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PRACTICAL STABILITY OF THE SOLUTIONS OF IMPULSIVE SYSTEMS OF DIFFERENTIAL-DIFFERENCE EQUATIONS VIA THE METHOD OF COMPARISON AND SOME APPLICATIONS TO POPULATION DYNAMICS

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Abstract

In this paper we consider an initial value problem for systems of impulsive differential-difference equations is considered. Making use of the method of comparison and differential inequalities for piecewise continuous functions, sufficient conditions for practical stability of the solutions of such systems are obtained. Applications to population dynamics are also given.

1. Introduction

One of the most important aspects of the theory of stability of the solutions of differential equations is practical stability. The main results in this area are due to A. A. Martynyuk ([16–18]).

The main problem in the theory of practical stability consists of studying the solutions of systems of differential equations, given in advance the domain where the initial conditions change, and the domain where the solutions should remain when the independent variable changes over a fixed interval (finite or infinite). In recent years this theory has been developed very intensively ([13–15, 18]).

The practical stability of the solutions of a system of the form

$$\dot{x} = f(t, x) \tag{1.1}$$

can be considered by studying the relations between this system and a system

$$\dot{u} = F(t, u) \tag{1.2}$$

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so that the practical stability of the solutions of System (1.2) should imply the practical stability of the solutions of System (1.1). These relations are obtained by employing differential inequalities. System (1.2) is usually of lower order and its right-hand side possesses a certain type of monotonicity which considerably simplifies the study of its solutions. Actually, this is the essence of the method of comparison in the theory of practical stability.

In recent years the mathematical theory of impulsive systems has also been intensively advancing (see [13–15, 20]).

Impulsive differential-difference equations are a generalization of impulsive differential equations. They are adequate mathematical models of processes and phenomena which undergo changes in state by jumps and for which a dependence on the history of the process is observed at each moment.

At the present time the theory of such equations is undergoing rapid development (see [2–7]).

In this paper we study the practical stability of the zero solution of a system of impulsive differential-difference equations by means of piecewise continuous Lyapunov functions [20] and the comparison principle coupled with the Razumikhin technique ([14, 19]).

2. Statement of the problem, preliminary notes and definitions

Let R^n be the n-dimensional Euclidean space with norm $|\cdot|$, Ω be a domain in R^n containing the origin and let $h \geq 0$. Let $\varphi_0 \in C[[t_0 - h, t_0], \Omega]$. Consider the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-h)), & t > t_0,\ t \neq \tau_k, \\ x(t) = \varphi_0(t), & t \in [t_0 - h, t_0], \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k) = I_k(x(\tau_k)), & \tau_k > t_0,\ k = 1, 2, \dots, \end{cases} \tag{2.1}$$

where $t_0 \in R$, $f : (t_0, \infty) \times \Omega \times \Omega \to R^n$, $I_k : \Omega \to R^n$, $k = 1, 2, \dots$, $t_0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$.

Together with this problem consider the problem

$$\begin{cases} \dot{u}(t) = F(t, u), & t > t_0,\ t \neq \tau_k, \\ u(t_0 + 0) = u_0, & \\ \Delta u(\tau_k) = J_k(u(\tau_k)), & \tau_k > t_0,\ k = 1, 2, \dots, \end{cases} \tag{2.2}$$

where $F : (t_0, \infty) \times G \to R^m$, $J_k : G \to R^m$, $k = 1, 2, \dots$, G is a domain in R^m containing the origin and $u_0 \in G$.

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Introduce the following notation: $R^+ = [0, \infty)$; $x(t) = x(t; t_0, \varphi_0)$ is the solution of Problem (2.1); $J^+(t_0, \varphi_0)$ is the maximal interval of type $[t_0, \beta)$ in which the solution $x(t) = x(t; t_0, \varphi_0)$ is defined; $u(t) = u(t; t_0, u_0)$ is the solution of Problem (2.2); $J^+(t_0, u_0)$ is the maximal interval in which the solution $u(t) = u(t; t_0, u_0)$ is defined; $\mathbf{K}$ is the class of all continuous and monotone increasing functions $a : R^+ \to R^+$ such that $a(0) = 0$ and $a(r) \to \infty$ as $r \to \infty$; $\mathbf{K}^*$ is the class of all continuous functions $a : (t_0, \infty) \times R^+ \to R^+$ which are monotone increasing with respect to their second argument and such that $a(t, 0) = 0$ and $a(t, r) \to \infty$ as $r \to \infty$; $C_0 = C[[t_0 - h, t_0], \Omega]$; $\|\varphi\| = \max_{t\in[t_0-h,t_0]} |\varphi(t)|$ is the norm of the function $\varphi \in C_0$; $\Omega_k = \{(t, x) \in (t_0, \infty) \times \Omega : \tau_{k-1} < t < \tau_k\}$, $k = 1, 2, \ldots$.

The solutions $x(t)$ of problems of the form (2.1) are piecewise continuous functions with points of discontinuity of the first kind $\tau_k > t_0$, $k = 1, 2, \ldots$, at which they are continuous from the left, that is, at the moments of impulse effect τ_k the following relations are valid:

$$x(\tau_k - 0) = x(\tau_k), \quad x(\tau_k + 0) = x(\tau_k) + I_k(x(\tau_k)), \quad k = 1, 2, \ldots .$$

If for some positive integer j we have $\tau_k < \tau_j + h < \tau_{k+1}$, $k = 0, 1, 2, \ldots$, then in the interval $[\tau_j + h, \tau_{k+1}]$ the solution $x(t)$ of Problem (2.1) coincides with the solution of the problem

$$\begin{cases} \dot{y}(t) = f(t, y(t), x(t - h + 0)), \\ y(\tau_j + h) = x(\tau_j + h), \end{cases}$$

and if $\tau_j + h \equiv \tau_k$ for $j = 0, 1, 2, \ldots$, $k = 1, 2, \ldots$, then in the interval $[\tau_j + h, \tau_{k+1}]$ the solution $x(t)$ coincides with the solution of the problem

$$\begin{cases} \dot{y}(t) = f(t, y(t), x(t - h + 0)), \\ y(\tau_j + h) = x(\tau_j + h) + I_k(x(\tau_j + h)). \end{cases}$$

If the point $x(\tau_k) + I_k(x(\tau_k)) \notin \Omega$, then the solution $x(t)$ of Problem (2.1) is not defined for $t > \tau_k$.

DEFINITION 1. Let λ, A, and B be positive constants ($\lambda < A$, $B < A$). The trivial solution of Problem (2.1) is said to be:

(1) *Practically stable with respect to* λ, A if

$$(\forall t_0 \in R)(\forall \varphi_0 \in C_0 : \|\varphi_0\| < \lambda)(\exists \varphi \in \mathbf{K}^*)(\forall t \in J^+(t_0, \varphi_0)) : \\ |x(t; t_0, \varphi_0)| \le \varphi(t_0, \|\varphi_0\|) \quad \text{and} \quad \varphi(t_0, \lambda) < A.$$

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(2) *Uniformly practically stable with respect to* λ, A if

$$(\forall \varphi_0 \in C_0 : \|\varphi_0\| < \lambda)(\exists \varphi \in \mathbf{K})(\forall t_0 \in R)(\forall t \in J^+(t_0, \varphi_0)) :$$
$$|x(t; t_0, \varphi_0)| \le \varphi(\|\varphi_0\|) \quad \text{and} \quad \varphi(\lambda) < A.$$

(3) *Contractively practically stable with respect to* λ, A, B if

$$(\forall t_0 \in R)(\forall \varphi_0 \in C_0 : \|\varphi_0\| < \lambda)(\exists \varphi \in \mathbf{K}^*)(\exists \psi : (t_0, \infty) \to R^+)(\forall t \in J^+(t_0, \varphi_0)) :$$
$$|x(t; t_0, \varphi_0)| \le \varphi(t_0, \|\varphi_0\|)\psi(t), \varphi(t_0, \lambda)\psi(t) < A \quad \text{and}$$
$$\varphi(t_0, \lambda)\psi(t_0 + \tau) < B \quad \text{for some } \tau > 0.$$

(4) *Contractively uniformly practically stable with respect to* λ, A, B if

$$(\forall \varphi_0 \in C_0 : \|\varphi_0\| < \lambda)(\exists \varphi \in \mathbf{K})(\exists \psi : (t_0, \infty) \to R^+)(\forall t_0 \in R)(\forall t \in J^+(t_0, \varphi_0)) :$$
$$|x(t; t_0, \varphi_0)| \le \varphi(\|\varphi_0\|)\psi(t), \varphi(\lambda)\psi(t) < A \quad \text{and}$$
$$\varphi(\lambda)\psi(t_0 + \tau) < B \quad \text{for some } \tau > 0.$$

Introduce in R^m a partial ordering defined in the following natural way: for $u, v \in R^m$ we will write $u \ge v (u > v)$ if and only if $u_j \ge v_j (u_j > v_j)$ for any $j = 1, 2, \dots, m$.

DEFINITION 2. The function $\psi : G \to R^m$, $G \subset R^m$ is said to be *monotone increasing* in G if $\psi(u) > \psi(v)$ for $u > v$ and $\psi(u) \ge \psi(v)$ for $u \ge v (u, v \in G)$.

DEFINITION 3. The function $F : (t_0, \infty) \times G \to R^m$ is said to be *quasi-monotone increasing* in $(t_0, \infty) \times G$ if for each pair of points (t, u) and (t, v) from $(t_0, \infty) \times G$ and for $j \in \{1, 2, \dots, m\}$ the inequality $F_j(t, u) \ge F_j(t, v)$ holds whenever $u_j = v_j$ and $u_i \ge v_i$ for $i = 1, 2, \dots, m, i \ne j$, that is, for any fixed $t \in (t_0, \infty)$ and any $j \in \{1, 2, \dots, m\}$ the function $F_j(t, u)$ is nondecreasing with respect to $(u_1, u_2, \dots, u_{j-1}, u_{j+1}, \dots, u_m)$.

In the case when the function $F : (t_0, \infty) \times G \to R^m$ is continuous and quasi-monotone increasing all solutions of Problem (2.2) starting from the point $(\bar{t}_0, u_0) \in (t_0, \infty) \times G$ lie between two singular solutions—the maximal and the minimal ones.

DEFINITION 4. The solution $u^+ : [t_0, \beta) \to R^m$ of Problem (2.2) is said to be a *maximal solution* if for any other solution $u : [t_0, \alpha) \to R^m$ of Problem (2.2) the inequality $u(t) \le u^+(t)$ holds for $t \in [t_0, \beta) \cap [t_0, \alpha)$.

The minimal solution of Problem (2.2) is defined analogously.

Let $e \in R^m$, $e = (1, 1, \dots, 1)$ and $\{u : 0 \le u \le e\} \subset G$. Further on we shall consider only such solutions of (2.2) for which $u(t) \ge 0$. Hence, the following modification of Definition 1 seems to be the most appropriate.

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DEFINITION 5. The trivial solution of Problem (2.2) is said to be:

(1) *Practically u-stable with respect to* λ, A if

$$(\forall t_0 \in R)(\forall u_0 \in G \text{ and } 0 \le u_0 \le \lambda e)(\exists \varphi \in \mathbf{K}^*)(\exists a \in \mathbf{K})(\forall t \in J^+(t_0, u_0)) :$$
$$u^+(t; t_0, u_0) \le \varphi(t_0, |u_0|)e \quad \text{and} \quad \varphi(t_0, \lambda) < a(A).$$

(2) *Uniformly practically u-stable with respect to* λ, A if

$$(\forall u_0 \in G \text{ and } 0 \le u_0 \le \lambda e)(\exists \varphi \in \mathbf{K})(\exists a \in \mathbf{K})(\forall t_0 \in R)(\forall t \in J^+(t_0, u_0)) :$$
$$u^+(t; t_0, u_0) \le \varphi(|u_0|)e \quad \text{and} \quad \varphi(\lambda) < a(A).$$

(3) *Contractively practically u-stable with respect to* λ, A, B if

$$(\forall t_0 \in R)(\forall u_0 \in G \text{ and } 0 \le u_0 \le \lambda e)(\exists \varphi \in \mathbf{K}^*)$$
$$(\exists a \in \mathbf{K})(\exists \psi : (t_0, \infty) \to R^+)(\forall t \in J^+(t_0, u_0)) :$$
$$u^+(t; t_0, u_0) \le \varphi(t_0, |u_0|)\psi(t)e, \varphi(t_0, \lambda)\psi(t) < A \quad \text{and}$$
$$\varphi(t_0, \lambda)\psi(t_0 + \tau) < a(B) \quad \text{for some } \tau > 0.$$

(4) *Contractively uniformly practically u-stable with respect to* λ, A, B if

$$(\forall u_0 \in G \text{ and } 0 \le u_0 \le \lambda e)(\exists \varphi \in \mathbf{K})(\exists a \in \mathbf{K})$$
$$(\exists \psi : (t_0, \infty) \to R^+)(\forall t_0 \in R)(\forall t \in J^+(t_0, u_0)) :$$
$$u^+(t; t_0, u_0) \le \varphi(|u_0|)\psi(t)e, \varphi(\lambda)\psi(t) < a(A) \quad \text{and}$$
$$\varphi(\lambda)\psi(t_0 + \tau) < a(B) \quad \text{for some } \tau > 0.$$

In what follows we shall use the class V_0 of piecewise continuous auxiliary functions $V : [t_0, \infty) \times \Omega \to G$ which are analogues of Lyapunov's functions [20].

DEFINITION 6. We shall say that the function $V : [t_0, \infty) \times \Omega \to G$ belongs to the class V_0 if:

(1) The function V is continuous in each of the sets Ω_k, $k = 1, 2, \dots$ and $V(t, 0) = 0$ for $t \in [t_0, \infty)$.

(2) The function V is locally Lipschitz continuous with respect to its second argument x in $\bigcup_{k=1}^{\infty} \Omega_k$.

(3) For each $k = 1, 2, \dots$ there exist the finite limits

$$V(\tau_k - 0, x) = \lim_{\substack{t \to \tau_k \\ (t,x) \in \Omega_k}} V(t, x), \qquad V(\tau_k + 0, x) = \lim_{\substack{t \to \tau_k \\ (t,x) \in \Omega_{k+1}}} V(t, x)$$

and

$$V(\tau_k - 0, x) = V(\tau_k, x), \quad x \in \Omega.$$

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We also introduce the following classes of functions:

$$PC[[t_0,\infty),\Omega] = \{x : [t_0,\infty) \to \Omega : x(t) \text{ is a piecewise continuous function with points of discontinuity of the first kind } \tau_1, \tau_2, \dots \text{ at which it is continuous from the left}\};$$

$$\Omega_t = \{x \in PC[[t_0,\infty),\Omega] : V(s,x(s)) \le V(t,x(t)),\ t-h \le s \le t,\ t \ge t_0,\ V \in V_0\}.$$

Let $V \in V_0$, $t \ge t_0$, $t \ne \tau_k$, $k = 1,2,\dots$ and $x \in PC[[t_0,\infty),\Omega]$. Introduce the function

$$D_-V(t,x(t)) = \lim_{\sigma\to 0^-} \inf \sigma^{-1}[V(t+\sigma, x(t)+\sigma f(t,x(t),x(t-h))) - V(t,x(t))].$$

Introduce the following conditions:

H1. $f \in C[(t_0,\infty)\times\Omega\times\Omega, R^n]$.

H2. $f(t,0,0) = 0$, $t \in (t_0,\infty)$.

H3. The function f is Lipschitz continuous with respect to its second and third arguments in $(t_0,\infty)\times\Omega\times\Omega$ uniformly on $t \in (t_0,\infty)$.

H4. $I_k \in C[\Omega, R^n]$, $k = 1,2,\dots$.

H5. The functions $(I + I_k) : \Omega \to \Omega$, $k = 1,2,\dots$, where I is the identity in Ω.

H6. $I_k(0) = 0$, $k = 1,2,\dots$.

H7. $t_0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots$.

H8. $\lim_{k\to\infty} \tau_k = \infty$.

In the proof of the main results we shall use the following lemma:

LEMMA 1 ([5, 6]). *Let the following conditions hold*:

(1) *Conditions* H1–H8 *are met.*

(2) *The function F is quasi-monotone increasing, continuous in the sets $(\tau_k, \tau_{k+1}] \times G$, $k \in N \cup \{0\}$ and $u^+ : J^+(t_0, u_0) \to R^m$ is the maximal solution of Problem* (2.2).

(3) *For each $k \in N \cup \{0\}$ and $v \in G$ there exists the limit*

$$\lim_{\substack{(t,u)\to(t,v)\\ t>\tau_k}} F(t,u).$$

(4) *The functions $\psi_k : G \to R^m$, $\psi_k(u) = u + J_k(u)$, $k = 1,2,\dots$, are monotone increasing in G.*

(5) *The function $V \in V_0$ is such that $V(t_0, \varphi_0(t_0)) \le u_0$ and the inequalities*

$$D_-V(t,x(t)) \le F(t, V(t,x(t))), \quad t \ne \tau_k,\ k = 1,2,\dots,$$
$$V(t+0, x(t) + I_k(x(t))) \le \psi_k(V(t,x(t))), \quad t = \tau_k,\ k = 1,2,\dots,$$

are valid for $t \in J^+(t_0, \varphi_0)$ and $x \in \Omega_t$.

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Then $V(t, x(t; t_0, \varphi_0)) \leq u^+(t; t_0, u_0)$, *where* $t \in J^+(t_0, \varphi_0) \cap J^+(t_0, u_0)$.

COROLLARY 1. *Let the following conditions hold*:

(1) *The conditions of Lemma* 1 *are satisfied.*

(2) *There exists a function* $a \in \mathbf{K}$ *such that*

$$a(|x|) \leq \max_{1 \leq j \leq m} V_j(t, x), \quad (t, x) \in [t_0, \infty) \times \Omega. \tag{2.3}$$

Then for $t \in J^+(t_0, \varphi_0) \cap J^+(t_0, u_0)$ *the following inequality is valid*:

$$|x(t; t_0, \varphi_0)| \leq a^{-1}\big[\max_j u_j^+(t; t_0, u_0)\big]. \tag{2.4}$$

PROOF. Since for the function $V \in V_0$ the conditions of Lemma 1 are satisfied, then from (2.3) we deduce the inequalities

$$a(|x(t; t_0, \varphi_0)|) \leq \max_{1 \leq j \leq m} V_j(t, x(t; t_0, \varphi_0)) \leq \max_{1 \leq j \leq m} u_j^+(t; t_0, u_0),$$

where $t \in J^+(t_0, \varphi_0) \cap J^+(t_0, u_0)$, which imply Inequality (2.4).

3. Main results

THEOREM 1. *Let the following conditions be fulfilled*:

(1) *The conditions of Corollary* 1 *are met.*

(2) *There exists a function* $b \in \mathbf{K}^*$ *such that*

$$V(t_0, x) \leq b(t_0, |x|)e, \quad x \in \Omega. \tag{3.1}$$

(3) $J_k(0) = 0$, $k = 1, 2, \ldots$.

(4) $F(t, 0) = 0$ *for* $t \in [t_0, \infty)$.

(5) $J^+(t_0, u_0) = [t_0, \infty)$.

(6) $J^+(t_0, \varphi_0) = [t_0, \infty)$.

Then the following assertions are valid:

(1) *If the trivial solution of* (2.2) *is practically u-stable, then the trivial solution of* (2.1) *is practically stable.*

(2) *If the trivial solution of* (2.2) *is uniformly practically u-stable, then the trivial solution of* (2.1) *is uniformly practically stable.*

(3) *If the trivial solution of* (2.2) *is contractively practically u-stable, then the trivial solution of* (2.1) *is contractively practically stable.*

(4) *If the trivial solution of* (2.2) *is contractively uniformly practically u-stable, then the trivial solution of* (2.1) *is contractively uniformly practically stable.*

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PROOF. Inequality (3.1) implies the inequalities

$$V(t_0, \varphi_0(t_0)) \leq b(t_0, |\varphi_0(t_0)|)e \leq b(t_0, \|\varphi_0\|)e$$

and, applying Corollary 1 for $u_0 = b(t_0, \|\varphi_0\|)e$, we get the estimate

$$|x(t; t_0, \varphi_0)| \leq a^{-1}\big[\max_j u_j^+(t; t_0, b(t_0, \|\varphi\|)e)\big]. \tag{3.2}$$

(1) If the trivial solution of (2.2) is practically u-stable, then a function $\varphi_1 \in \mathbf{K}^*$ exists so that $u^+(t; t_0, b(t_0, \|\varphi_0\|)e) \leq \varphi_1(t_0, \|\varphi_0\|)e$ and $\varphi_1(t_0, \lambda) < a(A)$. Then (3.2) yields

$$|x(t; t_0, \varphi_0)| \leq a^{-1}[\varphi_1(t_0, \|\varphi_0\|)] = \varphi(t_0, \|\varphi_0\|)$$

and

$$\varphi(t_0, \lambda) = a^{-1}[\varphi_1(t_0, \lambda)] < a^{-1}(a(A)) = A.$$

Since $\varphi_1 \in \mathbf{K}^*$ and $a \in \mathbf{K}$, then $\varphi \in \mathbf{K}^*$. The practical stability of the solution $x(t) \equiv 0$ of (2.1) is proved.

(2) If the solution $u(t) \equiv 0$ of (2.2) is uniformly practically u-stable, then a function $\varphi_2 \in \mathbf{K}$ exists so that $u^+(t; t_0, b(t_0, \|\varphi_0\|)e) \leq \varphi_2(\|\varphi_0\|)e$ and $\varphi_2(\lambda) < a(A)$. Then (3.2) yields

$$|x(t; t_0, \varphi_0)| \leq a^{-1}[\varphi_2(\|\varphi_0\|)] = \varphi(\|\varphi_0\|)$$

and

$$\varphi(\lambda) = a^{-1}(\varphi_2(\lambda)) < a^{-1}(a(A)) = A.$$

Since $\varphi_2, a \in \mathbf{K}$, then $\varphi \in \mathbf{K}$. Therefore the solution $x(t) \equiv 0$ of (2.1) is uniformly practically stable.

(3) If the solution $u(t) \equiv 0$ of (2.2) is contractively practically u-stable, then a function $\varphi_3 \in \mathbf{K}^*$ and a function $d : (t_0, \infty) \to R^+$ exist so that

$$u^+(t; t_0, b(t_0, \|\varphi_0\|)e) \leq \varphi_3(t_0, \|\varphi_0\|)d(t)e,$$

$\varphi_3(t_0, \lambda)d(t) < a(A)$ and $\varphi_3(t_0, \lambda)d(t_0 + \tau) < a(B)$ for some $\tau > 0$. Then (3.2) yields

$$|x(t; t_0, \varphi_0)| \leq a^{-1}[\varphi_3(t_0, \|\varphi_0\|)d(t)] = \varphi(t_0, \|\varphi_0\|)\psi(t),$$

where $\varphi \in \mathbf{K}^*$ and $\psi : (t_0, \infty) \to R^+$. Moreover,

$$\varphi(t_0, \lambda)\psi(t) = a^{-1}[\varphi_3(t_0, \lambda)d(t)] < a^{-1}(a(A)) = A$$

and

$$\varphi(t_0, \lambda)\psi(t_0 + \tau) = a^{-1}[\varphi_3(t_0, \lambda)d(t_0 + \tau)] < a^{-1}(a(B)) = B$$

for some $\tau > 0$. Hence the trivial solution of (2.1) is contractively practically stable.

Assertion (4) of Theorem 1 is proved in the same way as Assertion (3) of the same theorem.

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In studying the practical stability of the solutions of Problem (2.1) it can sometimes be suitable to represent the differential inequality used in Lemma 1 and Corollary 1 in the form $D_-V(t, x(t)) \le g(t, V(t, x(t)), x(t))$, where $g : (t_0, \infty) \times G \times \Omega \to R^m$.

Here we consider the auxiliary problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-h)), & t > t_0,\ t \ne \tau_k, \\ \dot{v}(t) = g(t, v(t), x(t)), & t > t_0,\ t \ne \tau_k, \\ x(t) = \varphi_0(t), & t \in [t_0 - h, t_0], \\ v(t_0 + 0) = v_0, & \\ \Delta x(\tau_k) = I_k(x(\tau_k)), & \tau_k > t_0,\ k = 1, 2, \dots, \\ \Delta v(\tau_k) = J_k(v(\tau_k)), & \tau_k > t_0,\ k = 1, 2, \dots, \end{cases} \tag{3.3}$$

where $v : (t_0, \infty) \to G$, $x : (t_0, \infty) \to \Omega$, $v_0 \in G$, $\varphi_0 \in C_0$.

We now introduce some notation: let $(x(t; t_0, \varphi_0, v_0), v(t; t_0, \varphi_0, v_0)) = (x(t), v(t))$ be the solution of Problem (3.3); and let $J^+(t_0, \varphi_0, v_0)$ be the maximal interval of type $[t_0, \beta)$ in which the solution $(x(t; t_0, \varphi_0, v_0), v(t; t_0, \varphi_0, v_0))$ is defined.

DEFINITION 7. The solution $(x(t), v^+(t)) = (x(t; t_0, \varphi_0, v_0), v^+(t; t_0, \varphi_0, v_0))$ of Problem (3.3) is said to be the *v-maximal solution* of (3.3) if for any other solution $(x(t), v(t)) = (x(t; t_0, \varphi_0, v_0), v(t; t_0, \varphi_0, v_0))$ of (3.3) the inequality $v(t) \le v^+(t)$ holds for all t for which both solutions $(x(t), v^+(t))$ and $(x(t), v(t))$ are defined.

The v-minimal solution of Problem (3.3) is defined analogously. In this case the method of comparison is based on a lemma whose proof is analogous to the proof of Lemma 1.

LEMMA 2. *Let the following conditions hold*:

(1) *Conditions* H1–H8 *are met.*

(2) *The function g is quasi-monotone increasing with respect to v, continuous in the sets $(\tau_k, \tau_{k+1}] \times G \times \Omega$, $k \in N \cup \{0\}$ and $(x(t), v^+(t))$ is the v-maximal solution of Problem* (3.3) *defined in the interval $J^+(t_0, \varphi_0, v_0)$.*

(3) *For each $k \in N \cup \{0\}$ and $(v, x) \in G \times \Omega$ there exists the limit*

$$\lim_{\substack{(t,u,x)\to(t,v,x)\\ t>\tau_k}} g(t, u, x).$$

(4) *The functions $\psi_k : G \to R^m$, $\psi_k(u) = u + J_k(u)$, $k = 1, 2, \dots$, are monotone increasing in G.*

(5) *The function $V \in V_0$ is such that $V(t_0, \varphi_0(t_0)) \le v_0$ and the inequalities*

$$D_-V(t, x(t)) \le g(t, V(t, x(t)), x(t)), \quad t \ne \tau_k,\ k = 1, 2, \dots,$$
$$V(t + 0, x(t) + I_k(x(t))) \le \psi_k(V(t, x(t))), \quad t = \tau_k,\ k = 1, 2, \dots,$$

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are valid for $t \in J^+(t_0, \varphi_0, v_0)$ *and* $x \in \Omega_t$.

Then $V(t, x(t; t_0, \varphi_0)) \le v^+(t; t_0, \varphi_0, v_0)$ *for* $t \in J^+(t_0, \varphi_0, v_0)$.

COROLLARY 2. *Let the following conditions hold*:

(1) *The conditions of Lemma 2 are satisfied.*

(2) *There exists a function* $a \in \mathbf{K}$ *such that* $a(|x|) \le \max_{1 \le j \le m} V_j(t, x)$, *where* $(t, x) \in [t_0, \infty) \times \Omega$.

Then for $t \in J^+(t_0, \varphi_0, v_0)$ *the following inequality is valid*:

$$|x(t; t_0, \varphi_0)| \le a^{-1}\big[\max_j v_j^+(t; t_0, \varphi_0, v_0)\big]. \tag{3.4}$$

Let $\mathbf{P}$ be the class of all continuous and monotone increasing functions $\varphi : R^m \to R^+$ such that $\varphi(0) = 0$ and $\varphi(v) \to \infty$ as $v \to \infty$, while $\mathbf{P}^*$ is the class of all continuous functions $\varphi : [t_0, \infty) \times R^m \to R^+$ monotone increasing along v, $v(t, 0) = 0$ and $\varphi(t, v) \to \infty$ as $|v| \to \infty$.

DEFINITION 8. The trivial solution of Problem (3.3) is said to be:

(1) *Practically v-stable with respect to* λ, A if

$$(\forall t_0 \in R)(\forall v_0 \in G \text{ and } 0 \le v_0 \le \lambda e)(\exists \varphi \in \mathbf{P}^*)$$
$$(\exists a \in \mathbf{K})(\forall \varphi_0 \in C_0)(\forall t \in J^+(t_0, \varphi_0, v_0)) :$$
$$v^+(t; t_0, \varphi_0, v_0) \le \varphi(t_0, v_0)e \quad \text{and} \quad \varphi(t_0, \lambda e) < a(A).$$

(2) *Uniformly practically v-stable with respect to* λ, A if

$$(\forall v_0 \in G \text{ and } 0 \le v_0 \le \lambda e)(\exists \varphi \in \mathbf{P})(\exists a \in \mathbf{K})$$
$$(\forall t_0 \in R)(\forall \varphi_0 \in C_0)(\forall t \in J^+(t_0, \varphi_0, v_0)) :$$
$$v^+(t; t_0, \varphi_0, v_0) \le \varphi(v_0)e \quad \text{and} \quad \varphi(\lambda e) < a(A).$$

(3) *Contractively practically v-stable with respect to* λ, A, B if

$$(\forall t_0 \in R)(\forall v_0 \in G \text{ and } 0 \le v_0 \le \lambda e)(\exists \varphi \in \mathbf{P}^*)(\exists a \in \mathbf{K})$$
$$(\exists \psi : (t_0, \infty) \to R^+)(\forall \varphi_0 \in C_0)(\forall t \in J^+(t_0, \varphi_0, v_0)) :$$
$$v^+(t; t_0, \varphi_0, v_0) \le \varphi(t_0, v_0)\psi(t)e, \varphi(t_0, \lambda e)\psi(t) \le a(A) \quad \text{and}$$
$$\varphi(t_0, \lambda e)\psi(t_0 + \tau) < a(B) \quad \text{for some } \tau > 0.$$

(4) *Contractively uniformly practically v-stable with respect to* λ, A, B if

$$(\forall v_0 \in G \text{ and } 0 \le v_0 \le \lambda e)(\exists \varphi \in \mathbf{P})(\exists a \in \mathbf{K})(\exists \psi : (t_0, \infty) \to R^+)$$
$$(\forall t_0 \in R)(\forall \varphi_0 \in C_0)(\forall t \in J^+(t_0, \varphi_0, v_0)) :$$
$$v^+(t; t_0, \varphi_0, v_0) \le \varphi(v_0)\psi(t)e, \varphi(\lambda e)\psi(t) \le a(A) \quad \text{and}$$
$$\varphi(\lambda e)\psi(t_0 + \tau) < a(B) \quad \text{for some } \tau > 0.$$

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THEOREM 2. *Let the following conditions be fulfilled*:

(1) *The conditions of Corollary* 2 *are met.*

(2) *There exists a function* $b \in \mathbf{K}^*$ *such that* $V(t_0, x) \le b(t_0, |x|)e$ *for* $x \in \Omega$.

(3) $g(t, 0, 0) = 0$ *for* $t \in [t_0, \infty)$.

(4) $J_k(0) = 0$, $k = 1, 2, \dots$.

(5) $J^+(t_0, \varphi_0, v_0) = [t_0, \infty)$.

Then the following assertions are valid:

(1) *If the trivial solution of* (3.3) *is practically* v*-stable, then the trivial solution of* (2.1) *is practically stable.*

(2) *If the trivial solution of* (3.3) *is uniformly practically* v*-stable, then the trivial solution of* (2.1) *is uniformly practically stable.*

(3) *If the trivial solution of* (3.3) *is contractively practically* v*-stable, then the trivial solution of* (2.1) *is contractively practically stable.*

(4) *If the trivial solution of* (3.3) *is contractively uniformly practically* v*-stable, then the trivial solution of* (2.1) *is contractively uniformly practically stable.*

The proof of Theorem 2 is analogous to the proof of Theorem 1; however, Definition 8 is used instead of Definition 5, and (3.4) is applied instead of (2.4).

4. Applications

4.1. The delay differential equation

$$\dot{u}(t) = ru(t)\left[1 - \frac{u(t-\tau)}{K}\right], \quad t \ge 0 \tag{4.1}$$

called Hutchinson's equation [11], is a single species population growth model, where r, τ and K are positive constants. This equation has been studied by many authors; see for example Cunningham [8], Gopalsamy [9], Kuang [12], Zhang and Gopalsamy [21, 22].

In this section we consider the case where at certain moments biotic and anthropogeneous factors act on the population "momentarily" so that the population number varies by jumps. Precisely we are concerned with the practical stability of the zero solution of the equation of the form

$$\begin{cases} \dot{N}(t) = rN(t)[1 - N(t-\tau)/K], & t \ge 0,\ t \ne \tau_k, \\ \Delta N(\tau_k) = N(\tau_k + 0) - N(\tau_k) = \alpha_k N(\tau_k), & k = 1, 2, \dots, \end{cases} \tag{4.2}$$

where $0 < \tau_1 < \tau_2 < \cdots$; $\lim_{k\to\infty} \tau_k = \infty$ and α_k are constants which characterize the magnitude of the impulse effect at the moments τ_k.

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Let $\varphi_1 \in C[[-\tau, 0], R^+]$ and let $N(t, \varphi_1)$ be the solution of (4.2) for which $N(s, \varphi_1) = \varphi_1(s)$, $s \in [-\tau, 0]$, $\varphi_1(0) > 0$.

We consider the function $V(t, x) = (1 - x/K)^2$. Then the set Ω_t is

$$\Omega_t = \left\{ N \in PC[R^+, (0, \infty)] : \left(1 - \frac{N(s)}{K}\right)^2 \le \left(1 - \frac{N(t)}{K}\right)^2, t - \tau \le s \le t \right\}.$$

THEOREM 3. *Let the following conditions be fulfilled*:

(1) $0 < \tau_1 < \tau_2 < \cdots$; $\lim_{k\to\infty} \tau_k = \infty$.

(2) *There exists a constant* $K_0 < K$ *such that* $0 < K_0 < (1 + \alpha_k)K_0$, $k = 1, 2, \ldots$.

(3) *For each* $k = 1, 2, \ldots$ *and* $N \in R^+$ *the following condition holds*: $(1+\alpha_k)N < K$.

Then the zero solution of (4.2) *is uniformly practically stable.*

PROOF. For $t \ge 0$ and $N \in \Omega_t$ we have

$$\begin{aligned} D_- V(t, N(t)) &= 2(1 - N(t)/K)(-1/K)rN(t)(1 - N(t - \tau)/K) \\ &\le -(2r/K)N(t)V(t, N(t)) \le -2rV(t, N(t)), \end{aligned}$$

where $t \ne \tau_k$, $k = 1, 2, \ldots$. From Conditions (2) and (3) of Theorem 3 it follows that

$$V(\tau_k+0, N(\tau_k)+\alpha_k N(\tau_k)) = \left[1 - \frac{(1+\alpha_k)N(\tau_k)}{K}\right]^2 < V(\tau_k, N(\tau_k)), \quad k = 1, 2, \ldots .$$

It is easy to see that the trivial solution of the problem given by

$$\begin{cases} \dot{u}(t) = -2ru(t), & t \ge 0,\ t \ne \tau_k, \\ \Delta u(\tau_k) = 0, & k = 1, 2, \ldots, \end{cases}$$

is uniformly practically u-stable. Then by Theorem 1 we obtain that the trivial solution of (4.2) is uniformly practically stable.

4.2. Gopalsamy and Ladas [10] proposed a model of a single species population exhibiting the so-called Allee effect [1] in which the per-capita growth rate is a quadratic function of the density and subject to delays. In particular, they studied the solutions of

$$\dot{N}(t) = N(t)[a + bN(t - \tau) - cN^2(t - \tau)], \tag{4.3}$$

where a, $c \in (0, \infty)$, $b \in R$ and $\tau \in [0, \infty)$. If $a > 0$, $b < 0$ and $c = 0$, then (4.3) reduces to an equation of Hutchinson's type (4.1).

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The purpose of this section is to consider the following model:

$$\begin{cases} \dot{N}(t) = N(t)[a(t) + b(t)N(t-\tau) - c(t)N^2(t-\tau)], & t \geq 0, t \neq \tau_k, \\ \Delta N(\tau_k) = I_k(N(\tau_k)), & k = 1, 2, \dots, \end{cases} \tag{4.4}$$

where a, b, c are continuous functions, a and c are positive functions, $b \in R$; $0 < \tau_1 < \tau_2 < \cdots$, $\lim_{k\to\infty} \tau_k = \infty$ and I_k are functions which characterize the magnitude of the impulse effect at the moments τ_k.

Let $a^0 = \max_{t\in[-\tau,\infty)} a(t)$, $b^0 = \max_{t\in[-\tau,\infty)} b(t)$ and $c_0 = \min_{t\in[-\tau,\infty)} c(t)$. Introduce the notation:

$$g(u) = a^0 + b^0 u - c_0 u^2, \quad u_1 = \frac{b^0 - \sqrt{b^{02} + 4a^0 c_0}}{2c_0}, \quad u_2 = \frac{b^0 + \sqrt{b^{02} + 4a^0 c_0}}{2c_0}.$$

THEOREM 4. *Let the following conditions be fulfilled*:

(1) $0 < \tau_1 < \tau_2 < \dots$; $\lim_{k\to\infty} \tau_k = \infty$.
(2) $F(u) = 2u^2 \dot{g}(u_1)(u - u_1)$ *is the function such that* $F \in \mathbf{K}$ *for* $u \in R^+$.
(3) $[u + I_k(u)]^2 \leq G_k(u)$, *where* $G_k : [0, K_0] \to [0, K)$ *and* $G_k \in \mathbf{K}$, $k = 1, 2, \dots$.
(4) *The zero solution of the equation*

$$\begin{cases} \dot{u}(t) = F(u(t)), & t \geq 0, \ t \neq \tau_k, \\ \Delta u(\tau_k) = G_k(u(\tau_k)) - u(\tau_k), & k = 1, 2, \dots, \end{cases} \tag{4.5}$$

is uniformly practically u-stable.

Then the zero solution of (4.4) *is uniformly practically stable.*

PROOF. Let $V(t, N) = N^2$. Then the set Ω_t is given by

$$\Omega_t = \{N \in PC[R^+, (0, \infty)] : N^2(s) \leq N^2(t), t - \tau \leq s \leq t\}.$$

For $t \geq 0$, $t \neq \tau_k$ and $N \in \Omega_t$ we have

$$\begin{aligned} D_- V(t, N(t)) &= 2N^2(t)[a(t) + b(t)N(t-\tau) - c(t)N^2(t-\tau)] \\ &\leq 2N^2(t)[a^0 + b^0 N(t-\tau) - c_0 N^2(t-\tau)] \\ &\leq 2N^2(t)\dot{g}(u_1)[N(t-\tau) - u_1] \\ &\leq 2N^2(t)\dot{g}(u_1)[N(t) - u_1] = F(N(t)). \end{aligned}$$

From Condition (3) of Theorem 4 it follows that

$$V(\tau_k + 0, N + I_k(N)) = [N + I_k(N)]^2 \leq G_k(N), \quad k = 1, 2, \dots.$$

Since by Condition (4) of Theorem 4 the zero solution of (4.5) is uniformly practically u-stable then Theorem 1 implies that the zero solution of (4.4) is uniformly practically stable.

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