constants C_ν of such measures (see (2.14)) are uniformly bounded with respect to z_n. Here $[\lambda_n, z_n]$ is the straight line interval with the endpoints λ_n and z_n, and $\delta_{[\lambda_n, z_n]}$ is the Lebesgue measure on the interval.

For $(\lambda_n)_{n\geq1} \subset \mathbb{C}_+$ to be such that the sequence $(\kappa^b_{\lambda_n})_{n\geq1}$ is an AOS in $\mathcal{H}(b)$, we show that there always exist non-trivial admissible sets $G = \bigcup_n G_n$. More precisely, we can take

$$G_n := \{z \in \mathbb{C}_+ : |z - \lambda_n| < \varepsilon_n \Im\lambda_n\},$$

where $(\varepsilon_n)_n$ is any sequence of positive numbers tending to 0. We first begin with a technical lemma.

Lemma 3.1 *Let $b \in H^\infty_1$, $(\varepsilon_n)_n$ be a sequence of positive numbers tending to* 0 *and let $(\lambda_n)_n$ and $(\mu_n)_n$ be two sequences in $\mathbb{C}_+$ satisfying*

$$|\lambda_n - \mu_n| \leq \varepsilon_n \Im\lambda_n, \qquad n \geq 1. \tag{3.2}$$

Then

$$\lim_{n\to\infty} \frac{\|k^b_{\mu_n}\|_b}{\|k^b_{\lambda_n}\|_b} = 1.$$

Proof We easily check from (3.2) that

$$1 - \varepsilon_n \leq \frac{\Im\mu_n}{\Im\lambda_n} \leq 1 + \varepsilon_n, \qquad n \geq 1. \tag{3.3}$$

Since

$$\|k^b_z\|^2_b = \frac{1 - |b(z)|^2}{4\pi\Im z},$$

it is sufficient to prove that

$$\frac{1-\varepsilon_n}{1+\varepsilon_n} \leq \frac{1 - |b(\lambda_n)|}{1 - |b(\mu_n)|} \leq \frac{1+\varepsilon_n}{1-\varepsilon_n}. \tag{3.4}$$

Using the Schwarz–Pick inequality, we have

$$\Big|\frac{b(\lambda_n) - b(\mu_n)}{1 - \overline{b(\lambda_n)}b(\mu_n)}\Big| \leq \Big|\frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu_n}}\Big| \leq \frac{|\lambda_n - \mu_n|}{\Im\lambda_n} \leq \varepsilon_n,$$

and (3.4) follows from [18, Lemma 7], which says that if $\lambda, \mu \in \mathbb{D}$ and satisfies

$$\Big|\frac{\lambda - \mu}{1 - \overline{\lambda}\mu}\Big| \leq \varepsilon,$$

then

$$\frac{1-\varepsilon}{1+\varepsilon} \leq \frac{1-|\lambda|}{1-|\mu|} \leq \frac{1+\varepsilon}{1-\varepsilon}.$$ ■

Corollary 3.5 *Let $b \in H_1^\infty$, $(\lambda_n)_{n\geq 1} \subset \mathbb{C}_+$ be such that $(\kappa^b_{\lambda_n})_n$ is an AOS in $\mathcal{H}(b)$, and let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive numbers tending to 0. Define*

$$G_n := \{z \in \mathbb{C}_+ : |z - \lambda_n| < \varepsilon_n \Im\lambda_n\}, \qquad n \geq 1.$$

Then the set $G = \bigcup_n G_n$ is an admissible set for $(\lambda_n)_{n\geq 1}$.

Proof It is obvious that the sets G_n satisfy (i) and that condition (ii) follows from Lemma 3.1. According to Proposition 2.8, there exists a constant $C > 0$ such that for every $n \geq 1$, we have

$$\sum_{p\geq 1} |\Gamma_{n,p}|^2 \leq C,$$

where $\Gamma_{n,p} = \langle \kappa^b_{\lambda_n}, \kappa^b_{\lambda_p}\rangle_b$. Since

$$|\Gamma_{n,p}|^2 = \frac{4\Im\lambda_n \Im\lambda_p}{|\lambda_p - \overline{\lambda_n}|^2} \frac{|1 - \overline{b(\lambda_n)} b(\lambda_p)|^2}{(1 - |b(\lambda_n|^2)(1 - |b(\lambda_p)|^2} \geq \frac{\Im\lambda_n \Im\lambda_p}{|\lambda_p - \overline{\lambda_n}|^2},$$

we obtain

$$\sum_{p\geq 1} \frac{\Im\lambda_n \Im\lambda_p}{|\lambda_p - \overline{\lambda_n}|^2} \leq C.$$

It is known (see, for instance, [14, Lecture VII]) that this condition implies that the measure $\nu = \sum_n \Im\lambda_n \delta_{\lambda_n}$ is a Carleson measure. Therefore, the sets G_n also satisfy (iii). ■

Note that in [2, 3], similar sets were considered in connection with the stability of the Riesz basis property. In that situation, condition (ii) can be replaced by the weaker condition that there exist two positive constants $c, C > 0$ such that

$$c \leq \frac{\|k^b_{z_n}\|_b}{\|k^b_{\lambda_n}\|_b} \leq C, \qquad z_n \in G_n, n \geq 1,$$

and the set G_n can be taken as $G_n := \{z \in \mathbb{C}_+ : |z - \lambda_n| < r\Im\lambda_n\}$, for sufficiently small $r > 0$.

Theorem 3.6 *Let $b \in H_1^\infty$, $1 < p < 2$, and $(\lambda_n)_n \subset \mathbb{C}_+ \cup E_2(b)$ be such that $(\kappa^b_{\lambda_n})_{n\geq 1}$ is an AOS in $\mathcal{H}(b)$. Assume that $G = \cup_{n\geq 1} G_n$ is an admissible set for $(\lambda_n)_{n\geq 1}$, and let $\mu_n \in G_n$, $n \geq 1$. If there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ there is $\varepsilon_N > 0$ verifying*

$$\sup_{n\geq N} \frac{1}{\|k^b_{\lambda_n}\|_b^2} \int_{[\lambda_n, \mu_n]} w_p^{-2}(z)|dz| \leq \varepsilon_N \tag{3.7}$$

and $\lim_{N\to\infty} \varepsilon_N = 0$, then the sequence $(\kappa^b_{\mu_n})_{n\geq 1}$ is an AOS in $\mathcal{H}(b)$. Moreover, if $(\kappa^b_{\lambda_n})_{n\geq 1}$ is a complete AOB for $\mathcal{H}(b)$ and if we can take $N_0 = 1$ and ε_1 sufficiently small, then $(\kappa^b_{\mu_n})_{n\geq 1}$ is also a complete AOB for $\mathcal{H}(b)$.

Proof Let

$$h^b_n = \frac{k^b_{\mu_n}}{\|k^b_{\lambda_n}\|_b}, \qquad n \geq 1.$$

Since by condition (ii), $\|k^b_{\mu_n}\|_b/\|k^b_{\lambda_n}\|_b \to 1$ as $n \to \infty$, we easily see that $(\kappa^b_{\mu_n})_{n\geq 1}$ is an AOS if and only if $(h^b_n)_{n\geq 1}$ is an AOS. In view of Proposition 2.10, it is then sufficient to check the estimate

$$\sum_{n\geq N} |\langle f, \kappa^b_{\lambda_n} - h^b_n\rangle|^2 \lesssim \varepsilon_N \|f\|^2_b, \qquad f \in \mathcal{H}(b). \tag{3.8}$$

It follows from (3.7) and [3, Corollary 5.4] that every function $f \in \mathcal{H}(b)$ is differentiable on $]\lambda_n, \mu_n[$, and the set of all functions in $\mathcal{H}(b)$ that are continuous on $[\lambda_n, \mu_n]$ is dense in $\mathcal{H}(b)$. Therefore, it is sufficient to prove (3.8) for functions $f \in \mathcal{H}(b)$ continuous on $[\lambda_n, \mu_n]$. Then

$$|\langle f, \kappa^b_{\lambda_n} - h^b_n\rangle|^2 = \frac{|f(\lambda_n) - f(\mu_n)|^2}{\|k^b_{\lambda_n}\|^2_b} = \frac{1}{\|k^b_{\lambda_n}\|^2_b}\Big|\int_{[\lambda_n,\mu_n]} f'(z)\,dz\Big|^2.$$

By the Cauchy–Schwartz inequality and (3.7), we get

$$|\langle f, \kappa^b_{\lambda_n} - h^b_n\rangle|^2 \leq \varepsilon_N \int_{[\lambda_n,\mu_n]} |f'(z) w_p(z)|^2\,|dz|.$$

for $n \geq N$. It follows from condition (iii) that $\nu = \sum_n \delta_{[\lambda_n,\mu_n]}$ is a Carleson measure with a constant C_ν that does not exceed some absolute constant depending only on G. Hence, according to (2.15), we have

$$\sum_{n\geq N} |\langle f, \kappa^b_{\lambda_n} - h^b_n\rangle|^2 \leq \varepsilon_N \sum_{n\geq N}\int_{[\lambda_n,\mu_n]} |f'(z) w_p(z)|^2\,|dz| \leq \varepsilon_N \|f' w_p\|^2_{L^2(\nu)} \leq K\varepsilon_N \|f\|^2_b.$$

Since $\varepsilon_N \to 0$ as $N \to \infty$, Proposition 2.10 implies that $(h^b_n)_{n\geq 1}$ is an AOS, and so is $(\kappa^b_{\mu_n})_{n\geq 1}$. The second part for complete AOB follows also from Proposition 2.10. ∎

Remark 3.9 If $(\lambda_n)_{n\geq 1} \subset \mathbb{C}_+$ and $(\kappa^b_{\lambda_n})_{n\geq 1}$ is a complete AOB for $\mathcal{H}(b)$, then it is sufficient to have (3.7) for N large enough to get that $(\kappa^b_{\mu_n})_{n\geq 1}$ is a complete AOB for $\mathcal{H}(b)$. Indeed, apply Theorem 3.6 with the sequence

$$\gamma_n = \begin{cases} \lambda_n & \text{if } n \leq N, \\ \mu_n & \text{if } n > N, \end{cases}$$

and part (i) of the following lemma which shows that we can replace a finite number of terms keeping the minimality and completeness.

Lemma 3.10 *Let $b \in H^\infty_1$ and $\Lambda = (\lambda_n)_{n\geq 1} \subset \mathbb{C}_+$.*

(i) *Assume that $(k^b_{\lambda_n})_{n\geq 1}$ is a minimal and complete sequence in $\mathcal{H}(b)$. Then for every $\mu \in \mathbb{C}_+ \setminus \Lambda$, the system $\{k^b_{\lambda_n}\}_{n\geq 2} \cup \{k^b_\mu\}$ is still minimal and complete in $\mathcal{H}(b)$.*

(ii) *Assume that $(k^b_{\lambda_n})_{n\geq 1}$ is not complete in $\mathcal{H}(b)$. Then, for every $\mu \in \mathbb{C}_+ \setminus \Lambda$, the system $\{k^b_{\lambda_n}\}_{n\geq 1} \cup \{k^b_\mu\}$ is minimal.*

This result was proved in [12] for the inner case. The general version is proved similarly; see [9, Lemma 31.2]. We also need a version of this result for real frequencies. We do not know if it is true in general, but we prove it when $b = \Theta$ is an inner function. The proof is based on the following key lemma.

Lemma 3.11 *Let Θ be an inner function, $x_0 \in \mathbb{R} \setminus (\sigma(\Theta) \cap \mathbb{R})$, and let $f \in K_\Theta$ be such that $f(x_0) = 0$. Then there exists a Blaschke factor J such that $\{J = -1\} = \{x_0\}$ and $f/(1+J) \in K_\Theta$.*

Proof Fix any $a > 0$, define

$$\gamma = x_0 + ia \in \mathbb{C}_+ \quad \text{and} \quad J(z) = b_\gamma(z) = \frac{z-\gamma}{z-\overline{\gamma}}.$$

Then

$$1 + J(z) = \frac{2(z-x_0)}{z-\overline{\gamma}} \quad \text{and} \quad \{J=-1\} = \{x_0\}.$$

To check that $f/(1+J) \in K_\Theta$, first note that

$$\frac{f(z)}{1+J(z)} = \frac{1}{2}\Big(f(z) + ia\frac{f(z)}{z-x_0}\Big).$$

Since $x_0 \in \mathbb{R} \setminus (\sigma(\Theta) \cap \mathbb{R})$, the function f extends analytically through a neighbourhood V_{x_0} of x_0, and we have

$$|f(z)| \le C|z-x_0|, \qquad z \in V_{x_0}.$$

Hence, $f/(z-x_0) \in L^2(\mathbb{R}) \cap \mathcal{N}^+ = H^2$, where $\mathcal{N}^+$ is the Smirnov class. We deduce that $f/(1+J) \in H^2$. It remains to note that

$$\frac{\Theta\overline{f}}{1+\overline{J}} = \frac{J\Theta\overline{f}}{1+J},$$

and since $f \in K_\Theta$, we have $\Theta\overline{f} \in H^2$. Thus $\Theta\overline{f}/(1+\overline{J}) \in L^2(\mathbb{R}) \cap \mathcal{N}^+ = H^2$. Finally, $f/(1+J) \in H^2 \cap \Theta\overline{H^2} = K_\Theta$. ∎

Lemma 3.12 *Let Θ be an inner function and let $(t_n)_{n\ge1} \subset E_2(\Theta)$.*

(i) *Assume that $t_1 \notin \sigma(\Theta)$ and $(k_{t_n}^\Theta)_{n\ge1}$ is a minimal and complete sequence in K_Θ. Then for every $t \in \mathbb{R} \setminus (\sigma(\Theta) \cap \mathbb{R})$ and $t \ne t_n$, $n \ge 1$, the system $\{k_{t_n}^\Theta\}_{n\ge2} \cup \{k_t^\Theta\}$ is still minimal and complete in K_Θ.*

(ii) *Assume that $t_n \notin \sigma(\Theta)$, $n \ge 1$, and $(k_{t_n}^\Theta)_{n\ge1}$ is not complete in K_Θ. Then for every $t \in \mathbb{R} \setminus (\sigma(\Theta) \cap \mathbb{R})$ and $t \ne t_n$, $n \ge 1$, the system $\{k_{t_n}^\Theta\}_{n\ge1} \cup \{k_t^\Theta\}$ is minimal.*

Proof (i) First, let us prove that the system $\{k_{t_n}^\Theta\}_{n\ge2} \cup \{k_t^\Theta\}$ is complete. Let $f \in K_\Theta$ such that $f(t_n) = 0$, $n \ge 2$ and $f(t) = 0$. According to Lemma 3.11, there is an inner function J such that $\{J = -1\} = \{t\}$ and $f/(1+J) \in K_\Theta$. Define

$$g = \frac{J - J(t_1)}{1+J} f = f - (J(t_1)+1)\frac{f}{1+J}.$$

The function g belongs to K_Θ and it vanishes at every point t_n, $n \ge 1$. Hence, the completeness of $(k_{t_n}^\Theta)_{n\ge1}$ implies that $g \equiv 0$ and thus $f \equiv 0$. This proves the completeness of $\{k_{t_n}^\Theta\}_{n\ge2} \cup \{k_t^\Theta\}$. As far as the minimality is concerned, note that for every $n \ge 1$, there exists a function $f_n \in K_\Theta$ such that $f_n(t_\ell) = \delta_{n,\ell}$, $\ell \ge 1$. By the completeness of $\{k_{t_n}^\Theta\}_{n\ge2} \cup \{k_t^\Theta\}$, we necessarily have $f_1(t) \ne 0$ and thus $k_t^\Theta \notin \operatorname{span}(k_{t_n}^\Theta : n \ge 2)$. Now fix $n \ge 2$. Using Lemma 3.11 one more time, there is an inner function J_1 such that $\{J_1 = -1\} = \{t_1\}$ and $f_n/(1+J_1) \in K_\Theta$. Now consider the function $g_n =$

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$((J_1 - J_1(t))f)/(1 + J_1)$. It is clear that $g_n \in K_\Theta$. Moreover, we have $g_n(t) = 0$, $g_n(t_\ell) = 0$, $\ell \neq n$, and $g_n(t_n) = (J_1(t_n) - J_1(t))/(1 + J_1(t_n)) \neq 0$ (since J_1 is a Blaschke factor and thus is one-to-one). Hence, we get that $k_{t_n}^\Theta \notin \operatorname{span}(\{k_{t_\ell}^\Theta\}_{\ell \geq 2, \ell \neq n} \cup \{k_t^\Theta\})$. This proves the minimality of $\{k_{t_n}^\Theta\}_{n \geq 2} \cup \{k_t^\Theta\}$.

(ii) Since $(k_{t_n}^\Theta)_{n \geq 1}$ is not complete in K_Θ, there exists a function $f \in K_\Theta$, $f \not\equiv 0$, such that $f(t_n) = 0$, $n \geq 1$. Fix $n \geq 1$. By Lemma 3.11, there is a Blaschke factor J_n such that $\{J_n = -1\} = \{t_n\}$ and $f/(1 + J_n) \in K_\Theta$. Now consider the function $f_n = ((J_n - J_n(t))f)/(1 + J_n)$. Then $f_n \in K_\Theta$, and we have $f_n(t) = 0$, $f_n(t_\ell) = 0$, $\ell \neq n$. Dividing one more time by $1 + J_n$ if necessary, we can assume that $f_n(t_n) \neq 0$. Hence, we deduce that $k_{t_n}^\Theta \notin \operatorname{span}(\{k_{t_\ell}^\Theta\}_{\ell \geq 1, \ell \neq n} \cup \{k_t^\Theta\})$. On the other hand, if $f(t) \neq 0$, we immediately get that $k_t^\Theta \notin \operatorname{span}(k_{t_n}^\Theta : n \geq 1)$. If $f(t) = 0$, then we can use Lemma 3.11 one more time to drop that extra zero. This proves the minimality of $\{k_{t_n}^\Theta\}_{n \geq 1} \cup \{k_t^\Theta\}$. ∎

Let Θ be an inner function, $(\lambda_n)_n \subset \mathbb{C}_+$ satisfying $\sup_{n \geq 1} |\Theta(\lambda_n)| < 1$. It is proved in [4] that if $(\kappa_{\lambda_n}^\Theta)_{n \geq 1}$ is an AOS, there exists $\varepsilon > 0$ such that $(\kappa_{\mu_n}^b)_{n \geq 1}$ is an AOS for all sequences $(\mu_n)_{n \geq 1} \in \mathbb{C}_+$ satisfying

$$\Big|\frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu_n}}\Big| \leq \varepsilon.$$

It is easy to see that this can be generalized to the general case when the inner function Θ is replaced by a function $b \in H_1^\infty$; see [9]. Without the additional hypothesis that $\sup_{n \geq 1} |b(\lambda_n)| < 1$, we obtain the following stability result concerning pseudo-hyperbolic perturbations.

Corollary 3.13 *Let $b \in H_1^\infty$ and $(\lambda_n)_{n \geq 1} \subset \mathbb{C}_+$ be such that $(\kappa_{\lambda_n}^b)_{n \geq 1}$ is an AOS in $\mathcal{H}(b)$. Let $\gamma > 1/3$ and $(\varepsilon_n)_{n \geq 1}$ a sequence of positive numbers tending to 0. For every sequence $(\mu_n)_{n \geq 1}$ satisfying*

$$\Big|\frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu_n}}\Big| \leq \varepsilon_n (1 - |b(\lambda_n)|)^\gamma, \qquad n \geq 1, \tag{3.14}$$

the sequence $(\kappa_{\mu_n}^b)_{n \geq 1}$ is an AOS. Moreover, if $(\kappa_{\lambda_n}^b)_{n \geq 1}$ is a complete AOB for $\mathcal{H}(b)$, then $(\kappa_{\mu_n}^b)_{n \geq 1}$ is also a complete AOB for $\mathcal{H}(b)$.

Proof According to Corollary 3.5, if we define the sets

$$G_n = \{z \in \mathbb{C}_+ : |z - \lambda_n| \leq \varepsilon_n \Im \lambda_n\},$$

then $G = \bigcup_n G_n$ is an admissible set for $(\lambda_n)_{n \geq 1}$. Let $(\mu_n)_{n \geq 1}$ satisfy (3.14). Then we have

$$|\lambda_n - \mu_n| \leq \varepsilon_n \big(1 - |b(\lambda_n)|\big)^\gamma \Im \lambda_n \leq \varepsilon_n \Im \lambda_n. \tag{3.15}$$

Therefore, $\mu_n \in G_n$. Without loss of generality, we can assume that $\gamma < 1$, and since $\gamma > 1/3$, there exists $1 < p < 2$ such that $2\frac{p-1}{p+1} = 1 - \gamma$. Let q be the conjugate exponent of p and note that $\frac{2p}{q(p+1)} = 1 - \gamma$. Using (2.13), (3.3), and (3.4), we have

$$w_p^{-2}(z) \leq C_1 \frac{(1 - |b(\lambda_n)|)^{1-\gamma}}{(\Im \lambda_n)^2},$$

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for $z \in [\lambda_n, \mu_n]$. Hence,

$$\frac{1}{\|k^b_{\lambda_n}\|_b^2}\int_{[\lambda_n,\mu_n]} w_p(z)^{-2}|dz| \leq C_2\frac{\Im\lambda_n}{1-|b(\lambda_n)|}|\lambda_n-\mu_n|\frac{(1-|b(\lambda_n)|)^{1-\gamma}}{(\Im\lambda_n)^2}.$$

Using (3.15), we obtain

$$\frac{1}{\|k^b_{\lambda_n}\|_b^2}\int_{[\lambda_n,\mu_n]} w_p(z)^{-2}|dz| \leq C_3\varepsilon_n.$$

The conclusion for AOS now follows from Theorem 3.6.

For complete AOB, we argue as follows. Let

$$\gamma_n = \begin{cases}\lambda_n & \text{if } n < N_0,\\ \mu_n & \text{if } n \geq N_0,\end{cases}$$

where N_0 will be chosen later. Since $(\gamma_n)_{n\geq1}$ satisfies (3.14), we get from the first part that

$$\sup_{n\geq1}\frac{1}{\|k^b_{\lambda_n}\|_b^2}\int_{[\lambda_n,\gamma_n]} w_p(z)^{-2}|dz| \leq C_3\sup_{n\geq N_0}\varepsilon_n.$$

Using that $\lim_{n\to\infty}\varepsilon_n = 0$, we can choose N_0 such that $C_3\sup_{n\geq N_0}\varepsilon_n$ is sufficiently small so that, according to Theorem 3.6, we will get that $(\kappa^b_{\gamma_n})_{n\geq1}$ is a complete AOB for $\mathcal{H}(b)$. Then we can apply Lemma 3.10 to get that $(\kappa^b_{\mu_n})_{n\geq1}$ is a complete and minimal sequence in $\mathcal{H}(b)$. Since it is also an AOS, it is finally a complete AOB for $\mathcal{H}(b)$. ∎

Remark 3.16 Note that in the case when $\lim_{n\to\infty}|b(\lambda_n)| = 1$, condition (3.14) can be replaced by the existence of a constant $C > 0$ such that

$$\Big|\frac{\lambda_n-\mu_n}{\lambda_n-\overline{\mu_n}}\Big| \leq C(1-|b(\lambda_n)|)^\gamma, \qquad n\geq1.$$

Indeed, it is sufficient to take $\gamma > \gamma_0 > 1/3$ and note that

$$C\big(1-|b(\lambda_n)|\big)^\gamma = \varepsilon_n\big(1-|b(\lambda_n)|\big)^{\gamma_0},$$

with $\varepsilon_n = C(1-|b(\lambda_n)|)^{\gamma-\gamma_0} \to 0$ as $n\to\infty$.

In the inner case, we can also give a stability result when the sequences $(\lambda_n)_n$ and $(\mu_n)_n$ are on the real line. We first need a result on the construction of admissible sets.

Lemma 3.17 *Let Θ be an inner function, $(t_n)_{n\geq1} \subset E_2(\Theta)$ such that $(\kappa^\Theta_{t_n})_{n\geq1}$ is a Riesz sequence in K_Θ and $(\varepsilon_n)_{n\geq1}$ a sequence of positive numbers tending to 0. Define*

$$G_n = \big\{t\in\mathbb{R} : |t-t_n| \leq \varepsilon_n v_0(t_n)\big\}, \qquad n\geq1, \tag{3.18}$$

where $v_0(t) = \min(d_0(t), |\Theta'(t)|^{-1})$ and $d_0(t) = \operatorname{dist}(t,\sigma(\Theta))$. Then the set $G = \bigcup_n G_n$ is an admissible set for $(\lambda_n)_{n\geq1}$.

Proof Consider the nontrivial case when $v_0(t_n) > 0$. In particular, we have

$$|t - t_n| \le \varepsilon_n d_0(t_n), \qquad t \in G_n.$$

Hence,

$$(1 - \varepsilon_n) d_0(t_n) \le d_0(t) \le (1 + \varepsilon_n) d_0(t_n), \qquad t \in G_n. \tag{3.19}$$

Now remember that when $t \in \mathbb{R}$, $k_t^\Theta \in K_\Theta$ if and only if

$$|\Theta'(t)| = a + \sum_{\ell=1}^{\infty} \frac{2\Im z_\ell}{|t - z_\ell|^2} + \int_{\mathbb{R}} \frac{d\sigma(x)}{|t - x|^2} < \infty,$$

and in that case,

$$\|k_t^\Theta\|_2^2 = |\Theta'(t)|. \tag{3.20}$$

Here, $(z_\ell)_\ell$ is the sequence of zeros of Θ, and σ is its associated singular measure. Using (3.19), it is not difficult to check that for every $\ell \ge 1$ and $t \in G_n$,

$$1 - \varepsilon_n \le \frac{|t - z_\ell|}{|t_n - z_\ell|} \le 1 + \varepsilon_n,$$

and for any $x \in \operatorname{supp} \sigma$,

$$1 - \varepsilon_n \le \frac{|t - x|}{|t_n - x|} \le 1 + \varepsilon_n.$$

Hence,

$$\frac{1}{(1 + \varepsilon_n)^2} |\Theta'(t_n)| \le |\Theta'(t)| \le \frac{1}{(1 - \varepsilon_n)^2} |\Theta'(t_n)|. \tag{3.21}$$

It then follows from (3.20) that

$$\frac{1}{1 + \varepsilon_n} \le \frac{\|k_t^\Theta\|_2}{\|k_{t_n}^\Theta\|_2} \le \frac{1}{1 - \varepsilon_n},$$

and we get that G_n satisfies condition (ii). Condition (i) is trivial, and condition (iii) follows along the same line as in [2, Lemma 5.1]. More precisely, using an increasing continuous branch of the argument of Θ on G_n (note that $\sigma(\Theta) \cap G_n = \varnothing$), it can be proved that for $t \in G_n$, we have

$$k_t^\Theta(t_n) \ge \frac{|\Theta'(t_n)|}{8\pi^2}. \tag{3.22}$$

Now using the fact that

$$\sum_{n \ge 1} \frac{|k_t^\Theta(t_n)|^2}{|\Theta'(t_n)|} = \sum_{n \ge 1} |\langle k_t^\Theta, \kappa_{t_n}^\Theta \rangle|^2 \le C \|k_t^\Theta\|_2^2 = C |\Theta'(t)|$$

we see that the number of integers n such that $t \in G_n$ is uniformly bounded. Hence, condition (iii) is also satisfied. ∎

Remark 3.23 It is natural to ask if Lemma 3.17 is satisfied when we replace the inner function Θ by a general function b in the unit ball of H^∞. The difficulty is indeed to get estimate (3.22).

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Theorem 3.24 *Let Θ be an inner function, let $(t_n)_{n\geq 1} \subset E_2(\Theta)$ such that $(\kappa^{\Theta}_{t_n})_{n\geq 1}$ is a complete AOB for K_Θ, and let $(s_n)_{n\geq 1}$ be a sequence of real numbers. Suppose there exists N_0 such that for all $n \geq N_0$, there is $\varepsilon_n > 0$ verifying*

$$\int_{[t_n,s_n]} \big(|\Theta'(t)| + |\Theta'(t)|^{-1} d_0^{-2}(t)\big)\, dt \leq \varepsilon_n \tag{3.25}$$

or

$$|s_n - t_n| \leq \varepsilon_n |\Theta'(t_n)| \min(d_0^2(t_n), |\Theta'(t_n)|^{-2}), \tag{3.26}$$

and $\lim_{n\to\infty} \varepsilon_n = 0$. Then $(\kappa^{\Theta}_{s_n})_{n\geq 1}$ is a complete AOB for K_Θ.

Proof We can of course assume that $s_n \neq t_n$ and $\varepsilon_n < 1/2$. Both (3.25) and (3.26) imply that there exists a point $u_n \in [s_n, t_n]$ such that $|s_n - t_n| \leq \varepsilon_n v_0(u_n)$. Then $v_0(u_n) \leq 4v_0(t_n)$ and $|s_n - t_n| \leq 4\varepsilon_n v_0(t_n)$. In particular, $s_n \in G_n$, where G_n is defined as in (3.18) (replacing ε_n by $4\varepsilon_n$). Moreover, using (2.13) and (3.21), we can write

$$\begin{aligned}\frac{1}{\|k^{\Theta}_{t_n}\|_2^2} \int_{[t_n,s_n]} w_p^{-2}(z)\,|dz| &\lesssim \int_{[t_n,s_n]} |\Theta'(t)|^{-1} \max(d_0^{-2}(t), |\Theta'(t)|^2)\, dt \\ &\lesssim \int_{[t_n,s_n]} (|\Theta'(t)|^{-1} d_0^{-2}(t) + |\Theta'(t)|)\, dt \lesssim \varepsilon_n.\end{aligned}$$

Applying Lemma 3.17 and Theorem 3.6, we get that $(\kappa^{\Theta}_{s_n})_{n\geq 1}$ is an AOS. It remains to prove the completeness and the minimality of $(\kappa^{\Theta}_{s_n})_{n\geq 1}$. We argue as in the proof of Corollary 3.13 replacing Lemma 3.10 by Lemma 3.12. More precisely, define

$$x_n = \begin{cases} t_n & \text{if } n < N_0, \\ s_n & \text{if } n \geq N_0, \end{cases}$$

for some positive integer N_0. Then we have

$$\sup_{n\geq 1} \frac{1}{\|k^{\Theta}_{t_n}\|_2^2} \int_{[t_n,x_n]} w_p^{-2}(z)\,|dz| \lesssim \sup_{n\geq N_0} \varepsilon_n,$$

and we can find N_0 such that, according to Theorem 3.6, the sequence $(k^{\Theta}_{x_n})_{n\geq 1}$ is a complete AOB for K_Θ. Note that if $t_n \in \sigma(\Theta)$, then $v_0(t_n) = 0$ and then $s_n = t_n$, and if $t_n \notin \sigma(\Theta)$, then $G_n \subset \mathbb{R} \setminus \sigma(\Theta)$ and then $s_n \notin \sigma(\Theta)$. Hence, we can apply Lemma 3.12 to get that $(\kappa^{\Theta}_{s_n})_{n\geq 1}$ is minimal and complete in K_Θ. ∎

We also give an analogue of a result of Cohn [5] who studied small perturbations with respect to the change of the argument of the inner function Θ. First, we need to introduce some more definitions. An inner function Θ in $\mathbb{C}_+$ is said to be a *meromorphic inner function* if it has a meromorphic extension to $\mathbb{C}$. In that case, we know that the argument of Θ is a real analytic increasing function on $\mathbb{R}$. Moreover, we say that an inner function Θ satisfies the *connected level set condition* (abbreviated $\Theta \in (CLS)$) if there is $\delta \in (0,1)$ such that the set $\Omega(\Theta,\delta) = \{z \in \mathbb{C}_+ : |\Theta(z)| < \delta\}$ is connected.

Corollary 3.27 *Let Θ be a meromorphic inner function such that $\Theta \in (CLS)$, let φ be its argument, and let $(t_n)_{n\geq 1} \subset \mathbb{R}$ be such that $(\kappa^{\Theta}_{t_n})_{n\geq 1}$ is a complete AOB for K_Θ.*

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Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive numbers tending to 0*. If*

$$|\varphi(s_n) - \varphi(t_n)| \leq \varepsilon_n,$$

then $(\kappa^{\Theta}_{s_n})_{n\geq 1}$ is a complete AOB for K_Θ.

Proof As noted in [2, Remark 1, p. 2419], since Θ is (CLS) and $(\kappa^{\Theta}_{t_n})_n$ is a Riesz sequence, there exits a constant $C > 0$ such that

$$|\Theta'(t)|^{-1} \leq C d_0(t), \qquad t \in G_n.$$

Therefore,

$$\int_{[t_n,s_n]} (|\Theta'(t)| + |\Theta'(t)|^{-1} d_0^{-2}(t))\, dt \lesssim \int_{[t_n,s_n]} |\Theta'(t)|\, dt = |\varphi(t_n) - \varphi(s_n)| \leq \varepsilon_n.$$

Then apply Theorem 3.24. ∎

Example 3.28 Let $\Theta_a(z) = e^{iaz}$, $a > 0$, and $\alpha \in [0, 2\pi)$. Then

$$\Theta_a^{-1}(\{e^{i\alpha}\}) = \{t_n = (\alpha + 2n\pi)/a : n \in \mathbb{Z}\},$$

and $(\kappa^{\Theta_a}_{t_n})_{n\in\mathbf{Z}}$ is an orthonormal basis for K_{Θ_a}, the so-called Clark basis. If $(s_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ is a sequence satisfying

$$\lim_{n\to\pm\infty} \Big| s_n - \frac{\alpha + 2n\pi}{a} \Big| = 0,$$

then Corollary 3.27 implies that $(\kappa^{\Theta_a}_{s_n})_{n\in\mathbf{Z}}$ is a complete AOB for K_{Θ_a}.

4 Example of Exponential Systems

In the particular case where $\Theta_a(z) = e^{iaz}$, the Fourier transform $\mathcal{F}$ maps unitarily K_{Θ_a} onto $L^2(0, a)$ and $\mathcal{F}(\kappa^{\Theta_a}_\lambda) = \chi^a_\lambda$, where

$$\chi^a_\lambda(t) = \Big(\frac{2\Im\lambda}{1 - e^{-2a\Im\lambda}} \Big)^{1/2} e^{i\lambda t}, \qquad \lambda \in \mathbb{C}_+.$$

Thus, the geometric properties (completeness, minimality, Riesz basis, AOS, AOB, ...) of system of normalized reproducing kernels $(\kappa^{\Theta_a}_{\lambda_n})_n$ in K_{Θ_a} and of normalized exponentials system $(\chi^a_{\lambda_n})_n$ in $L^2(0, a)$ are the same. In [4], AOS (or AOB) formed by reproducing kernels $k^{\Theta}_{\lambda_n}$ are studied under the additional condition that

$$\sup_{n\geq 1} |\Theta(\lambda_n)| < 1. \tag{4.1}$$

In the particular case when $\Theta = \Theta_a$, condition (4.1) is equivalent to

$$\inf_{n\geq 1} (\Im\lambda_n) > 0. \tag{4.2}$$

Under that assumption, it is proved in [4, Proposition 7.2] that $(\chi^a_{\lambda_n})_n$ is an AOB in $L^2(0, a)$ if and only if $(\lambda_n)_n$ is a thin sequence, which means that

$$\lim_{n\to\infty} \prod_{k\neq n} \Big| \frac{\lambda_k - \lambda_n}{\lambda_k - \overline{\lambda_n}} \Big| = 1.$$

Using Proposition 2.8, we construct a class of example of AOS where (4.1) (or equivalently (4.2)) is not necessarily satisfied.

Proposition 4.3 *Let* $(\lambda_n)_{n\geq 1}\subset\mathbb{C}$ *be a sequence such that*

(i) $\sup_n |\Im\lambda_n| < \infty$;

(ii) *there exists a* $q > 1$ *such that* $|\frac{\lambda_{n+1}}{\lambda_n}| > q$ *for all* $n \geq 1$.

Then the sequence $(\chi^a_{\lambda_n})_{n\geq 1}$ *is an AOS in* $L^2(0,a)$ *for all* $a > 0$.

Proof We apply Proposition 2.8. Observe that

$$\Gamma_{n,m} = \langle \chi^a_{\lambda_n}, \chi^a_{\lambda_m}\rangle = \Big(\frac{4\Im\lambda_n\Im\lambda_m}{(1-e^{-2a\Im\lambda_n})(1-e^{-2a\Im\lambda_m})}\Big)^{1/2}\frac{e^{i(\lambda_n-\overline{\lambda_m})a}-1}{i(\lambda_n-\overline{\lambda_m})},$$

$$\sup_{n,m\geq 1}\Big|\frac{4\Im\lambda_n\Im\lambda_m}{(1-e^{-2a\Im\lambda_n})(1-e^{-2a\Im\lambda_m})}\Big| < \infty,$$

provided $\sup_n \Im\lambda_n < \infty$. If $\Im\lambda_n = 0$ (that is $\lambda_n \in \mathbb{R}$), the normalized factor $\Im\lambda_n/(1-e^{-2a\Im\lambda_n})$ should be understood as a^{-1} and corresponds to $\|\chi^a_{\lambda_n}\|^2_{L^2(0,a)} = a$. It follows from (ii) that for $m > n$, we have $|\lambda_m| > q^{m-n}|\lambda_n|$. Since $q > 1$ that implies that $\lim_{n\to\infty}|\lambda_n| = \infty$. In particular, we can pick an integer N such that for all $n \geq N$, we have $|\lambda_n| \geq 1$. For $n \geq N$, write

$$\begin{aligned}\sum_{\substack{m\geq N\\ m\neq n}}|\Gamma_{n,m}| \lesssim \sum_{\substack{m\geq N\\ m\neq n}}\Big|\frac{e^{i(\lambda_n-\overline{\lambda_m})a}-1}{i(\lambda_n-\overline{\lambda_m})}\Big| &\lesssim \sum_{N\leq m<n}\frac{1}{|\lambda_m||\frac{\lambda_n}{\overline{\lambda_m}}-1|} + \sum_{n<m}\frac{1}{|\lambda_n||1-\frac{\overline{\lambda_m}}{\lambda_n}|}\\ &\leq \sum_{N\leq m<n}\frac{1}{|\lambda_m|(|\frac{\lambda_n}{\lambda_m}|-1)} + \sum_{n<m}\frac{1}{|\lambda_n|(|\frac{\lambda_m}{\lambda_n}|-1)}\\ &\leq \frac{1}{q-1}\sum_{N\leq m<n}\frac{1}{|\lambda_m|} + \frac{1}{|\lambda_n|}\sum_{n<m}\frac{1}{q^{m-n}|\lambda_n|-1}\\ &\leq \frac{1}{q-1}\frac{1}{|\lambda_N|}\sum_{N\leq m}\frac{1}{q^{m-N}} + \frac{1}{|\lambda_N|}\sum_{n<m}\frac{1}{q^{m-n}-1}.\end{aligned}$$

Thus,

$$\sup_{n\geq N}\sum_{\substack{m\geq N\\ m\neq n}}|\Gamma_{n,m}| \lesssim \frac{1}{|\lambda_N|}\to 0, \quad \text{as } N\to\infty.$$

Proposition 2.8 implies now that $(\chi^a_{\lambda_n})_{n\geq 1}$ is an AOS in $L^2(0,a)$. ∎

Example 4.4 The sequence $\lambda_n = r^n + i/n$, $(r > 1)$ satisfies the assumptions of Proposition 4.3 and $\Im\lambda_n \to 0$ as n goes to ∞.

5 Projecting onto a Closed Subspace

Let $b_1, b_2 \in H^\infty_1$ such that $b_2|b_1$, in the sense that $b_1 = b_2 b$ where $b \in H^\infty_1$. In this case, we know that $\mathcal{H}(b_2)\subset\mathcal{H}(b_1)$, and more precisely, we have

$$\mathcal{H}(b_1) = \mathcal{H}(b_2) + b_2\mathcal{H}(b).$$

See [17, I.10–I.11] or [9, Section 18.7] for details on this decomposition.

It should be noted that, in general, the above decomposition is not orthogonal. However, for reproducing kernels, we do have such an orthogonal decomposition.

Lemma 5.1 *Let $b_1 = b_2 b$ with $b_2, b \in H_1^\infty$. Let Λ be a finite subset in $\mathbb{C}_+$. Then for every $a_\lambda \in \mathbb{C}$, $\lambda \in \Lambda$, we have*

$$\Big\| \sum_{\lambda\in\Lambda} a_\lambda k_\lambda^{b_1} \Big\|_{b_1}^2 = \Big\| \sum_{\lambda\in\Lambda} a_\lambda k_\lambda^{b_2} \Big\|_{b_2}^2 + \Big\| \sum_{\lambda\in\Lambda} a_\lambda \overline{b_2(\lambda)} k_\lambda^{b} \Big\|_{b}^2. \tag{5.2}$$

Proof First note that

$$k_\lambda^{b_1} = k_\lambda^{b_2} + b_2 \overline{b_2(\lambda)} k_\lambda^b. \tag{5.3}$$

Now if LH and RH denote the left-hand side and right-hand side of (5.2), we have

$$LH = \sum_{\lambda,\mu\in\Lambda} a_\lambda \overline{a_\mu} k_\lambda^{b_1}(\mu),$$

$$RH = \sum_{\lambda,\mu\in\Lambda} a_\lambda \overline{a_\mu} k_\lambda^{b_2}(\mu) + \sum_{\lambda,\mu\in\Lambda} a_\lambda \overline{a_\mu}\, \overline{b_2(\lambda)} b_2(\mu) k_\lambda^{b}(\mu).$$

It remains to use (5.3) to get (5.2). ∎

Let $(\lambda_n)_{n\ge1} \subset \mathbb{C}_+$ and assume that $(\kappa_{\lambda_n}^{b_1})_{n\ge1}$ is a complete AOB for $\mathcal{H}(b_1)$. It is very natural to ask if the sequence $(\kappa_{\lambda_n}^{b_2})_{n\ge1}$ remains an AOB in $\mathcal{H}(b_2)$. The answer depends on the following ratio:

$$\mathcal{R}_{b_1,b_2}(n) := \frac{\|k_{\lambda_n}^{b_1}\|_{b_1}^2}{\|k_{\lambda_n}^{b_2}\|_{b_2}^2} = \frac{1-|b_1(\lambda_n)|^2}{1-|b_2(\lambda_n)|^2}.$$

The following result says that if the behavior of $b_1(\lambda_n)$ and $b_2(\lambda_n)$ are comparable as $n \to \infty$, then we can transfer AOBs between the respective de Branges–Rovnyak spaces.

Theorem 5.4 *Let $b_1 = b_2 b$, where $b_2, b \in H_1^\infty$, and $(\lambda_n)_{n\ge1} \subset \mathbb{C}_+$ satisfying*

$$\sum_n \big| \mathcal{R}_{b_1,b_2}(n) - 1 \big| < \infty.$$

If the sequence $(\kappa_{\lambda_n}^{b_1})_{n\ge1}$ is a complete AOB for $\mathcal{H}(b_1)$, then there is an integer $p \ge 1$ such that $(\kappa_{\lambda_n}^{b_2})_{n\ge p}$ is a complete AOB for $\mathcal{H}(b_2)$. Conversely, if $(\kappa_{\lambda_n}^{b_2})_{n\ge1}$ is an AOB in $\mathcal{H}(b_2)$, then $(\kappa_{\lambda_n}^{b_1})_{n\ge1}$ is an AOB in $\mathcal{H}(b_1)$.

Proof First note that $(k_{\lambda_n}^{b_2})_{n\ge1}$ is complete in $\mathcal{H}(b_2)$. Indeed, let $f \in \mathcal{H}(b_2)$, $f \perp k_{\lambda_n}^{b_2}$, $n \ge 1$. Since $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$, we can write

$$0 = \langle f, k_{\lambda_n}^{b_2}\rangle_{b_2} = f(\lambda_n) = \langle f, k_{\lambda_n}^{b_1}\rangle_{b_1}.$$

Thus, f is orthogonal to $k_{\lambda_n}^{b_1}$, $n \ge 1$, and the completeness of $(k_{\lambda_n}^{b_1})_{n\ge1}$ in $\mathcal{H}(b_1)$ implies that $f \equiv 0$.

Since $(\kappa_{\lambda_n}^{b_1})_{n\ge1}$ is an AOB in $\mathcal{H}(b_1)$, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$(1-\varepsilon)\sum_{n\ge N} |a_n|^2 \le \Big\| \sum_{n\ge N} a_n \kappa_{\lambda_n}^{b_1} \Big\|_{b_1} \le (1+\varepsilon)\sum_{n\ge N} |a_n|^2. \tag{5.5}$$

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Moreover, since the sequence $(\mathcal{R}_{b_1,b_2}(n)-1)_n$ is in ℓ^1, we can also assume that N satisfies

$$\sum_{n\geq N}\left(\frac{\|k_{\lambda_n}^{b_1}\|_{b_1}^2}{\|k_{\lambda_n}^{b_2}\|_{b_2}^2}-1\right)<\varepsilon. \tag{5.6}$$

In particular, this guarantees that

$$1-\varepsilon<\frac{\|k_{\lambda_n}^{b_1}\|_{b_1}^2}{\|k_{\lambda_n}^{b_2}\|_{b_2}^2}<1+\varepsilon. \tag{5.7}$$

We now prove that $(\kappa_{\lambda_n}^{b_2})_{n\geq1}$ is an AOS in $\mathcal{H}(b_2)$. Using Lemma 5.1, we have

$$\Big\|\sum_{n\geq N}a_n\frac{k_{\lambda_n}^{b_1}}{\|k_{\lambda_n}^{b_2}\|_{b_2}}\Big\|_{b_1}^2=\Big\|\sum_{n\geq N}a_n\frac{k_{\lambda_n}^{b_2}}{\|k_{\lambda_n}^{b_2}\|_{b_2}}\Big\|_{b_2}^2+\Big\|\sum_{n\geq N}a_n\frac{\overline{b_2(\lambda_n)}k_{\lambda_n}^{b}}{\|k_{\lambda_n}^{b_2}\|_{b_2}}\Big\|_{b}^2.$$

Thus,

$$\begin{aligned}\Big\|\sum_{n\geq N}a_n\kappa_{\lambda_n}^{b_2}\Big\|_{b_2}^2&=\Big\|\sum_{n\geq N}a_n\frac{\|k_{\lambda_n}^{b_1}\|_{b_1}}{\|k_{\lambda_n}^{b_2}\|_{b_2}}\kappa_{\lambda_n}^{b_1}\Big\|_{b_1}^2-\Big\|\sum_{n\geq N}a_n\frac{\overline{b_2(\lambda_n)}k_{\lambda_n}^{b}}{\|k_{\lambda_n}^{b_2}\|_{b_2}}\Big\|_{b}^2.\\&=I_1-I_2.\end{aligned}$$

For I_1, use estimates (5.5) and (5.7) to get

$$\begin{aligned}(1-\varepsilon)^2\sum_{n\geq N}|a_n|^2&\leq(1-\varepsilon)\sum_{n\geq N}|a_n|^2\frac{\|k_{\lambda_n}^{b_1}\|_{b_1}^2}{\|k_{\lambda_n}^{b_2}\|_{b_2}^2}\leq\Big\|\sum_{n\geq N}a_n\frac{\|k_{\lambda_n}^{b_1}\|_{b_1}}{\|k_{\lambda_n}^{b_2}\|_{b_2}}\kappa_{\lambda_n}^{b_1}\Big\|_{b_1}^2=I_1\\&\leq(1+\varepsilon)\sum_{n\geq N}|a_n|^2\frac{\|k_{\lambda_n}^{b_1}\|_{b_1}^2}{\|k_{\lambda_n}^{b_2}\|_{b_2}^2}\leq(1+\varepsilon)^2\sum_{n\geq N}|a_n|^2.\end{aligned}$$

For I_2, we use (5.2), (5.6), and Cauchy–Schwarz inequality to obtain

$$\begin{aligned}I_2=\Big\|\sum_{n\geq N}a_n\frac{\overline{b_2(\lambda_n)}k_{\lambda_n}^{b}}{\|k_{\lambda_n}^{b_2}\|_{b_2}}\Big\|_{b}^2&\leq\Big(\sum_{n\geq N}|a_n|^2\Big)\Big(\sum_{n\geq N}\frac{\|\overline{b_2(\lambda_n)}k_{\lambda_n}^{b}\|_{b}^2}{\|k_{\lambda_n}^{b_2}\|_{b_2}^2}\Big)\\&=\Big(\sum_{n\geq N}|a_n|^2\Big)\sum_{n\geq N}\Big(\frac{\|k_{\lambda_n}^{b_1}\|_{b_1}^2}{\|k_{\lambda_n}^{b_2}\|_{b_2}^2}-1\Big)\leq\varepsilon\sum_{n\geq N}|a_n|^2.\end{aligned}$$

It follows that $(\kappa_{\lambda_n}^{b_2})_n$ is an AOS. Now let p be the smallest integer such that $(\kappa_{\lambda_n}^{b_2})_{n\geq p}$ is an AOB in $\mathcal{H}(b_2)$. If $p=1$, then since $(\kappa_{\lambda_n}^{b_2})_{n\geq1}$ is complete in $\mathcal{H}(b_2)$, we have the result. Otherwise combining Lemma 3.10(ii) and the fact that a sequence is an AOB if and only if it is a minimal AOS, we conclude that $(\kappa_{\lambda_n}^{b_2})_{n\geq p}$ is a complete AOB for $\mathcal{H}(b_2)$.

Conversely, assume that $(\kappa_{\lambda_n}^{b_2})_{n\geq1}$ is an AOB in $\mathcal{H}(b_2)$. We note that

$$(\mathcal{R}_{b_2,b_1}(n)-1)_n=(1/\mathcal{R}_{b_1,b_2}(n)-1)_n\in\ell^1.$$

Then using similar computations as before, we see that $(\kappa_{\lambda_n}^{b_1})_n$ is an AOS in $\mathcal{H}(b_1)$. It remains to check the minimality of $(k_{\lambda_n}^{b_1})_{n\geq1}$. Since $(k_{\lambda_n}^{b_2})_{n\geq1}$ is minimal in $\mathcal{H}(b_2)$, there exists a sequence of functions $\psi_n \in \mathcal{H}(b_2)$, $n \geq 1$, such that $\langle \psi_n, k^{b_2}_{\lambda_\ell} \rangle_{b_2} = \delta_{n,\ell}$. From the inclusion $\mathcal{H}(b_2) \subset \mathcal{H}(b_1)$, we can write

$$\langle \psi_n, k^{b_1}_{\lambda_\ell} \rangle_{b_1} = \psi_n(\lambda_\ell) = \langle \psi_n, k^{b_2}_{\lambda_\ell} \rangle_{b_2} = \delta_{n,\ell},$$

which proves that $(k^{b_1}_{\lambda_n})_{n\geq 1}$ is a minimal sequence in $\mathcal{H}(b_1)$. ∎

Corollary 5.8 *Let b_1 and b_2 be two functions in H^∞_1 such that they have a common factor b,* i.e., *both b_1/b and b_2/b are in H^∞_1. Moreover, assume that $(\mathcal{R}_{b_1,b}(n)-1)_n \in \ell^1$ and $(\mathcal{R}_{b_2,b}(n)-1)_n \in \ell^1$. If $(\kappa^{b_1}_{\lambda_n})_{n\geq 1}$ is an AOB in $\mathcal{H}(b_1)$, then there is an integer $p \geq 1$ such that $(\kappa^{b_2}_{\lambda_n})_{n\geq p}$ is an AOB in $\mathcal{H}(b_2)$.*

The assumption that $(\mathcal{R}_{b_1,b_2}(n)-1)_n \in \ell^1$ may appear very restrictive. However, as the following result shows, in some particular cases, it is indeed also necessary.

Corollary 5.9 *Let $b_1 = \Theta_2 b$ where $b \in H^\infty_1$ and Θ_2 is an inner function such that $\infty \notin \sigma(\Theta_2)$. Let $(\lambda_n)_{n\geq 1}$ be a sequence of points in $\mathbb{C}_+$ such that $(\kappa^{b_1}_{\lambda_n})_{n\geq 1}$ is a complete AOB for $\mathcal{H}(b_1)$ and*

$$\sup_{n\geq 1} \|k^b_{\lambda_n}\|_b < \infty. \tag{5.10}$$

Then the following are equivalent:

(i) *There is an integer $p \geq 1$ such that $(\kappa^{\Theta_2}_{\lambda_n})_{n\geq p}$ is a complete AOB for K_{Θ_2}.*

(ii) $(\mathcal{R}_{b_1,\Theta_2}(n)-1)_n \in \ell^1$.

Proof (ii) ⇒ (i): Follows from Theorem 5.4.
(i) ⇒ (ii): We recall a well known fact (see [2, Lemma 4.4]) that $\sup_n |\lambda_n| < \infty$, provided $\infty \notin \sigma(\Theta_2)$ and $(\kappa^{\Theta_2}_{\lambda_n})_{n\geq p}$ is an AOB in K_{Θ_2} (in fact, it is sufficient that $(\kappa^{\Theta_2}_{\lambda_n})_n$ is a frame).

Let $\gamma \in \mathbb{C}_+$. Then the function

$$f(z) := \Theta_2(z)\frac{1-\overline{b(\gamma)}b(z)}{z-\overline{\gamma}}$$

belongs to $\Theta_2\mathfrak{H}(b) \subset K_{\Theta_2} + \Theta_2\mathfrak{H}(b) = \mathfrak{H}(b_1)$. Since $(\kappa^{b_1}_{\lambda_n})_n$ is an AOB in $\mathcal{H}(b_1)$, we must have

$$\sum_{n\geq 1} |\langle f, \kappa^{b_1}_{\lambda_n} \rangle|^2 < \infty$$

$$\text{i.e.,} \qquad \sum_{n\geq 1} |\Theta_2(\lambda_n)|^2 \Big|\frac{1-\overline{b(\gamma)}b(\lambda_n)}{\lambda_n-\overline{\gamma}}\Big|^2 \frac{2\Im\lambda_n}{1-|b_1(\lambda_n)|^2} < \infty.$$

We observe that, since $\sup_n |\lambda_n| < \infty$, when $|\gamma|$ is large enough, we have

$$\Big|\frac{1-\overline{b(\gamma)}b(\lambda_n)}{\lambda_n-\overline{\gamma}}\Big| \gtrsim \frac{1-|b(\gamma)|}{|\gamma|}.$$

Thus,

$$\sum_{n\geq 1} |\Theta_2(\lambda_n)|^2 \frac{2\Im\lambda_n}{1-|b_1(\lambda_n)|^2} < \infty.$$

Since $1-|b(\lambda_n)|^2 \lesssim \Im\lambda_n$, we have

$$\sum_{n\geq1}|\Theta_2(\lambda_n)|^2\frac{1-|b(\lambda_n)|^2}{1-|b_1(\lambda_n)|^2}<\infty,$$

i.e.,

$$\sum_{n\geq1}|\Theta_2(\lambda_n)|^2\frac{\|k^b_{\lambda_n}\|^2_b}{\|k^{b_1}_{\lambda_n}\|^2_{b_1}}<\infty.$$

Finally, we get

$$\sum_{n\geq1}(1-\mathcal{R}_{\Theta_2,b_1}(n))=\sum_{n\geq1}\frac{\|k^{b_1}_{\lambda_n}\|^2_{b_1}-\|k^{\Theta_2}_{\lambda_n}\|^2_2}{\|k^{b_1}_{\lambda_n}\|^2_{b_1}}<\infty.$$

In other words, $(\mathcal{R}_{\Theta_2,b_1}(n)-1)_n\in\ell^1$. Since $\mathcal{R}_{b_1,\Theta_2}(n)=1/\mathcal{R}_{\Theta_2,b_1}(n)$, it follows that the sequence $(\mathcal{R}_{b_1,\Theta_2}(n)-1)_n$ is in ℓ^1. ■

Example 5.11 Note that (5.10) is, in particular, satisfied in the case when $b=\Theta$ is an inner function such that $\Theta'\in L^\infty(\mathbb{R})$. Indeed, as was shown in [1, Corollary 4.7], we have

$$\|k^\Theta_{\lambda_n}\|_2\leq\|k^\Theta_{x_n}\|_2=|\Theta'(x_n)|^{1/2},$$

where $x_n=\Re\lambda_n$.

Remark 5.12 The results given in that section can also be proved when $b_1=\Theta_1$ is an inner function and the sequence $(\lambda_n)_{n\geq1}$ belongs to $\mathbb{C}_+\cup\mathbb{R}\setminus(\sigma(\Theta)\cap\mathbb{R})$.

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Laboratoire Paul Painlevé, Université Lille 1, 59 655 Villeneuve d'Ascq Cédex
e-mail: emmanuel.fricain@math.univ-lille1.fr rishika.rupam@math.univ-lille1.fr