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RESEARCH ARTICLE

Finding product sets in some classes of amenable groups

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Abstract
In [15], using methods from ergodic theory, a longstanding conjecture of Erdős (see [5, Page 305]) about sumsets in large subsets of the natural numbers was resolved. In this paper, we extend this result to several important classes of amenable groups, including all finitely generated virtually nilpotent groups and all abelian groups $(G, +)$ with the property that the subgroup $2G := \{g + g : g \in G\}$ has finite index. We prove that in any group G from the above classes, any $A \subset G$ with positive upper Banach density contains a shifted product set of the form $\{tb_ib_j : i < j\}$, for some infinite sequence $(b_n)_{n\in\mathbb{N}}$ and some $t \in G$. In fact, we show this result for all amenable groups that posses a property which we call square absolute continuity. Our results provide answers to several questions and conjectures posed in [13].

Contents

1. Introduction

In [15], Bryna Kra, Joel Moreira, Florian K. Richter and Donald Robertson, using methods from ergodic theory, proved that every subset A of the positive integers with positive upper Banach density contains $\{b_1 + b_2 + t : b_1 \neq b_2 \in B\}$ for some infinite set $B \subset A$ and some $t \in \mathbb{N}$. This resolved a longstanding conjecture of Erdős (see [5, Page 305]).

A natural question to ask is whether this result generalizes to other countable groups, such as $\mathbb{Z}^d$ for $d \geq 2$ or the discrete Heisenberg group, for example. The purpose of this paper is to extend the result in [15] to several important classes of amenable groups, including all finitely generated virtually nilpotent groups and all abelian groups $(G, +)$ with the property that the subgroup consisting of the elements $2g := g + g$, where $g \in G$, has finite index. To this end, we first extend the result to all amenable groups satisfying a property that we call square absolute continuity (see Definition 1.4). Then we show that our result applies to the aforementioned classes of groups by showing that they are (virtually) square absolutely continuous. Our main results provide partial answers to some questions and conjectures posed in [13] regarding product sets in large subsets of amenable groups.

Throughout, let G denote a countable group. Let us start with some basic definitions.

Definition 1.1. Let $(G, \cdot)$ be a group. A sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ of finite subsets of G is

- a *left Følner sequence* if it satisfies

$$\lim_{N\to\infty} \frac{|g\Phi_N \cap \Phi_N|}{|\Phi_N|} = 1 \quad \text{or equivalently,} \quad \lim_{N\to\infty} \frac{|g\Phi_N \triangle \Phi_N|}{|\Phi_N|} = 0$$

  for any $g \in G$, and
- a *right Følner sequence* if it satisfies

$$\lim_{N\to\infty} \frac{|\Phi_N g \cap \Phi_N|}{|\Phi_N|} = 1 \quad \text{or equivalently,} \quad \lim_{N\to\infty} \frac{|\Phi_N g \triangle \Phi_N|}{|\Phi_N|} = 0$$

  for any $g \in G$.

If both conditions are satisfied, then Φ is a *two-sided Følner sequence*.

We remark that if a group admits a left (or right) Følner sequence, then it admits a two-sided Følner sequence. Amenable groups, which are the central object of our study, are defined as follows:

Definition 1.2. A group $(G, \cdot)$ is called amenable if it admits a left Følner sequence.

The most common example of an amenable group is $\mathbb{Z}^d$, for any $d \in \mathbb{N}$. Other examples of amenable groups are finite groups, abelian groups, solvable groups and finitely generated groups of subexponential growth. In addition, products of amenable groups and virtually amenable groups are amenable. Følner sequences are useful to define notions of density in amenable groups.

Definition 1.3. Let $(G, \cdot)$ be an amenable group, Φ a left (right) Følner sequence, and let $A \subset G$. Then the *left (right) upper density* of A with respect to Φ is defined as

$$\overline{d}_\Phi(A) := \limsup_{N\to\infty} \frac{|A \cap \Phi_N|}{|\Phi_N|}.$$

We say that A has *positive left (right) upper Banach density* if it has positive left (right) upper density with respect to some left (right) Følner sequence. We also say that A has *positive upper Banach density* if it has positive upper density with respect to some two-sided Følner sequence.

Note that if G is an amenable group, and $A \subset G$ has positive left upper Banach density, then this does not necessarily mean that A has positive right upper Banach density.

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Given a group $(G,\cdot)$ and a sequence $B=(b_n)_{n\in\mathbb{N}}\subset G$, we define

$$B \blacktriangleleft B := \{b_i b_j : i < j\},$$

$$B \blacktriangleright B := \{b_i b_j : i > j\},$$

and

$$B \odot B := \{b_i b_j : i \neq j\}.$$

If G is abelian, then $B\blacktriangleleft B = B\blacktriangleright B = B\odot B$, which we also denote by $B\oplus B$ if the group operation in G is written using additive notation. We refer to the map $s_G : G\to G, s_G(g)=g^2$ as the *squaring map on* G. The image of this map is the subset of G consisting of all the elements of the form g^2, where $g\in G$. We denote this by G^2 (i.e., $s_G(G)=G^2$), and we often refer to it as the *subset of squares*.

Definition 1.4. Let G be an amenable group and $\phi : G\to G$ be a map. We say that G is *ϕ-absolutely continuous* if G admits two left Følner sequences $\Phi=(\Phi_N)_{N\in\mathbb{N}}$ and $\Psi=(\Psi_N)_{N\in\mathbb{N}}$ satisfying the following: for any $\varepsilon>0$, there exists some $\delta>0$ such that for any $u : G\to[0,1]$ satisfying

$$\limsup_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N}u(g)<\delta,$$

we have that

$$\limsup_{N\to\infty}\frac{1}{|\Psi_N|}\sum_{g\in\Psi_N}u(\phi(g))<\varepsilon.$$

If, in particular, $\phi=s_G$, then we say that G is *square absolutely continuous*.

Remark 1.5. For a set $C\subset G$, we denote by C^{-1} the set $C^{-1}:=\{g^{-1} : g\in C\}$. Note that when ϕ is the squaring map $\phi=s_G$, the existence of two left Følner sequences in Definition 1.4 is equivalent to the existence of two right Følner sequences: for any pair Φ,Ψ of left Følner sequences, the pair $\Phi^{-1}=(\Phi_N^{-1})_{N\in\mathbb{N}}$, $\Psi^{-1}=(\Psi_N^{-1})_{N\in\mathbb{N}}$ is a pair of right Følner sequences, and if Φ,Ψ satisfy the conditions of Definition 1.4, then so do Φ^{-1},Ψ^{-1}. More precisely, given $\varepsilon>0$, there is some $\delta>0$ so that the conditions of Definition 1.4 are satisfied for Φ,Ψ. Let $u : G\to[0,1]$ satisfy $\limsup_{N\to\infty}\frac{1}{|\Phi_N^{-1}|}\sum_{g\in\Phi_N^{-1}}u(g)<\delta$ and consider the map $w : G\to[0,1]$, $w(g):=u(g^{-1})$. Then $\limsup_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N}w(g)<\delta$, and therefore, $\limsup_{N\to\infty}\frac{1}{|\Psi_N|}\sum_{g\in\Psi_N}w(g^2)<\varepsilon$, which in turn implies that $\limsup_{N\to\infty}\frac{1}{|\Psi_N^{-1}|}\sum_{g\in\Psi_N^{-1}}u(g^2)<\varepsilon$.

1.1. Main results

Throughout, we say that a sequence $(b_n)_{n\in\mathbb{N}}$ in G is *infinite* if the set $\{b_n : n\in\mathbb{N}\}$ is infinite. The first main theorem of this paper is the following:

Theorem 1.6. *Let G be a square absolutely continuous group and $A\subset G$ with positive left upper Banach density. Then there exist an infinite sequence $B=(b_n)_{n\in\mathbb{N}}\subset A$ and some $t\in G$ such that*

$$B\blacktriangleleft B\subset t^{-1}A.$$

Theorem 1.6 provides a positive answer to [13, Question 5.16], with the extra assumption that G is square absolutely continuous, and under the weaker assumption that the set A has positive left upper Banach density, instead of positive upper Banach density.

Remark 1.7. Note that Theorem 1.6 immediately implies an analogous result for right upper Banach density instead of left upper Banach density. Indeed, since by Remark 1.5, square absolute continuity

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is preserved through the map $g \mapsto g^{-1}$, Theorem 1.6 is equivalent to the assertion that for any $A \subset G$ with positive right upper Banach density, there exists an infinite sequence $C = (c_n)_{n\in\mathbb{N}} \subset A$ and some $r \in G$ such that

$$C \blacktriangleright C \subset Ar^{-1}.$$

Before continuing, let us recall the following definitions for a group G:

- G is *nilpotent* if its lower central series is finite, that is to say there is $n \in \mathbb{N}$ such that

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e_G\},$$

  where $G_{i+1} := [G_i, G]$ is the commutator group of G_i and G, that is, the subgroup of G generated by the elements of the form $hgh^{-1}g^{-1}$, where $h \in G_i, g \in G$.
- G is *finitely generated* if there exist $g_1, \dots, g_n \in G$ such that any element of G can be written as product of $g_1, \dots, g_n$.
- G is *torsion-free* if it does not have any nontrivial element of finite order, that is to say, for any $g \in G$ with $g \neq e_G$ and any $n \in \mathbb{N}$, we have $g^n \neq e_G$.
- If P is a property of groups, then we say that a group G is *virtually* P if it has a finite-index subgroup that has the property P.

In any finitely generated nilpotent group G, there exist some $s \in \mathbb{N}$ (depending on the degree of nilpotency and the number of generators of G), some $a_i \in G$ and some functions $t_i : G \longrightarrow \mathbb{Z}$, for $1 \leq i \leq s$, such that any $x \in G$ can be written as $x = a_1^{t_1(x)} \cdots a_s^{t_s(x)}$. The s-tuple $(a_1, \dots, a_s)$ is a *Mal'cev basis*, and the s-tuple $(t_1, \dots, t_s)$ is a *Mal'cev coordinate system* with respect to this Mal'cev basis. If G is also torsion-free, then the coordinate maps are injective, and hence, we can identify G with $\mathbb{Z}^s$ and it is convenient to also identify any $x \in G$ with its coordinates $(t_1(x), \dots, t_s(x)) \in \mathbb{Z}^s$. The above facts about Mal'cev bases can be found in [11, Chapter 17.2].

Theorem 1.8. *Every torsion-free finitely generated nilpotent group is square absolutely continuous.*

Combining Theorems 1.6 and 1.8, we have that every torsion-free finitely generated nilpotent group satisfies the conclusion of Theorem 1.6. In fact, we prove the following slight strengthening:

Corollary 1.9. *Let G be a torsion-free finitely generated nilpotent group and $A \subset G$ with positive left upper Banach density. Then there exist an infinite sequence $B = (b_n)_{n\in\mathbb{N}} \subset A$ and some $t \in G$ such that*

$$B \blacktriangleleft B \subset t^{-1}A.$$

Moreover, given a Mal'cev coordinate system $(t_1, \dots, t_s)$ on G, we can choose B so that the following holds: for any finite set $C \subset \mathbb{Z}$ and any $1 \leq i \leq s$, the set $\{b \in B \colon t_i(b) \in C\}$ is finite.

Furthermore, we are able to extend the first statement of Corollary 1.9 to all finitely generated virtually nilpotent groups.

Corollary 1.10. *Let G be a finitely generated virtually nilpotent group and $A \subset G$ with positive left upper Banach density. Then there exist $g \in G$, an infinite sequence $B = (b_n)_{n\in\mathbb{N}} \subset g^{-1}A$, and some $t \in G$ such that*

$$B \blacktriangleleft B \subset t^{-1}A.$$

In particular, this holds for the group $UT(n, F)$,[1] where $n \in \mathbb{N}$ and $F = (F, +, \cdot)$ is any infinite commutative unital ring with the property that the additive group $(F, +)$ is finitely generated.

[1] $UT(n, F)$ is the unitriangular $n \times n$ matrix group with entries from F. Note that $U(3, \mathbb{Z})$ is the well-known 3×3 Heisenberg group.

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Remark 1.11. Note that the class of finitely generated virtually nilpotent groups coincides, in view of Gromov's theorem [8], with the class of finitely generated groups of polynomial growth.

The final part of this subsection is concerned with sumsets in abelian groups. Let $G = (G, +)$ be an abelian group. We write $2g$ to denote the element $g + g$ for any $g \in G$. Moreover, we refer to the map $s_G : G \longrightarrow G$ as the *doubling map*, and its image is now the subgroup of G consisting of all elements of the form $2g$, where $g \in G$. We denote this subgroup by $2G$ (i.e., $s_G(G) = 2G$), and we often refer to it as the *doubling subgroup*.

In [13, Conjecture 5.14], it is conjectured that in any countable abelian group G, every set of positive upper Banach density contains a set of the form $B \oplus B + t = \{b_1 + b_2 + t : b_1 \neq b_2 \in B\}$ for an infinite set $B \subset G$ and some $t \in G$. It follows from Corollary 1.10 that this conjecture holds under the additional assumption that G is finitely generated, and moreover, it extends [15, Theorem 1.2] from $(\mathbb{N}, +)$ to all finitely generated abelian groups. In fact, we extend [15, Theorem 1.2] to an even larger collection of abelian groups that contains all the finitely generated abelian groups along with some infinitely generated ones. The following theorem allows us to do so:

Theorem 1.12. *Every abelian group whose doubling subgroup has finite index is square absolutely continuous.*

The following corollary is an obvious consequence of Theorems 1.6 and 1.12:

Corollary 1.13. *Let $(G, +)$ be an abelian group such that $2G$ is a finite-index subgroup of G, and let $A \subset G$ with positive upper Banach density. Then there exist an infinite set $B \subset A$ and some $t \in G$ such that*

$$B \oplus B \subset A - t.$$

In particular, this holds for

- *all finitely generated abelian groups,* [2] *and*
- $(\mathbb{F}_p^\omega, +)$, [3] *where p is any odd prime.*

Corollary 1.13 is in fact optimal, in the sense that $2G$ being a finite-index subgroup of G is a necessary assumption. As shown in a recent paper of Ethan Ackelsberg [1], if $2G$ has infinite index in G, then one can always find a set A with upper Banach density arbitrarily close to 1 which does not contain any shifted sumset $t + B \oplus B$ of some infinite set B. Therefore, Corollary 1.13 along with the work in [1] fully resolve [13, Conjecture 5.14].

We remark that Corollary 1.13 can also be proved independently of Theorem 1.6, meaning that by slightly modifying the proof of Theorem 1.6, one can directly obtain the result for abelian groups with finite-index doubling subgroup without showing that such groups are square absolutely continuous.

1.2. More product sets and open questions

The following remark shows that in the formulation of Theorem 1.6 one can replace left shifts with right shifts and the statement remains true.

Remark 1.14. Let G and A be as in Theorem 1.6. Then there exist some $t \in G$ and some $B = (b_n)_{n\in\mathbb{N}} \subset tAt^{-1}$ such that

$$B \blacktriangleleft B \subset At^{-1}.$$

To see why, let $B' \subset A$ and $t \in G$ such that $B' \blacktriangleleft B' \subset t^{-1}A$, as guaranteed by Theorem 1.6, and then let $B = tB't^{-1}$.

[2] It is not hard to check that in finitely generated abelian groups, the doubling subgroup has finite-index.
[3] $\mathbb{F}_p^\omega$ is the direct product of infinitely many copies of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and it is clearly infinitely generated abelian.

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Aside from replacing left shifts with right shifts, it is also natural to ask whether one can replace product sets of the form $B \blacktriangleleft B$ with those of the form $B \blacktriangleright B$, and additionally when the restriction $B \subset A$ can be imposed. The following table addresses this question in the case when G is a square absolutely continuous group.

$B \blacktriangleleft B \subset t^{-1}A$, for $B \subset A$	True (Theorem 1.6; for $\overline{d}_\Phi(A) > 0$ for some left Følner Φ)
$B \blacktriangleleft B \subset At^{-1}$, for $B \subset G$	True (Remark 1.14; for $\overline{d}_\Phi(A) > 0$ for some left Følner Φ)
$B \blacktriangleleft B \subset At^{-1}$, for $B \subset A$	False (Example 6.1; with $\overline{d}_\Phi(A) > 0$ for some left Følner Φ)
$B \blacktriangleright B \subset t^{-1}A$, for $B \subset G$	False (Example 6.2; with $\overline{d}_\Phi(A) = 1$ for some left Følner Φ)
$B \blacktriangleright B \subset At^{-1}$, for $B \subset G$	False (Example 6.3; with $\overline{d}_\Phi(A) = 1$ for some left Følner Φ)

The above table shows that Theorem 1.6 is optimal for sets of positive left upper Banach density in noncommutative groups, in the sense that it is not necessarily true that one can find a product set of the form $B \odot B$ (or even $B \blacktriangleright B$) inside shifts of such sets. In addition, we remark that the table above provides a partial answer to [13, Question 5.19].

It remains interesting to ask whether product sets of the form $B \odot B$ can be found in sets with positive upper Banach density. Unfortunately, our methods here are insufficient to handle this case. In this spirit, we conclude this section with the two questions below. We remark that the second one is a special case of [13, Question 5.17].

Question 1.15. Let G be a square absolutely continuous group and $A \subset G$ be a set of positive upper Banach density. Is it true that there exists some infinite set $B \subset G$ such that

$$B \odot B \subset t^{-1}A \cup Ar^{-1}$$

for some $t, r \in G$?

Question 1.16. Let G and A be as in Question 1.15. Is it true that there exists some infinite set $B \subset G$ such that

$$B \odot B \subset t^{-1}Ar^{-1}$$

for some $t, r \in G$?

1.3. *Proof ideas*

To prove Theorem 1.6, we follow an ergodic-theoretic approach, and we employ ideas similar to the ones used in [15] in the setting of $(\mathbb{N}, +)$. This approach is based on methods that were introduced in [14] to generalize another sumset conjecture of Erdős, which was initially proved in [18] by Moreira, Richter and Robertson. However, the generality of the setting of amenable groups compared to $(\mathbb{N}, +)$ causes several issues and complications that we need to handle differently. These issues, along with the new ideas we develop to deal with them, are briefly discussed below.

After translating Theorem 1.6 into a dynamical statement (see Theorem 3.8), we reduce the problem to finding certain dynamical configurations given by limit points of orbits of ergodic measure-preserving G-actions, called *Erdős progressions* (see Definition 3.2). The natural environment in which one can study such progressions is the Kronecker factor of a system, as Erdős progressions are simply 3-term arithmetic progressions there.

One of the main obstructions we had to overcome in our proof is the lack of commutativity of G. The most notable among the issues that this leads to is that the Kronecker factor does not have the structure of an abelian group, but instead, it is a homogeneous space $Z = K/H$ for some compact group K. This makes the study of Erdős progressions more technically challenging. To be more precise, the abelian nature of the Kronecker factor in the setting of $(\mathbb{N}, +)$-actions is heavily used in [15]. Consequently, due to the absence of commutativity in our case, many of the techniques in [15] do not generalize easily to our setting. Another difficulty that arises in noncommutative groups is the erratic behavior of the set of squares G^2. In particular, orbits of points along G^2 may be trapped in zero-measure regions, which causes serious trouble in finding Erdős progressions. The assumption that G is square absolutely continuous is critical in avoiding this scenario. In addition, we need an extension of a result of Host and Kra ([9, Proposition 6.1]) concerning actions of $(\mathbb{N}, +)$, to the more general setting of amenable group actions (see Lemma 3.5, proof in Appendix A).

2. Preliminaries

In this section, we state all the preliminaries that will be useful in the rest of the paper regarding classic notions and theorems of ergodic theory of actions of amenable groups. So, for the rest of the section, G denotes an arbitrary countable and discrete amenable group.

Basics on G-systems: Given a compact metric space $X = (X, d_X)$, a *continuous action* $T = (T_g)_{g\in G}$ of G on X is a collection of continuous functions $T_g : X \to X$ such that for any $g_1, g_2 \in G$, $T_{g_1} \circ T_{g_2} = T_{g_1 g_2}$. Given such an action, we call the pair (X, T) a *topological G-system.*

Given a topological G-system (X, T) and a point $x \in X$, we define its *orbit* as $\mathcal{O}_T(x) = \{T_g x : g \in G\}$, and we say that the point is *transitive* if $\mathcal{O}_T(x)$ is dense in X.

Fix a topological G-system (X, T). Let $M(X)$ denote the space of Borel probability measures on X, equipped with the weak* topology, which is compact and metrizable. A measure $\mu \in M(X)$ is said to be T-invariant if it is invariant under T_g for all $g \in G$. Amenability of G implies that there are T-invariant measures in $M(X)$. The subset of $M(X)$ consisting of T-invariant measures is denoted by $M^T(X)$, and it is a nonempty closed and convex subset of $M(X)$. The Borel σ-algebra on X is denoted by $\mathscr{B}_X$ or just $\mathscr{B}$ if no confusion may arise.

For $\mu \in M^T(X)$, the action T on the Borel probability space (X, μ) is called a *measure-preserving G-action*, and (X, μ, T) is called a *measure-preserving G-system*. Note that we omit writing the symbol for the σ-algebra, and from now on, whenever this happens, the implied σ-algebra will be the Borel σ-algebra. For simplicity, we refer to the above as G-actions, and G-systems, respectively. Recall that all G-actions considered throughout are continuous.

Given a G-system (X, μ, T), one can define an action, which by abuse of notation will again be denoted by $T = (T_g)_{g\in G}$, of G on $L^2(X)$ by $T_g : L^2(X) \to L^2(X)$, $T_g f = f \circ T_g$. It is not hard see that for all $g \in G$, T_g is an isometry of $L^2(X)$. Note also that since G acts from the left on X, then G acts from the right on $L^2(X)$.

We remark that we are only considering G-systems where G acts on the prescribed space from the left, and then any associated Følner sequence will be considered left, without mentioning it, unless it is necessary.

Note that we could define G-systems more generally as follows: a G-system is a quadruple $(X, \mathcal{A}, \mu, T)$, where X is any set, $\mathcal{A}$ is a σ-algebra on X, μ is a probability measure on $(X, \mathcal{A})$ and T is a left action of G on X which is measurable and preserves μ. We chose to not define systems in that generality, as for our purposes, we will always work with the more specific G-systems defined above. The only occasion where we need this more general definition of G-systems is when we define the Kronecker factor right after Theorem 2.10, in which case $\mathcal{A}$ is a sub-σ-algebra of the Borel σ-algebra.

Product G-system: Given two G-systems (X, μ, T) and (Y, ν, S), we define the product G-system $(X \times Y, \mu \times \nu, T \times S)$, where the underlying σ-algebra is the product of the Borel σ-algebras on X and Y, which coincides with the Borel σ-algebra on $X \times Y$, and the action is $T \times S = (T_g \times S_g)_{g\in G}$.

Factors of G-systems: Given two G-systems (X, μ, T) and (Y, ν, S), we say that (Y, ν, S) is a *factor* of (X, μ, T) if there exists a measurable map $\pi : X \to Y$, which we call a *factor map*, satisfying $\mu(\pi^{-1}E) = \nu(E)$ for any measurable $E \subset Y$ and for any $g \in G$, $\pi \circ T_g = S_g \circ \pi$ μ-almost everywhere on X. When the former is true, we say that ν is the push-forward of μ under π, and we write $\pi\mu = \nu$. When

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additionally, the factor map π is continuous and $\pi \circ T_g = S_g \circ \pi$ holds everywhere on X for any $g \in G$, we say that π is a *continuous factor map* and (Y, ν, S) is a *continuous factor* of (X, μ, T).

Ergodicity and ergodic theorems for G-systems: A G-system (X, μ, T) is called *ergodic* if for any measurable set A the following holds:

$$T_g^{-1}A = A \text{ for all } g \in G \implies \mu(A) = 0 \text{ or } \mu(A) = 1.$$

Given a G-system (X, μ, T), let $\mathscr{A}$ be a sub-σ-algebra of $\mathscr{B}$. For $f \in L^2(X, \mu)$, the *conditional expectation of f on $\mathscr{A}$*, denoted by $\mathbb{E}_\mu(f \mid \mathscr{A})$, is defined as the orthogonal projection of f on the closed subspace $L^2(X, \mathscr{A}, \mu)$ of $L^2(X, \mu)$. We also denote the sub-σ-algebra of the T-invariant sets by $\mathcal{I} = \mathcal{I}(T)$; that is,

$$\mathcal{I} = \mathcal{I}(T) := \{E \in \mathscr{B} : T_g^{-1}E = E \text{ for all } g \in G\}.$$

Theorem 2.1 (Mean Ergodic Theorem for G-systems, see [7, Theorem 3.33]). *Let (X, μ, T) be a G-system, and let Φ be a Følner sequence. Then, for any $f \in L^2(X)$,*

$$\frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} T_g f \to \mathbb{E}_\mu(f \mid \mathcal{I})$$

as $N \to \infty$ in $L^2(X)$. In addition, if the system is ergodic, the ergodic averages above converge to $\int_X f \, \mathrm{d}\mu$.

Measure disintegration and ergodic decomposition: When X is a compact metric space, the space $M(X)$ of Borel probability measures on X can be endowed with a σ-algebra $\mathcal{M}$ such that the space $(M(X), \mathcal{M})$ is a standard Borel space. The following theorem about disintegrations of measures is very useful.

Theorem 2.2 (Disintegration of measures, see [10, Chapter 2, Section 2.5]). *Let X be a compact metric space, $\mathscr{B}$ the Borel σ-algebra on X and μ a probability measure on $(X, \mathscr{B})$. Let also $\mathscr{D}$ be a sub-σ-algebra of $\mathscr{B}$. Then there is a $(\mathscr{D}, \mu)$-almost everywhere defined and measurable map $(X, \mathscr{D}) \to (M(X), \mathcal{M})$, $x \mapsto \mu_x$ with the following properties:*

- *For every $f \in L^1(X, \mu)$, the function $x \mapsto \int_X f \, \mathrm{d}\mu_x$ is in $L^1(X, \mathscr{D}, \mu)$, and for all $D \in \mathscr{D}$, we have $\int_D f \, \mathrm{d}\mu = \int_D \left(\int_X f \, \mathrm{d}\mu_x \right) \mathrm{d}\mu(x)$. In particular, this implies that $\int_X f \, \mathrm{d}\mu_x = \mathbb{E}_\mu(f \mid \mathscr{D})(x)$ for $(\mathscr{D}, \mu)$-almost every $x \in X$.*
- *For $(\mathscr{D}, \mu)$-almost every $x \in X$, $\mu_x([x]_\mathscr{D}) = 1$, where $[x]_\mathscr{D} = \cap_{x \in D \in \mathscr{D}} D$.*

The map satisfying the above properties is unique modulo $(\mathscr{D}, \mu)$-null sets and is called the ***disintegration*** *of the measure μ over the sub-σ-algebra $\mathscr{D}$. In that case, we write $\mu = \int_X \mu_x \, \mathrm{d}\mu(x)$.*

In this paper, we will extensively make use of *disintegrations over (continuous) factor maps*. Let $\pi : (X, \mu, T) \to (Y, \nu, S)$ be a factor map between two G-systems. Given a function $f \in L^1(X, \mu)$, the conditional expectation $\mathbb{E}_\mu(f \mid Y)$ of f with respect to the factor Y is the function in $L^1(Y, \nu)$ defined by

$$\int_B \mathbb{E}_\mu(f \mid Y)(y) \, \mathrm{d}\nu(y) = \int_{\pi^{-1}(B)} f(x) \, \mathrm{d}\mu(x)$$

for every Borel measurable set $B \subset Y$. Then Theorem 2.2 gives a disintegration $y \mapsto \mu_y$ defined on Y, which is unique up to ν-null measure sets, and satisfies the following:

- for every $f \in L^1(X, \mu)$, for ν-almost every $y \in Y$,

$$\mathbb{E}_\mu(f \mid Y)(y) = \int_X f \, \mathrm{d}\mu_y, \tag{2.1}$$

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- for ν-almost every $y \in Y$, $\mu_y(\pi^{-1}(\{y\})) = 1$, and finally,
- for any $g \in G$ and for ν-almost every $y \in Y$, $(T_g)\mu_y = \mu_{S_g y}$.

Let (X, μ, T) be a G-system. Consider the (unique) disintegration of μ with respect to $\mathcal{I} = \mathcal{I}(T)$ given by Theorem 2.2. This disintegration is called the *ergodic decomposition of* μ. Equivalently, we say that the disintegration $x \mapsto \mu_x$ is the ergodic decomposition of μ if for any $f : X \to \mathbb{C}$ measurable and bounded,

$$\int_X f \, \mathrm{d}\mu_x = \mathbb{E}_\mu(f \mid \mathcal{I})(x) \tag{2.2}$$

holds for $(\mathcal{I}, \mu)$-almost every $x \in X$.

Theorem 2.3 (Ergodic decomposition of G-systems, see [7, Theorem 3.22]). *If (X, μ, T) is a G-system as above, then for $(\mathcal{I}, \mu)$-almost every $x \in X$, the measure μ_x is T-invariant and the system (X, μ_x, T) is ergodic.*

Generic points and the support of a measure: In addition, we will need the notion of generic points:

Definition 2.4. Let (X, μ, T) be a G-system and let Φ be a Følner sequence. A point $a \in X$ is called *generic for μ along* Φ if for all $f \in C(X)$, we have

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} f(T_g a) = \int_X f \, \mathrm{d}\mu$$

or equivalently if

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} \delta_{T_g a} = \mu,$$

where δ_x is the Dirac mass at $x \in X$ and the limit is in the weak* topology. If a is generic for μ along Φ, then we denote this by $a \in \mathtt{gen}(\mu, \Phi)$.

Moreover, we will need the notion of the support of a measure. The *support* of a Borel probability measure μ on a compact metric space X is the smallest closed full-measure subset of X and is denoted by $\mathtt{supp}(\mu)$. We will need the following lemma, which says that generic points for a measure have dense orbit in the support of the measure. Its proof is quite standard, and we only include it for completeness.

Lemma 2.5. *Let (Y, ν, S) be a G-system and let $y, w \in Y$. If $y \in \mathtt{gen}(\nu, \Phi)$ for some Følner sequence Φ, and $w \in \mathtt{supp}(\nu)$, then $S_{g_n} y \to w$, for some infinite sequence $(g_n)_{n\in\mathbb{N}}$ in G.*

Proof. Fix a compatible metric on X and let $\mathtt{B}(w, \varepsilon)$ be the open ball centered at w with radius $\varepsilon > 0$ with respect to this metric. By Urysohn's lemma, for every $\varepsilon > 0$, there exists a continuous function $f : X \to [0, 1]$ with $f = 1$ on $\mathtt{B}(w, \varepsilon/2)$ and $f = 0$ outside $\mathtt{B}(w, \varepsilon)$. Since $w \in \mathtt{supp}(\nu)$, it follows that $\int_Y f \, \mathrm{d}\nu > 0$. Now, using that $y \in \mathtt{gen}(\nu, \Phi)$, we have that

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} f(S_g y) = \int_X f \, \mathrm{d}\nu > 0,$$

which implies that $S_g y \in \mathtt{B}(w, \varepsilon)$ for infinitely many $g \in G$. The result then follows. □

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We will also make use of the following result of Lindenstrauss:

Proposition 2.6 (see [16, Theorem 1.2 and Proposition 1.4]). *Let (X, μ, T) be a G-system and Φ be a Følner sequence in G. Then there is a subsequence Ψ of Φ such that for all $f \in L^1(\mu)$,*

$$\lim_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} T_g f(x) = \mathbb{E}_\mu(f \mid \mathcal{I})(x)$$

for μ-almost every $x \in X$.

The next two lemmas follow easily from Proposition 2.6 and (2.2).

Lemma 2.7. *Let (X, μ, T) be an ergodic G-system. Then for any Følner sequence Φ, there exists some subsequence Ψ such that μ-almost every $x \in X$ is in $\mathtt{gen}(\mu, \Psi)$.*

Lemma 2.8. *Let (X, μ, T) be a G-system, let Φ be a Følner sequence, and let $x \mapsto \mu_x$ be the ergodic decomposition of μ. Then there exists some subsequence Ψ of Φ such that μ-almost every $x \in X$ is in $\mathtt{gen}(\mu_x, \Psi)$.*

Finally, it will be useful to have the following generalization of [6, Proposition 3.9], whose proof is again the same as for actions of $(\mathbb{N}, +)$, but we include it for the convenience of the reader.

Lemma 2.9. *Let G be an amenable group, let (X, μ, T) be an ergodic G-system, and let $a \in X$ be a point such that μ is supported on $\overline{\mathcal{O}_T(a)}$. Then there exists some Følner sequence Ψ such that $a \in \mathtt{gen}(\mu, \Psi)$.*

Proof. By Lemma 2.7, there exists some $x_0 \in \overline{\mathcal{O}_T(a)}$ that is generic for μ along some Følner sequence Φ. Let $\mathcal{F} = (f_k)_{k\in\mathbb{N}}$ be a dense subset of $(C(X), \|\cdot\|_\infty)$ and let $(\Phi_{N_n})_{n\in\mathbb{N}}$ be a subsequence of Φ such that for every $n \in \mathbb{N}$ and for every $j = 1, 2, \ldots, n$,

$$\left| \frac{1}{|\Phi_{N_n}|} \sum_{g\in\Phi_{N_n}} f_j(T_g x_0) - \int_X f_j \,\mathrm{d}\mu \right| < \frac{1}{n}.$$

Since $x_0 \in \overline{\mathcal{O}_T(a)}$, there exists some $(g_n)_{n\in\mathbb{N}} \subset G$ such that $T_{g_n} a \to x_0$, so that we may assume that the equation above holds if we substitute x_0 with $T_{g_n} a$. Consider the Følner sequence $\Psi = (\Psi_n)$ given by $\Psi_n = \Phi_{N_n} g_n$. Note that this is still a left Følner sequence. It follows that for every $n \in \mathbb{N}$ and any $j = 1, 2, \ldots, n$,

$$\left| \frac{1}{|\Psi_n|} \sum_{g\in\Psi_n} f_j(T_g a) - \int_X f_j \,\mathrm{d}\mu \right| < \frac{1}{n}.$$

Since $\mathcal{F}$ is dense in $C(X)$, the conclusion follows as before. □

Kronecker factor and the Jacobs-de Leeuw-Glicksberg decomposition: Let (X, μ, T) be a G-system. A function $f \in L^2(X)$ is called

- *compact* if $\overline{\{T_g f : g \in G\}}$ is compact with respect to the strong topology on $L^2(X)$.
- *weak-mixing* if for any Følner sequence Φ, and any $f' \in L^2(X)$,

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} |\langle T_g f, f'\rangle| = 0.$$

We define the *compact component* of $L^2(X)$ as $\mathcal{H}_c(T) = \overline{\mathrm{span}\{f \in L^2(X) : f \text{ is compact}\}}$, and the *weak-mixing component* of $L^2(X)$ as $\mathcal{H}_{wm}(T) = \{f \in L^2(X) : f \text{ is weak-mixing}\}$. When no confusion may arise, we simply write $\mathcal{H}_c$ and $\mathcal{H}_{wm}$, respectively.

In case that G is an amenable group, the Jacobs-de Leeuw-Glicksberg decomposition theorem applies, stating that these two components give a decomposition of $L^2(X)$.

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Theorem 2.10 (Jacobs-de Leeuw-Glicksberg decomposition, see [12, Theorem 2.24]). *If* (X, μ, T) *is a G-system, then*

$$L^2(X) = \mathcal{H}_c \oplus \mathcal{H}_{wm}.$$

Now we will give a description of the factor of (X, μ, T) corresponding to the subspace $\mathcal{H}_c$ of $L^2(X)$. Note that if $f \in \mathcal{H}_c$, then for all $g \in G$, $T_g f \in \mathcal{H}_c$, so $\mathcal{H}_c$ is invariant under the action of T on $L^2(X)$. Let $\mathcal{A}$ be the smallest σ-algebra with respect to which all functions in $\mathcal{H}_c$ are measurable. Then $\mathcal{A}$ is a T-invariant σ-algebra contained in the Borel σ-algebra of X. Therefore, the system $(X, \mathcal{A}, \mu, T)$ is a factor of the original system, with the factor map being the identity $\mathrm{id} : X \to X$. This factor is called the *Kronecker factor* of (X, μ, T).

Our goal now is to give a nice algebraic description of the Kronecker factor when the G-system (X, μ, T) is ergodic, but first we need the following definition.

Definition 2.11. Let K be a compact group, H be a closed subgroup of K, and $\alpha \colon G \to K$ be a group homomorphism. Consider the homogeneous space $Z = K/H$. Let also m be the normalized Haar measure on Z, and $R = (R_g)_{g \in G}$, where for each $g \in G$, $R_g : Z \to Z$ is given by $R_g(z) = \alpha(g)z$. Then the G-system (Z, m, R) is called *a rotation on the homogeneous space* Z *by* α.

Proposition 2.12. *[17, Theorem 1] Let* (X, μ, T) *be an ergodic G-system. Then its Kronecker factor is measurably isomorphic to a rotation on some homogeneous space* Z *by some* α *with dense image.*

From Proposition 2.12, we get that if $\mathcal{C}$ is the Borel σ-algebra on Z, then $\pi^{-1}(\mathcal{C})$ is equivalent to $\mathcal{A}$ (i.e., they are equal modulo sets that have zero μ measure). Proposition 2.12 allows us to identify the Kronecker factor with a rotation on a homogeneous space whenever (X, μ, T) is ergodic.

Characteristic factors for G-systems: The notion of *characteristic factors* will play a fundamental role later in one of our proofs. Here, we have the following theorem for the characteristic factors with respect to some double averages that will concern us.

Theorem 2.13. *Let* (X, μ, T) *be an ergodic G-system, let* (Z, m, R) *be its Kronecker factor and let* Φ *be a Følner sequence. Then for any* $f_1, f_2 \in L^\infty(X)$*, we have*

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} T_g f_1 \otimes T_g f_2 = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} T_g \mathbb{E}_\mu(f_1 \mid Z) \otimes T_g \mathbb{E}_\mu(f_2 \mid Z) \tag{2.3}$$

in $L^2(X \times X, \mu \times \mu)$.

Theorem 2.13 says that the Kronecker factor is the characteristic factor for the averages in the left-hand side of (2.3). The proof of Theorem 2.13 will follow easily from the next lemma.

Lemma 2.14. *Let* (X, μ, T) *be a G-system. Then*

$$\mathcal{H}_{wm}(T) \otimes L^2(X) \subset \mathcal{H}_{wm}(T \times T) \ \text{ and } \ L^2(X) \otimes \mathcal{H}_{wm}(T) \subset \mathcal{H}_{wm}(T \times T).$$

Proof. We will only prove the first inclusion, as the second follows in an analogous way. Let Φ be any Følner sequence, and let $f_1 \in \mathcal{H}_{\mathrm{wm}}(T)$ and $f_2 \in L^2(X)$. We may assume that $\|f_1\|_2, \|f_2\|_2 \le 1$. We want to show that

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} \left| \langle (T_g \times T_g)(f_1 \otimes f_2), F \rangle_{L^2(\mu\times\mu)} \right|^2 = 0 \tag{2.4}$$

for any $F \in L^2(X \times X, \mu \times \mu)$.

Let $F \in L^2(X \times X, \mu \times \mu)$ and $\varepsilon > 0$. We may assume that $\|F\|_{L^2(\mu\times\mu)} = 1$. Now, since finite linear combinations of functions of the form $f_1' \otimes f_2'$, where $f_1', f_2' \in L^2(X)$, form a dense subset of

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$L^2(X\times X,\mu\times\mu)$, we can find $F'=\sum_{i=1}^k c_i(f'_{1,i}\otimes f'_{2,i})$ with $\|F'\|_{L^2(\mu\times\mu)}\leq 1$ such that $\|F-F'\|_{L^2(\mu\times\mu)}<\varepsilon/2$. Then by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|\langle (T_g\times T_g)(f_1\otimes f_2),F\rangle_{L^2(\mu\times\mu)}\right|^2=\langle (T_g\times T_g)(f_1\otimes f_2),F\rangle_{L^2(\mu\times\mu)}\langle F,(T_g\times T_g)(f_1\otimes f_2)\rangle_{L^2(\mu\times\mu)}\\
&=\langle (T_g\times T_g)(f_1\otimes f_2),F'\rangle_{L^2(\mu\times\mu)}\langle F',(T_g\times T_g)(f_1\otimes f_2)\rangle_{L^2(\mu\times\mu)}\\
&\qquad+\langle (T_g\times T_g)(f_1\otimes f_2),F'\rangle_{L^2(\mu\times\mu)}\langle F-F',(T_g\times T_g)(f_1\otimes f_2)\rangle_{L^2(\mu\times\mu)}\\
&\qquad+\langle (T_g\times T_g)(f_1\otimes f_2),F-F'\rangle_{L^2(\mu\times\mu)}\langle F,(T_g\times T_g)(f_1\otimes f_2)\rangle_{L^2(\mu\times\mu)}\\
&<\sum_{1\leq i,j\leq k}\overline{c_i}c_j\langle (T_g\times T_g)(f_1\otimes f_2),f'_{1,i}\otimes f'_{2,i}\rangle_{L^2(\mu\times\mu)}\langle f'_{1,j}\otimes f'_{2,j},(T_g\times T_g)(f_1\otimes f_2)\rangle_{L^2(\mu\times\mu)}+\varepsilon\\
&\leq\sum_{1\leq i,j\leq k}\overline{c_i}c_j\|f'_{2,i}\|_2\|f'_{1,j}\|_2\|f'_{2,j}\|_2|\langle T_gf_1,f'_{1,i}\rangle|+\varepsilon.
\end{aligned}$$

Therefore, using that f_1 is a weak-mixing function, we have that

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N}\left|\langle (T_g\times T_g)(f_1\otimes f_2),F\rangle_{L^2(\mu\times\mu)}\right|^2\\
&=\sum_{1\leq i,j\leq k}\overline{c_i}c_j\|f'_{2,i}\|_2\|f'_{1,j}\|_2\|f'_{2,j}\|_2\limsup_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N}|\langle T_gf_1,f'_{1,i}\rangle|+\varepsilon=\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, then (2.4) follows. The proof is complete. □

Proof of Theorem 2.13. Let Φ be Følner sequence, and let $f_1,f_2\in L^2(X)$. Then we write

$$\begin{aligned}
\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N}T_gf_1\otimes T_gf_2&=\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N}T_g(f_1-\mathbb{E}_\mu(f_1\mid Z))\otimes T_g(f_2-\mathbb{E}_\mu(f_2\mid Z))\\
&+\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N}T_g(f_1-\mathbb{E}_\mu(f_1\mid Z))\otimes T_g\mathbb{E}_\mu(f_2\mid Z)\\
&+\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N}T_g\mathbb{E}_\mu(f_1\mid Z)\otimes T_g(f_2-\mathbb{E}_\mu(f_2\mid Z))\\
&+\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N}T_g\mathbb{E}_\mu(f_1|Z)\otimes T_g\mathbb{E}_\mu(f_2\mid Z).
\end{aligned}\tag{2.5}$$

Note that the limits of all the terms above exist by the mean ergodic theorem (Theorem 2.1) applied to $T\times T$. By Theorem 2.10, the functions $f_1-\mathbb{E}_\mu(f_1\mid Z)$, $f_2-\mathbb{E}_\mu(f_2\mid Z)$ are both weak-mixing. Then, by Lemma 2.14, the functions $(f_1-\mathbb{E}_\mu(f_1\mid Z))\otimes(f_2-\mathbb{E}_\mu(f_2\mid Z))$, $(f_1-\mathbb{E}_\mu(f_1\mid Z))\otimes\mathbb{E}_\mu(f_2\mid Z)$ and $\mathbb{E}_\mu(f_1\mid Z)\otimes(f_2-\mathbb{E}_\mu(f_2\mid Z))$ are weak-mixing with respect to $T\times T$. Hence, the limits of the first three terms in the right-hand side of (2.5) are 0 in $L^2(X\times X,\mu\times\mu)$. Then the theorem follows. □

A correspondence principle: Finally, we need the following instance of Furstenberg's correspondence principle.

Theorem 2.15 (Cf. [4, Theorem 2.8]). *Let G be an amenable group, let $A\subset G$ and assume that there exists a left Følner sequence Φ such that $d_\Phi(A)=\lim_{N\to\infty}\frac{|A\cap\Phi_N|}{|\Phi_N|}$ exists. Then there exist an ergodic G-system (X,μ,T), a clopen set $E\subset X$, a Følner sequence Ψ and a point $a\in\mathsf{gen}(\mu,\Psi)$ such that $\mu(E)\geq d_\Phi(A)$ and $A=\{h\in G: T_ha\in E\}$.*

Proof. Consider the compact metric space $X:=\{0,1\}^G=\{x=(x_g)_{g\in G}: x_g\in\{0,1\}\ \forall\ g\in G\}$, equipped with the Borel σ-algebra. We define the continuous action T on X by $T_h(x_g)_{g\in G}=(x_{gh})_{g\in G}$, for any $h\in G$ and $(x_g)_{g\in G}\in X$. Now, consider the point $a=(\mathbb{1}_A(g))_{g\in G}\in X$ and the clopen set

$E = \{x = (x_g)_{g\in G} \in X : x_{e_G} = 1\}$. By the choice of a, for $h \in G$, we have that $h \in A$ if and only if $T_h a \in E$, and therefore, $A = \{h \in G : T_h a \in E\}$. Consider the sequence of Borel probability measures on X defined by

$$N \mapsto \mu_N = \frac{1}{|\Phi_N|} \sum_{h\in\Phi_N} \delta_{T_h a},$$

and let μ' be a weak* limit point of that sequence. Then $\mu'(E) = d_\Phi(A)$, and μ' is T-invariant, but not necessarily ergodic. Let $x \mapsto \mu'_x$ be the ergodic decomposition of μ'. Then $\mu' = \int_X \mu'_x \, \mathrm{d}\mu'(x)$, so there exists $x_0 \in X$ such that for the measure $\mu = \mu'_{x_0}$, we have that (X, μ, T) is ergodic and $\mu(E) \geq \mathrm{d}_\Phi(A)$. For all $N \in \mathbb{N}$, μ_N gives full measure to the orbit closure of a, and hence, μ' also gives full measure to the orbit closure of a. Therefore, we may assume that μ is also supported on the orbit closure of a (as this is the case with μ'_x for μ'-almost every $x \in X$). Then it follows by Lemma 2.9, that $a \in \mathtt{gen}(\mu, \Psi)$ for some Følner sequence Ψ. □

3. Reduction of Theorem 1.6 to dynamical statements

In this section, we translate our first main theorem – namely, Theorem 1.6 – in a dynamical language. This will allow us to approach the problem through ergodic theoretic techniques.

3.1. Dynamical reformulation via correspondence principle

Usually in ergodic theory, correspondence principles serve as bridges between combinatorial and dynamical statements. In our situation, we can use the correspondence principle (Theorem 2.15) to show that Theorem 1.6 follows from Theorem 3.1 below, which is more dynamical in nature.

Theorem 3.1 (First dynamical reformulation of Theorem 1.6). *Let G be a square absolutely continuous group and (X, μ, T) be an ergodic G-system. Let $a \in \mathtt{gen}(\mu, \Phi)$ for some Følner sequence Φ and $E \subset G$ be clopen with $\mu(E) > 0$. Then there exist an infinite sequence $B = (b_n)_{n\in\mathbb{N}} \subset \{h \in G : T_h a \in E\}$ and some $t \in G$ such that*

$$t \cdot B \blacktriangleleft B \subset \{h \in G : T_h a \in E\}.$$

Proof that Theorem 3.1 implies Theorem 1.6. Let $A \subset G$ have positive left upper Banach density, so that there exists some Følner sequence Φ such that $d_\Phi(A) = \lim_{N\to\infty} \frac{|A\cap\Phi_N|}{|\Phi_N|} > 0$, (where we have passed to a subsequence). Then consider (X, μ, T), E, Ψ and a, as ensured by Theorem 2.15, satisfying $\mu(E) \geq d_\Phi(A) > 0$ and $\{h \in G : T_h a \in E\} = A$. It follows then by Theorem 3.1 that there exist an infinite sequence $B = (b_n)_{n\in\mathbb{N}} \subset A$ and some $t \in G$ such that $t \cdot B \blacktriangleleft B \subset A$. □

3.2. Erdős progressions

The conclusion of Theorem 3.1 is still a rather combinatorial statement, so we need to reformulate it again into a dynamical statement. For this to be achieved, we will use the notion of Erdős progressions, as defined in [15], which in $\mathbb{Z}$ is a dynamical variant of 3-term arithmetic progressions. In our case, Erdős progressions are a dynamical variant of progressions of the form (z, kz, k^2z), $k \in K$, $z \in Z$, which are the natural generalization of 3-term arithmetic progressions in our setting.

Definition 3.2. Given a topological G-system (X, T), a point $(x_0, x_1, x_2) \in X^3$ is a 3-*term Erdős progression* if there exists an infinite sequence $(g_n)_{n\in\mathbb{N}}$ in G such that

$$(T_{g_n} \times T_{g_n})(x_0, x_1) \to (x_1, x_2). \tag{3.1}$$

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We refer to 3-term Erdős progressions simply as Erdős progressions. Through the notion of Erdős progressions, we are able to reformulate Theorem 3.1 as follows:

Theorem 3.3 (Second dynamical reformulation). *Let G be a square absolutely continuous group, and let (X,μ,T) be an ergodic G-system and $a \in \mathsf{gen}(\mu,\Phi)$ for some Følner sequence Φ. If $E \subset X$ is a clopen set with $\mu(E) > 0$, then there exist $t \in G$, $x_1 \in E$ and $x_2 \in T_t{}^{-1}E$ such that $(a,x_1,x_2) \in X^3$ forms an Erdős progression.*

For the reduction of Theorem 3.1 to Theorem 3.3, we provide the following lemma.

Lemma 3.4. *Let G be a group. Let (X,T) be a topological G-system, and let $E,F \subset X$ be open sets. Assume that there exists an Erdős progression $(a,x_1,x_2) \in X^3$ with $x_1 \in E$ and $x_2 \in F$. Then there exists an infinite sequence $B = (b_n)_{n\in\mathbb{N}} \subset \{g \in G : T_g a \in E\}$ such that $B \blacktriangleleft B \subset \{g \in G : T_g a \in F\}$.*

To see how Theorem 3.1 follows from Theorem 3.3, just take $F = T_t{}^{-1}E$ in the above lemma.

Proof of Lemma 3.4. By assumption, there exists an infinite sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $(T_{g_n} \times T_{g_n})(a,x_1) \to (x_1,x_2)$. Since $T_{g_n}a \to x_1 \in E$ and E is open, we get that $T_{g_n}a \in E$ for n sufficiently large, so we may assume without loss of generality that $(g_n)_{n\in\mathbb{N}} \subset \{g \in G : T_g a \in E\}$. Therefore, we will construct the sequence B to be a subset of $(g_n)_{n\in\mathbb{N}}$.

We construct the sequence $B = (b_n)_{n\in\mathbb{N}}$ inductively.

- Since $T_{g_n}x_1 \to x_2 \in F$ and F is open, we can pick $b_1 \in (g_n)_{n\in\mathbb{N}}$ such that $T_{b_1}x_1 \in F$. Then $(x_1,x_2) \in (T_{b_1}^{-1}F) \times F$ which is open.
- Since $(T_{g_n} \times T_{g_n})(a,x_1) \to (x_1,x_2)$, we pick $b_2 \in (g_n)_{n\in\mathbb{N}}$ such that $(T_{b_2} \times T_{b_2})(a,x_1) \in (T_{b_1}^{-1}F) \times F$ and $b_2 \neq b_1$ (this is possible since there are infinitely many choices). Then

$$a \in T_{b_1 b_2}^{-1}F \quad \text{and} \quad x_1 \in T_{b_2}^{-1}F.$$

- Induction step: Assume we have found $b_1, b_2, \ldots, b_n \in (g_n)_{n\in\mathbb{N}}$ all distinct to each other such that

$$a \in \bigcap_{1 \le i < j \le n} T_{b_i b_j}^{-1}F \quad \text{and} \quad x_1 \in \bigcap_{1 \le m \le n} T_{b_m}^{-1}F. \tag{3.2}$$

  Since $(T_{g_n} \times T_{g_n})(a,x_1) \to (x_1,x_2) \in \left(\bigcap_{1\le m\le n} T_{b_m}^{-1}F\right) \times F$ and this set is open, we can pick $b_{n+1} \in (g_n)_{n\in\mathbb{N}}$ such that

$$(T_{b_{n+1}} \times T_{b_{n+1}})(a,x_1) \in \left(\bigcap_{1\le m\le n} T_{b_m}^{-1}F\right) \times F$$

  and $b_{n+1} \notin \{b_m : 1 \le m \le n\}$ (since there are infinitely many choices). Combining this with the inductive hypothesis, we obtain that

$$a \in \bigcap_{1 \le i < j \le n+1} T_{b_i b_j}^{-1}F \quad \text{and} \quad x_1 \in \bigcap_{1 \le m \le n+1} T_{b_m}^{-1}F.$$

Taking $B = (b_n)_{n\in\mathbb{N}}$, we clearly have an infinite subset of $(g_n)_{n\in\mathbb{N}}$, and since (3.2) holds for any $n \in \mathbb{N}$ by construction, we get that $B \blacktriangleleft B \subset \{g \in G : T_g a \in F\}$, as desired. □

3.3. Continuous factor maps to the Kronecker factor

On our way to show Theorem 3.3, it will be useful to have the extra assumption of the G-system having a continuous factor map to its Kronecker factor. The reason for that will become clear towards the proof of our main theorem. As we will see below, it is possible to make such an assumption.

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We begin by generalizing a result of Host and Kra in [9] from actions of $(\mathbb{N},+)$ to actions of amenable groups.

Lemma 3.5. [9, Proposition 6.1 for group actions] *Let G be an amenable group, let (X,μ,T) be an ergodic G-system, (Z,m,R) be its Kronecker factor and let $\rho:(X,\mu,T)\to(Z,m,R)$ be a factor map. If $a\in X$ is a transitive point, then there exists a point $z\in Z$ and a Følner sequence Ψ such that*

$$\lim_{N\to\infty}\frac{1}{|\Psi_N|}\sum_{g\in\Psi_N} f_1(T_g a)\cdot f_2(R_g z)=\int_X f_1\cdot(f_2\circ\rho)\,\mathrm{d}\mu \tag{3.3}$$

holds for any $f_1\in C(X)$ and $f_2\in C(Z)$.

We remark that the result still holds if we replace (Z,m,R) by any factor of (X,μ,T) that is distal as a topological system.

Proof. The proof of this lemma is given in the Appendix A. □

Remark 3.6. Let (X,μ,T) be an ergodic G-system, and let $a\in\mathtt{gen}(\mu,\Phi)$ for some Følner sequence Φ. From Lemma 2.5, we have that every point in $\mathtt{supp}(\mu)$ belongs to the orbit closure of a, and therefore, $\mu\big(\overline{\mathcal{O}_T(a)}\big)=1$. This implies that we can replace X with $\overline{\mathcal{O}_T(a)}$ without affecting the ergodic theoretic properties of the system, and then the generic point a is also transitive. Therefore, whenever we have a generic point in a system, we may assume without loss of generality that it is also transitive.

Proposition 3.7. *Let (X,μ,T) be an ergodic G-system, let $a\in\mathtt{gen}(\mu,\Phi)$ for some Følner sequence Φ, and let $E\subset X$ be a set with $\mu(E)>0$. Then there exists an ergodic extension $(\widetilde{X},\widetilde{\mu},\widetilde{T})$ of (X,μ,T), a Følner sequence $\widetilde{\Phi}$ and a point $\widetilde{a}\in\mathtt{gen}(\widetilde{\mu},\widetilde{\Phi})$ such that*

(i) *There exists a continuous factor map $\widetilde{\pi}:\widetilde{X}\to X$ with $\widetilde{\pi}(\widetilde{a})=a$.*
(ii) *$(\widetilde{X},\widetilde{\mu},\widetilde{T})$ has continuous factor map to its Kronecker factor.*
(iii) *The set $\widetilde{E}=\widetilde{\pi}^{-1}(E)$, has $\widetilde{\mu}(\widetilde{E})=\mu(E)>0$.*
(iv) *If $(\widetilde{a},\widetilde{x}_1,\widetilde{x}_2)\in\widetilde{X}^3$ is an Erdős progression, then $(a,x_1,x_2)\in X^3$ is an Erdős progression, where $x_i=\widetilde{\pi}(\widetilde{x}_i)$, for $i=1,2$.*

Proof. The proofs of (i) and (ii) are identical to those in the case G is the semigroup $(\mathbb{N},+)$, and they can be found in [14, Proposition 3.20]. Therefore, here we only provide a sketch of the proof.

Let (Z,m,R) be the Kronecker factor of (X,μ,T), and let $\pi:X\to Z$ be a factor map. Define $\widetilde{X}=X\times Z$ and $\widetilde{T}=T\times R$, consider the map $\rho:X\to\widetilde{X}$, given by $\rho(x)=(x,\pi(x))$, and then define $\widetilde{\mu}=\rho\mu$. Then the map $\rho:X\to\widetilde{X}$ is an isomorphism of the G-systems $(\widetilde{X},\widetilde{\mu},\widetilde{T})$ and (X,μ,T), and therefore, since (X,μ,T) is ergodic, we get that $(\widetilde{X},\widetilde{\mu},\widetilde{T})$ is also ergodic. In addition, the projection on the first coordinate $\widetilde{\pi}:\widetilde{X}\to X$ is a continuous factor map of the systems.

By Remark 3.6, we may assume that the point a is transitive. Then we can use Lemma 3.5 to find a point $z\in Z$ and Følner sequence $\widetilde{\Phi}$ such that (3.3) holds for all $f_1\in C(X)$ and $f_2\in C(Z)$. Using the definition of the measure $\widetilde{\mu}$, it is not too difficult to see that for any continuous function $F\in C(\widetilde{X})$,

$$\lim_{N\to\infty}\frac{1}{|\widetilde{\Phi}_N|}\sum_{g\in\widetilde{\Phi}_N} F(T_g\times R_g)(a,z)=\int_{\widetilde{X}} F\,\mathrm{d}\widetilde{\mu},$$

which means that the point $\widetilde{a}=(a,z)$ is in $\mathtt{gen}(\widetilde{\mu},\widetilde{\Phi})$. In addition, $\widetilde{\pi}(\widetilde{a})=a$.

Now, as the systems (X,μ,T) and $(\widetilde{X},\widetilde{\mu},\widetilde{T})$ are isomorphic, their Kronecker factors are also isomorphic, so we may assume that (Z,m,R) is the Kronecker factor of $(\widetilde{X},\widetilde{\mu},\widetilde{T})$. Then the projection on the second coordinate $p:\widetilde{X}\to Z$ is a continuous factor map from $(\widetilde{X},\widetilde{\mu},\widetilde{T})$ to its Kronecker factor.

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Note that (iii) is immediate from the definition of factors and factor maps. Hence, it remains to prove (iv). By assumption, $(\widetilde{T}_{g_n} \times \widetilde{T}_{g_n})(\widetilde{a}, \widetilde{x}_1) \to (\widetilde{x}_1, \widetilde{x}_2)$ for some $(g_n)_{n\in\mathbb{N}}$ in G. We now notice that

$$T_{g_n}(a) = T_{g_n}(\widetilde{\pi}(\widetilde{a})) = \widetilde{\pi}(\widetilde{T}_{g_n}) \to \widetilde{\pi}(\widetilde{x}_1) = x_1,$$

since $\widetilde{\pi}$ is a continuous factor map. Similarly, we get $T_{g_n}(x_1) \to x_2$, and the result follows. □

This proposition allows us to reduce Theorem 3.3 to the case of ergodic G-systems with continuous factor maps to their Kronecker factor, as desired. Evidently, Theorem 3.3 follows from the following:

Theorem 3.8 (Reduction to G-systems with continuous factor maps to the Kronecker factor). *Let G be a square absolutely continuous group, let (X, μ, T) be an ergodic G-system admitting a continuous factor map to its Kronecker factor and let $a \in \mathtt{gen}(\mu, \Phi)$ for some Følner sequence Φ. If $E \subset X$ is clopen and $\mu(E) > 0$, then there exist $t \in G$, $x_1 \in E$ and $x_2 \in T_t^{-1}E$ such that $(a, x_1, x_2) \in X^3$ forms an Erdős progression.*

The proof of Theorem 3.8 will be given in the next section.

4. Measures on Erdős progressions and the proof of Theorem 3.8

In this section, we prove Theorem 3.8 and, consequently, Theorem 1.6.

4.1. Measures on Erdős progressions

In what follows, we fix a square absolutely continuous group G. We also fix an ergodic G-system (X, μ, T). In addition, as per the assumptions of Theorem 3.8, we assume that (X, μ, T) admits a continuous factor map to its Kronecker factor. The Kronecker factor of (X, μ, T) is denoted by (Z, m, R), and $\pi : X \to Z$ stands for the continuous factor map. According to Proposition 2.12, $Z = K/H$, where K is a compact group and H is a closed subgroup of K. We denote by p the natural projection $p : K \to K/H$, $p(k) = kH$. We also fix a bi-invariant metric d_K on K, that is, a metric on K compatible with the topology on K such that for all $u, v, w \in K$, $d_K(uv, uw) = d_K(v, w) = d_K(vu, wu)$.

Moreover, m is the (left) Haar measure on Z, which is given as the push forward of the (left) Haar measure m_K in K by the natural projection $K \to Z = K/H$. We remark that since K is compact, it is unimodular, so m_K is two-sided invariant. Finally, the action $R = (R_g)_{g\in G}$ is given by $R_g(z) = \alpha(g)z$, where $\alpha : G \to K$ is a group homomorphism with dense image. Also, $\pi_i : X \times X \to X$ denotes the projection to the i-th coordinate, for $i = 1, 2$. Moreover, $z \mapsto \eta_z$ is a fixed disintegration of μ over the continuous factor map π.

Definition 4.1. Consider the squaring map $s_K : K \to K, s_K(k) = k^2$, on K. We define the Borel probability measure m_{K^2} on K as the push-forward of the Haar measure m_K under the map s_K, that is, the measure on K, given by $m_{K^2}(A) = m_K(s_K^{-1}(A))$, for each Borel $A \subset K$.

We will prove the following lemma, which is a key ingredient that will allow us to define the measures in order to study Erdős progressions.

Lemma 4.2. *The measure m_{K^2} is absolutely continuous with respect to m_K.*

Before we prove Lemma 4.2, let us state and prove an auxiliary lemma that will be used throughout this section:

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Lemma 4.3. *Let $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ be any Følner sequence in G. Then the sequence of measures $(\nu_N)_{N\in\mathbb{N}}$ defined as*

$$\nu_N := \frac{1}{|\Psi_N|}\sum_{g\in\Psi_N}\delta_{\alpha(g)}$$

converges in the weak topology to the Haar measure m_K on K.*

Proof. The space of Borel probability measures on K is weak* compact and metrizable, so in order to prove the result, it suffices to prove that if $(\nu_{N_j})_{j\in\mathbb{N}}$ is a convergent subsequence of $(\nu_N)_{N\in\mathbb{N}}$, then it converges to the Haar measure m_K.

Let $(\nu_{N_j})_{j\in\mathbb{N}}$ be a subsequence of $(\nu_N)_{N\in\mathbb{N}}$ which converges in the weak* topology to a measure ν on K. To prove that ν is the Haar measure on K, it suffices to prove that for all continuous functions h on K and all $k\in K$, we have that $\int_K h(ky)\,\mathrm{d}\nu(y) = \int_K h(y)\,\mathrm{d}\nu(y)$.

Let d_K be a translation invariant metric on K. Let also $h : K \to \mathbb{C}$ be continuous, $k\in K$ and $\varepsilon > 0$. Since K is compact, we have that h is uniformly continuous, so there is $\delta > 0$ such that if $d_K(y_1, y_2) < \delta$, then $|h(y_1) - h(y_2)| < \varepsilon$. Recall that $(\alpha(g))_{g\in G}$ is dense in K, so there is g_0 such that $d_K(k, \alpha(g_0)) < \delta$. We then have that for all $y\in K$, $d_K(ky, \alpha(g_0)y) < \delta$, so $|h(ky) - h(\alpha(g_0)y)| < \varepsilon$, and we obtain that

$$\begin{aligned}\left|\int_K h(y)\,\mathrm{d}\nu(y) - \int_K h(ky)\,\mathrm{d}\nu(y)\right| &\le \left|\int_K h(y)\,\mathrm{d}\nu(y) - \int_K h(\alpha(g_0)y)\,\mathrm{d}\nu(y)\right| \\ &\quad+ \left|\int_K h(\alpha(g_0)y)\,\mathrm{d}\nu(y) - \int_K h(ky)\,\mathrm{d}\nu(y)\right| \\ &\le \left|\int_K h(y)\,\mathrm{d}\nu(y) - \int_K h(\alpha(g_0)y)\,\mathrm{d}\nu(y)\right| + \varepsilon.\end{aligned}$$

From the continuity of h and the definition of ν, we have that

$$\begin{aligned}\int_K h(\alpha(g_0)y)\,\mathrm{d}\nu(y) &= \lim_{j\to\infty}\frac{1}{|\Psi_{N_j}|}\sum_{g\in\Psi_{N_j}} h(\alpha(g_0)\alpha(g)) = \lim_{j\to\infty}\frac{1}{|\Psi_{N_j}|}\sum_{g\in\Psi_{N_j}} h(\alpha(g_0g)) \\ &= \lim_{j\to\infty}\frac{1}{|\Psi_{N_j}|}\sum_{g\in g_0\Psi_{N_j}} h(\alpha(g)) = \lim_{j\to\infty}\frac{1}{|\Psi_{N_j}|}\sum_{g\in\Psi_{N_j}} h(\alpha(g)) \\ &= \int_K h(y)\,\mathrm{d}\nu(y),\end{aligned}$$

so combining with the previous, we get that $\left|\int_K h(y)\,\mathrm{d}\nu(y) - \int_K h(ky)\,\mathrm{d}\nu(y)\right| \le \varepsilon$, and since ε was arbitrary, we obtain that $\int_K h(y)\,\mathrm{d}\nu(y) = \int_K h(ky)\,\mathrm{d}\nu(y)$, which proves that $\nu = m_K$ and concludes the proof. □

Proof of Lemma 4.2. As K is compact and metrizable, the measures m_K, m_{K^2} are regular. In particular, for each Borel A, we have

$$m_K(A) = \sup_{\substack{C\subset A\\ C\text{ compact}}} m_K(C) = \inf_{\substack{O\supset A\\ O\text{ open}}} m_K(O) \quad\text{and}\quad m_{K^2}(A) = \sup_{\substack{C\subset A\\ C\text{ compact}}} m_{K^2}(C) = \inf_{\substack{O\supset A\\ O\text{ open}}} m_{K^2}(O).$$

Therefore, to prove that m_{K^2} is absolutely continuous with respect to m_K, it suffices to prove that for each compact set $C\subset K$, if $m_K(C) = 0$, then $m_{K^2}(C) = 0$.

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Let $C \subset K$ be a nonempty (for otherwise the result is trivial) compact with $m_K(C) = 0$ and let $\varepsilon > 0$. As G is square absolutely continuous, we know that there are two Følner sequences Φ and Ψ in G and a $\delta > 0$ such that for any $u : G \to [0,1]$ satisfying $\limsup_{N\to\infty} \frac{1}{|\Phi_N|}\sum_{g\in\Phi_N} u(g) < \delta$, we have that $\limsup_{N\to\infty} \frac{1}{|\Psi_N|}\sum_{g\in\Psi_N} u(g^2) < \varepsilon$. Since

$$0 = m_K(C) = \inf_{\substack{O\supset C\\ O \text{ open}}} m_K(O),$$

we can pick an open set $O \supset C$ with $m_K(O) < \delta$. By Urysohn's lemma, we know that there is a continuous function $f : K \to [0,1]$ such that $f = 1$ on C and $f = 0$ outside O. By Lemma 4.3, we then have that

$$\delta > m_K(O) \geq \int_K f(k)\, \mathrm{d}m_K(k) = \lim_{N\to\infty} \frac{1}{|\Phi_N|}\sum_{g\in\Phi_N} f(\alpha(g)),$$

and by the choice of δ, we get that

$$\limsup_{N\to\infty} \frac{1}{|\Psi_N|}\sum_{g\in\Psi_N} f(\alpha(g^2)) < \varepsilon.$$

From the definition of m_{K^2} and the continuity of $k \mapsto f(k^2)$, we then obtain

$$\begin{aligned}\int_K f(k)\, \mathrm{d}m_{K^2}(k) = \int_K f(k^2)\, \mathrm{d}m_K(k) &= \lim_{N\to\infty} \frac{1}{|\Psi_N|}\sum_{g\in\Psi_N} f(\alpha(g)^2)\\ &= \lim_{N\to\infty} \frac{1}{|\Psi_N|}\sum_{g\in\Psi_N} f(\alpha(g^2)) < \varepsilon,\end{aligned}$$

where for the third equality, we use that α is a group homomorphism. So after all, we have $m_{K^2}(C) \leq \int_K f(k)\, \mathrm{d}m_{K^2}(k) < \varepsilon$, and as ε was arbitrary, we have that $m_{K^2}(C) = 0$. This concludes the proof. □

Now, we define a measure σ on $X \times X$, and we want σ to be defined as a natural measure to study Erdős progressions. It is not hard to see that Erdős progressions on K/H are exactly the triplets of the form $(z, kz, k^2 z)$ for some $k \in K$ and $z \in K/H$. Therefore, following the definition given in [15], we will define these measures as the natural measures on points $(x_1, x_2) \in X \times X$ to find Erdős progressions on K/H starting at $\pi(a)$, namely, $(\pi(a), k\pi(a), k^2\pi(a))$, for $k \in K$.

Definition 4.4. We define the measure σ on $X \times X$, given by

$$\sigma := \int_K \eta_{k\pi(a)} \times \eta_{k^2\pi(a)}\, \mathrm{d}m_K(k). \tag{4.1}$$

Let us comment on why the measure σ is well-defined. Let $k_0 \in K$ be such that $\pi(a) = k_0 H$. Since η is m-almost everywhere defined on Z and Borel measurable, we can consider a Borel measurable set $Z' \subset Z$ with $m(Z') = 1$ such that η_z is defined for all $z \in Z'$. Since p is a Borel measurable map and $pm_K = m$, the set $K' = p^{-1}(Z')$ is Borel measurable and has $m_K(K') = 1$. Then also $m_K(K'k_0^{-1}) = 1$, and then it is not difficult to check that the map $K \to M(X)$, $k \mapsto \eta_{k\pi(a)}$ is defined on $K'k_0^{-1}$. However, from Lemma 4.2, we have that m_{K^2} is absolutely continuous with respect to m_K, so $1 = m_{K^2}(K'k_0^{-1}) = m_K(s_K^{-1}(K'k_0^{-1}))$, and again, it is not too difficult to check that the map $K \to M(X)$, $k \mapsto \eta_{k^2\pi(a)}$ is defined on $s_K^{-1}(K'k_0^{-1})$. So, after all, the map $K \to M(X \times X)$, $k \mapsto \eta_{k\pi(a)} \times \eta_{k^2\pi(a)}$ is defined on $K'k_0^{-1} \cap s_K^{-1}(K'k_0^{-1})$ and $m_K(K'k_0^{-1} \cap s_K^{-1}(K'k_0^{-1})) = 1$ (i.e., $k \mapsto \eta_{k\pi(a)} \times \eta_{k^2\pi(a)}$ is m_K-almost everywhere defined). Also, since all the maps involved in the definition of $k \mapsto \eta_{k\pi(a)} \times \eta_{k^2\pi(a)}$

are Borel measurable, we have that $k \mapsto \eta_{k\pi(a)} \times \eta_{k^2\pi(a)}$ is also Borel measurable. Therefore, σ is indeed well-defined.

Using the invariance of m_K we can express σ as

$$\sigma = \int_K \eta_{kk_0H} \times \eta_{k^2k_0H} \,\mathrm{d}m_K(k) = \int_K \eta_{kH} \times \eta_{kk_0^{-1}kH} \,\mathrm{d}m_K(k). \tag{4.2}$$

Proposition 4.5. *The measure σ has the following properties:*

(i) $\pi_1\sigma = \mu$.

(ii) $\pi_2\sigma$ *is absolutely continuous with respect to* μ.

Proof of Proposition 4.5. (i) Using (4.2), we have that

$$\pi_1\sigma = \int_K \eta_{kH} \,\mathrm{d}m_K(k) = \int_Z \eta_z \,\mathrm{d}m(z) = \mu.$$

(ii) From the definition of σ, we have that $\pi_2\sigma = \int_K \eta_{k^2\pi(a)} \,\mathrm{d}m_K(k)$. Fix k_0 such that $\pi(a) = k_0H$.

Let $A \subset X$ with $\mu(A) = 0$. Then $\mu(A) = \int_Z \eta_z(A) \,\mathrm{d}m(z)$, so we get that there is a set $Z' \subset Z$ with $m(Z') = 1$ such that for all $z \in Z'$, $\eta_z(A) = 0$. As $pm_K = m$, we have $m_K(p^{-1}(Z')) = 1$ and then also $m_K((p^{-1}Z')k_0^{-1}) = 1$. Finally, using Lemma 4.2, we get that $m_{K^2}((p^{-1}Z')k_0^{-1}) = m_K(s_K^{-1}((p^{-1}Z')k_0^{-1})) = 1$. For each $k \in s_K^{-1}((p^{-1}Z')k_0^{-1})$, we have that $k^2\pi(a) \in Z'$, so $\eta_{k^2\pi(a)}(A) = 0$, and therefore, $\pi_2\sigma(A) = 0$. □

Theorem 4.6. *For any set $E \subset X$ with $\mu(E) > 0$, we have that*

$$\sigma\left(E \times \bigcup_{t\in G} T_t^{-1}E\right) > 0.$$

Proof. Let $E \subset X$ with $\mu(E) > 0$ and recall that we want to show that the set $E \times \bigcup_{t\in G} T_t^{-1}E$ has positive measure with respect to σ. We begin by expressing this set as

$$E \times \bigcup_{t\in G} T_t^{-1}E = (E \times X) \cap \left(X \times \bigcup_{t\in G} T_t^{-1}E\right).$$

By Proposition 4.5 (i), we have that $\sigma(E \times X) = \mu(E) > 0$. Therefore, it is enough to show that

$$\sigma\left(X \times \bigcup_{t\in G} T_t^{-1}E\right) = 1. \tag{4.3}$$

Notice that the set $\bigcup_{t\in G} T_t^{-1}E$ is clearly T-invariant, and since μ is ergodic and $\mu(E) > 0$, it follows that $\mu\big(\bigcup_{t\in G} T_t^{-1}E\big) = 1$. By Proposition 4.5 (ii), $\pi_2\sigma$ is absolutely continuous with respect to μ, so

$$1 = \pi_2\sigma\left(\bigcup_{t\in G} T_t^{-1}E\right) = \sigma\left(X \times \bigcup_{t\in G} T_t^{-1}E\right),$$

which concludes the proof. □

4.2. A continuous ergodic decomposition

In this subsection, we will define measures $\lambda_{(x_1,x_2)}$, for $(x_1,x_2) \in X\times X$ in a way that $(x_1,x_2) \mapsto \lambda_{(x_1,x_2)}$ will be a continuous ergodic decomposition of $\mu\times\mu$ (i.e., $(x_1,x_2) \mapsto \lambda_{(x_1,x_2)}$ will be both a continuous

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map and an ergodic decomposition of $\mu \times \mu$). We follow the definition given in [14, Eq. (3.10)] and [15, Eq. (3.1)].

Definition 4.7. For $(x_1, x_2) \in X \times X$, we define the measures $\lambda_{(x_1,x_2)}$ on $X \times X$ by

$$\lambda_{(x_1,x_2)} = \int_K \eta_{k\pi(x_1)} \times \eta_{k\pi(x_2)} \, \mathrm{d}m_K(k). \tag{4.4}$$

Given $x_1, x_2 \in X$, we let $k_1, k_2 \in K$ be such that $\pi(x_i) = k_i H$, for $i = 1, 2$. Then, using the invariance of m_K, we can write

$$\lambda_{(x_1,x_2)} = \int_K \eta_{kk_1H} \times \eta_{kk_2H} \, \mathrm{d}m_K(k) = \int_K \eta_{kH} \times \eta_{kk_1^{-1}k_2H} \, \mathrm{d}m_K(k). \tag{4.5}$$

Theorem 4.8. *The map $(x_1, x_2) \mapsto \lambda_{(x_1,x_2)}$ is a continuous ergodic decomposition of $\mu \times \mu$ in the following sense:*

(i) *It is a continuous map.*
(ii) *It satisfies $\int_{X\times X} \lambda_{(x_1,x_2)} \, \mathrm{d}(\mu \times \mu)(x_1, x_2) = \mu \times \mu$.*
(iii) *The G-system $(X \times X, \lambda_{(x_1,x_2)}, T \times T)$ is ergodic for $\mu \times \mu$-almost every $(x_1, x_2) \in X \times X$.*

In addition, for any $x_1, x_2 \in X$, we have that

$$\lambda_{(x_1,x_2)} = \lambda_{(T_g x_1, T_g x_2)} \tag{4.6}$$

for any $g \in G$.

Proof. For the proof of (i) and (ii), we refer to [14, Proposition 3.11], as the proof there can be directly adapted to our case. We will now prove (iii). It is not too difficult to see that for all $(x_1, x_2) \in X \times X$ and for all $g \in G$, $(T_g \times T_g)\lambda_{(x_1,x_2)} = \lambda_{(x_1,x_2)}$ (i.e., $\lambda_{(x_1,x_2)}$ is $T \times T$-invariant). Therefore, to prove (iii), it suffices to prove that there is some Følner sequence Ψ in G such that for $(\mu \times \mu)$-almost every $(x_1, x_2) \in X \times X$ and all bounded and measurable functions F on $X \times X$,

$$\lim_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} (T_g \times T_g)F = \int_{X\times X} F \, \mathrm{d}\lambda_{(x_1,x_2)},$$

in $L^2(X \times X, \lambda_{(x_1,x_2)})$.

Now, since X is a compact metric space, there is a countable family of continuous functions $(f_k)_{k\in\mathbb{N}}$ which is dense in $L^p(\nu)$ for all $p \in [1, +\infty)$ and all Borel probability measures ν on X. Then, it is not too difficult to see that the set consisting of finite linear combinations of functions the form $(f_{j_1} \otimes f_{j_2})_{j_1,j_2\in\mathbb{N}}$ is dense in $L^2(\rho)$ for all Borel probability measures ρ on $X \times X$. Hence, using an approximation argument, it suffices to prove that there is a Følner Ψ in G and a set $W \subset X \times X$ with $(\mu \times \mu)(W) = 1$ such that for all $(x_1, x_2) \in W$ and for all $j_1, j_2 \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} (T_g \times T_g)(f_{j_1} \otimes f_{j_2}) = \int_{X\times X} f_{j_1} \otimes f_{j_2} \, \mathrm{d}\lambda_{(x_1,x_2)},$$

in $L^2(X \times X, \lambda_{(x_1,x_2)})$.

Step 1. Let Φ be any Følner sequence in G. Then, using Theorem 2.13, we get that for each $j_1, j_2 \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} (T_g \times T_g)(f_{j_1} \otimes f_{j_2}) = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} (T_g \times T_g)(\mathbb{E}_\mu(f_{j_1} \mid Z) \otimes \mathbb{E}_\mu(f_{j_2} \mid Z))$$

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in $L^2(\mu \times \mu)$. Combining this with (ii) yields

$$\lim_{N\to\infty} \int_{X\times X} \int_{X\times X} \left| \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} (T_g \times T_g)(f_{j_1} \otimes f_{j_2})(y_1, y_2) - \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} (T_g \times T_g)(\mathbb{E}_\mu(f_{j_1} \mid Z) \otimes \mathbb{E}_\mu(f_{j_2} \mid Z))(y_1, y_2) \right|^2 \mathrm{d}\lambda_{(x_1,x_2)}(y_1, y_2)\, \mathrm{d}(\mu \times \mu)(x_1, x_2) = 0.$$

Then for each $j_1, j_2 \in \mathbb{N}$, we can find a sub-Følner sequence $\widetilde{\Phi}$ of Φ, depending on j_1, j_2, such that for $(\mu \times \mu)$-almost every $(x_1, x_2) \in X \times X$, we have

$$\lim_{N\to\infty} \int_{X\times X} \left| \frac{1}{|\widetilde{\Phi}_N|} \sum_{g\in\widetilde{\Phi}_N} (T_g \times T_g)(f_{j_1} \otimes f_{j_2}) - \frac{1}{|\widetilde{\Phi}_N|} \sum_{g\in\widetilde{\Phi}_N} (T_g \times T_g)(\mathbb{E}_\mu(f_{j_1} \mid Z) \otimes \mathbb{E}_\mu(f_{j_2} \mid Z)) \right|^2 \mathrm{d}\lambda_{(x_1,x_2)} = 0,$$

and since the limits of both averages above exist by Theorem 2.1, we have that, for $(\mu\times\mu)$-almost every $(x_1, x_2) \in X \times X$,

$$\lim_{N\to\infty} \frac{1}{|\widetilde{\Phi}_N|} \sum_{g\in\widetilde{\Phi}_N} (T_g \times T_g)(f_{j_1} \otimes f_{j_2}) = \lim_{N\to\infty} \frac{1}{|\widetilde{\Phi}_N|} \sum_{g\in\widetilde{\Phi}_N} (T_g \times T_g)(\mathbb{E}_\mu(f_{j_1} \mid Z) \otimes \mathbb{E}_\mu(f_{j_2} \mid Z))$$

in $L^2(X \times X, \lambda_{(x_1,x_2)})$. Since the family $(f_{j_1} \otimes f_{j_2})_{j_1,j_2\in\mathbb{N}}$ is countable, then using a diagonal argument, one can find a Følner sequence Ψ and a set $W_1 \subset X \times X$ with $(\mu \times \mu)(W_1) = 1$ such that for all $(x_1, x_2) \in W_1$ and $j_1, j_2 \in \mathbb{N}$, we have that

$$\lim_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} (T_g \times T_g)(f_{j_1} \otimes f_{j_2}) = \lim_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} (T_g \times T_g)(\mathbb{E}_\mu(f_{j_1} \mid Z) \otimes \mathbb{E}_\mu(f_{j_2} \mid Z)) \tag{4.7}$$

in $L^2(X \times X, \lambda_{(x_1,x_2)})$.

Step 2. Consider the sequence of probability measures on K defined by $\nu_N := \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} \delta_{\alpha(g)}$. From Lemma 4.3, we know that $\nu_N \to m_K$ as $N \to \infty$ in the weak* topology.

Let $\phi_1, \phi_2 : Z \to \mathbb{C}$ be continuous. For each $z_1, z_2 \in Z$, consider the function $\phi_{z_1,z_2} : K \to \mathbb{C}$ defined by $\phi_{z_1,z_2}(k) = \phi_1(kz_1)\phi_2(kz_2)$. Then, ϕ_{z_1,z_2} is continuous, so we have that

$$\begin{aligned}\int_K \phi_1(kz_1)\phi_2(kz_2)\, \mathrm{d}m_K(k) = \int_K \phi_{z_1,z_2}(k)\, \mathrm{d}m_K(k) &= \lim_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} \phi_{z_1,z_2}(\alpha(g)) \\ &= \lim_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} \phi_1(\alpha(g)z_1)\phi_2(\alpha(g)z_2) \\ &= \lim_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} (R_g \times R_g)(\phi_1 \otimes \phi_2)(z_1, z_2).\end{aligned}$$

Since the previous holds for all $z_1, z_2 \in Z$, using the dominated convergence theorem, we get that

$$\lim_{N\to\infty} \int_{Z\times Z} \left| \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} (R_g \times R_g)(\phi_1 \otimes \phi_2)(z_1, z_2) - \int_K \phi_1(kz_1)\phi_2(kz_2)\, \mathrm{d}m_K(k) \right|^2 \mathrm{d}(m \times m)(z_1, z_2) = 0$$

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(i.e., the sequence $\frac{1}{|\Psi_N|}\sum_{g\in\Psi_N}(R_g \times R_g)(\phi_1 \otimes \phi_2)$ converges in $L^2(Z \times Z, m \times m)$ as $N \to \infty$ to the function $(z_1, z_2) \mapsto \int_K \phi_1(kz_1)\phi_2(kz_2)\, \mathrm{d}m_K(k)$).

Step 3. Given two bounded measurable maps $h_1, h_2 : Z \to \mathbb{C}$, approximating them in $L^2(Z, m)$ by two continuous functions ϕ_1, ϕ_2 and using Step 2, one can prove that $\frac{1}{|\Psi_N|}\sum_{g\in\Psi_N}(R_g \times R_g)(h_1 \otimes h_2)$ converges in $L^2(Z \times Z, m \times m)$ as $N \to \infty$ to the function $(z_1, z_2) \mapsto \int_K h_1(kz_1)h_2(kz_2)\, \mathrm{d}m_K(k)$. Since $\pi : (X, \mu, T) \to (Z, m, R)$ is a factor map, it is not too difficult then to see that

$$\frac{1}{|\Psi_N|}\sum_{g\in\Psi_N}(T_g \times T_g)(h_1 \circ \pi \otimes h_2 \circ \pi) \to \left[(x_1, x_2) \mapsto \int_K h_1(k\pi(x_1))h_2(k\pi(x_2))\, \mathrm{d}m_K(k)\right] \tag{4.8}$$

as $N \to \infty$ in $L^2(X \times X, \mu \times \mu)$.

For each $j \in \mathbb{N}$, $\mathbb{E}_\mu(f_j \mid Z)$ can be viewed either as a function on Z or as a function on X measurable with respect to $\pi^{-1}(Z)$. In this proof, we always view $\mathbb{E}_\mu(f_j \mid Z)$ as a function on X measurable with respect to $\pi^{-1}(Z)$. For each $j \in \mathbb{N}$, let ψ_j be $\mathbb{E}_\mu(f_j \mid Z)$ when viewed as a function on Z, so we have that for μ-almost every $x \in X$, $\psi_j \circ \pi(x) = \mathbb{E}_\mu(f_j \mid Z)(x)$. Then for each $j_1, j_2 \in \mathbb{N}$ and $(x_1, x_2) \in X \times X$, we have

$$\begin{aligned}\int_{X\times X} f_{j_1}(y_1)f_{j_2}(y_2)\, \mathrm{d}\lambda_{(x_1,x_2)}(y_1, y_2) &= \int_K \int_{X\times X} f_{j_1}(y_1)f_{j_2}(y_2)\, \mathrm{d}(\eta_{k\pi(x_1)} \times \eta_{k\pi(x_2)})(y_1, y_2)\, \mathrm{d}m_K(k)\\ &= \int_K \int_X f_{j_1}(y_1)\, \mathrm{d}\eta_{k\pi(x_1)}(y_1) \int_X f_{j_2}(y_2)\, \mathrm{d}\eta_{k\pi(x_2)}(y_2)\, \mathrm{d}m_K(k)\\ &= \int_K \psi_{j_1}(k\pi(x_1))\psi_{j_1}(k\pi(x_1))\, \mathrm{d}m_K(k) \end{aligned} \tag{4.9}$$

where the last equality follows using (2.1). After all, combining (4.8) and (4.9), we get that for each $j_1, j_2 \in \mathbb{N}$,

$$\frac{1}{|\Psi_N|}\sum_{g\in\Psi_N}(T_g \times T_g)(\mathbb{E}_\mu(f_{j_1} \mid Z) \otimes \mathbb{E}_\mu(f_{j_2} \mid Z)) \to \left[(x_1, x_2) \mapsto \int_{X\times X} f_{j_1} \otimes f_{j_2}\, \mathrm{d}\lambda_{(x_1,x_2)}\right]$$

as $N \to \infty$ in $L^2(X \times X, \mu \times \mu)$. Now, since the family $f_{j_1} \otimes f_{j_2}$ is countable, using (ii) and a diagonal argument as in Step 1, one can find a sub-Følner sequence of Ψ, which by abuse of notation we again denote by Ψ, and a set $W_2 \subset X \times X$ with $(\mu \times \mu)(W_2) = 1$ such that for all $(x_1, x_2) \in W_2$ and $j_1, j_2 \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{|\Psi_N|}\sum_{g\in\Psi_N}(T_g \times T_g)(\mathbb{E}_\mu(f_{j_1} \mid Z) \otimes \mathbb{E}_\mu(f_{j_2} \mid Z)) = \int_{X\times X} f_{j_1} \otimes f_{j_2}\, \mathrm{d}\lambda_{(x_1,x_2)} \tag{4.10}$$

in $L^2(X \times X, \lambda_{(x_1,x_2)})$.

Let $W = W_1 \cap W_2$. Then $(\mu \times \mu)(W) = 1$, and combining (4.7) and (4.10), we get that for all $(x_1, x_2) \in W$ and all $j_1, j_2 \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{|\Psi_N|}\sum_{g\in\Psi_N}(T_g \times T_g)(f_{j_1} \otimes f_{j_2}) = \int_{X\times X} f_{j_1} \otimes f_{j_2}\, \mathrm{d}\lambda_{(x_1,x_2)}$$

in $L^2(X \times X, \lambda_{(x_1,x_2)})$, which was to be proved.

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To conclude the proof, we are left with showing (4.6). To this end, using the invariance of m_K, for any $g \in G$, we have that

$$\lambda_{(T_g x_1, T_g x_2)} = \int_Z \eta_{k\alpha(g)\pi(x_1)} \times \eta_{k\alpha(g)\pi(x_2)} \, \mathrm{d}m_K(k) = \int_Z \eta_{k\pi(x_1)} \times \eta_{k\pi(x_2)} \, \mathrm{d}m_K(k\alpha(g)^{-1})$$
$$= \int_Z \eta_{k\pi(x_1)} \times \eta_{k\pi(x_2)} \, \mathrm{d}m_K(k) = \lambda_{(x_1,x_2)}.$$

The proof of the theorem is now complete. □

Theorem 4.9. *We have that*

$$\sigma(\{(x_1,x_2) \in X \times X : (x_1,x_2) \in \mathtt{supp}(\lambda_{(x_1,x_2)})\}) = 1.$$

Theorem 4.10. *There exists some Følner sequence* Ψ *such that*

$$\sigma(\{(x_1,x_2) \in X \times X : (a,x_1) \in \mathtt{gen}(\lambda_{(x_1,x_2)}, \Psi)\}) = 1.$$

We first deal with Theorem 4.9. Let us first state and prove some results that will be useful in order to prove Theorem 4.9.

Let $\mathcal{F}(X)$ be the family of the closed, nonempty subsets of the compact metric space (X, d_X). We endow this family with the Hausdorff metric $\mathtt{D}$, defined by

$$\mathtt{D}(A,B) = \max\left\{\sup_{x\in A} d_X(x,B), \sup_{y\in B} d_X(y,A)\right\},$$

for any $A, B \in \mathcal{F}(X)$.

We will need the following two lemmas, the proofs of which are omitted, as they can be found in [15]:

Lemma 4.11. [15, Lemma 3.8] *Let W be a compact metric space,* $M(W)$ *the space of Borel probability measures on W endowed with the weak* topology, and* $\mathcal{F}(W)$ *the space of closed, nonempty subsets of W with the Hausdorff metric. Then*

- *The map* $\nu \mapsto \mathtt{supp}(\nu)$ *from* $M(W)$ *to* $\mathcal{F}(W)$ *is Borel measurable.*
- *If* $x \mapsto \rho_x$ *is a measurable map from W to* $M(W)$*, then* $\{x \in X : x \in \mathtt{supp}(\rho_x)\}$ *is a Borel set.*

Lemma 4.12. [15, Lemma 3.9] *The disintegration* $z \mapsto \eta_z$ *satisfies that the subset* $\{x \in X : x \in \mathtt{supp}(\eta_x)\}$ *of X is Borel measurable and*

$$\mu(\{x \in X : x \in \mathtt{supp}(\eta_x)\}) = 1.$$

Using those, we can now prove the following proposition, which is a variant of [15, Proposition 3.10].

Proposition 4.13. *There exists a sequence* $\delta_j \to 0$ *such that for* μ*-almost every* $x \in X$*, there exists* $w \in K$ *with* $\pi(x) = wH$ *such that for any open neighborhood U of x, we have*

$$\lim_{j\to\infty} \frac{m_K(\{k \in K : \eta_{kH}(U) > 0\} \cap \mathtt{B}_K(w,\delta_j))}{m_K(\mathtt{B}_K(w,\delta_j))} = 1, \tag{4.11}$$

where $\mathtt{B}_K(w,\delta_j)$ *denotes the ball centered at* $w \in K$ *and with radius* δ_j *in K with respect to the fixed metric* d_K.

Proof. Let $F : K \to \mathcal{F}(X)$ given by $F(k) = \mathtt{supp}(\eta_{kH})$. The natural projection $p : K \to Z$ is continuous, hence Borel measurable, the map $z \mapsto \eta_z$ is Borel measurable, as $(\eta_z)_{z\in Z}$ is a disintegration, and by Lemma 4.11, the map $\nu \mapsto \mathtt{supp}(\nu)$ is also Borel measurable; thus, we obtain that their

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composition F is also Borel measurable. By Lusin's theorem [2, Theorem 12.8], for any $j \in \mathbb{N}$, there exists a closed $K_j \subset K$ with

$$m_K(K_j) > 1 - 2^{-j} \tag{4.12}$$

such that $F|_{K_j}$ is continuous. For $j \in \mathbb{N}$, using the fact that K, and so K_j, is compact, we obtain that $F|_{K_j}$ is uniformly continuous. Therefore, for any $j \in \mathbb{N}$, there exists some $\delta_j > 0$ such that for any $k_1, k_2 \in K_j$ we have

$$d_K(k_1, k_2) \leq \delta_j \Longrightarrow \mathtt{D}(F(k_1), F(k_2)) < \frac{1}{j}.$$

Fix $j \in \mathbb{N}$. Then by the invariance of m_K, there is $c_j > 0$ such that for all $k \in K$, $m_K(\mathtt{B}_K(k, \delta_j)) = c_j$. The regularity of the measure m_K implies that there is a compact set $C_j \subset \mathtt{B}_K(e_K, \delta_j)$ such that $m_K(C_j) \geq c_j - \frac{c_j}{2^j}$. Now, by Urysohn's lemma, there is a continuous function $f_j : K \to [0, 1]$ such that $f_j = 1$ on C_j and $f_j = 0$ outside $\mathtt{B}_K(e_K, \delta_j)$.

Consider the set

$$W_j = \left\{k \in K_j : \int_{K_j} f_j(wk^{-1})\, \mathrm{d}m_K(w) \geq \left(1 - \frac{1}{j}\right) \int_K f_j(w)\, \mathrm{d}m_K(w)\right\}.$$

Note that

$$\int_K f_j(w)\, \mathrm{d}m_K(w) \geq m_K(C_j) \geq c_j - \frac{c_j}{2^j} > 0.$$

Then we can consider the function

$$\chi_j : K \to [0, 1], \ \ \chi_j(k) = \frac{\int_{K_j} f_j(wk^{-1})\, \mathrm{d}m_K(w)}{\int_K f_j(w)\, \mathrm{d}m_K(w)},$$

and moreover, we let

$$A_j = \left\{k \in K : \chi_j(k) \geq 1 - \frac{1}{j}\right\}.$$

Then we see that

$$W_j = K_j \cap A_j. \tag{4.13}$$

We will show that the set W_j is closed. Using the dominated convergence theorem and the fact that f_j is continuous, one can show that if $(k_\ell)_{\ell \in \mathbb{N}}$ is a sequence in K and $k_\ell \to k$, then $\int_{K_j} f_j(wk_\ell^{-1})\, \mathrm{d}m_K(w) \to \int_{K_j} f_j(wk^{-1})\, \mathrm{d}m_K(w)$, and this proves the continuity of χ_j. As a result, the set A_j is a closed subset of K, and since K_j is also closed, it follows that W_j is closed.

Now, using Fubini's theorem and the invariance of m_K, we deduce that

$$\begin{aligned}\int_K \chi_j(k)\, \mathrm{d}m_K(k) &= \frac{1}{\int_K f_j(w)\, \mathrm{d}m_K(w)} \int_K \int_K f_j(wk^{-1}) \mathbb{1}_{K_j}(w)\, \mathrm{d}m_K(k)\, \mathrm{d}m_K(w) \\ &= \frac{1}{\int_K f_j(w)\, \mathrm{d}m_K(w)} \int_K \mathbb{1}_{K_j}(w) \int_K f_j(k)\, \mathrm{d}m_K(k)\, \mathrm{d}m_K(w) \\ &= m_K(K_j) > 1 - \frac{1}{2^j},\end{aligned}$$

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and then we have that

$$1-\frac{1}{2^j}<\int_{A_j}\chi_j(k)\,\mathrm{d}m_K(k)+\int_{A_j^{\mathrm{c}}}\chi_j(k)\,\mathrm{d}m_K(k)\le m_K(A_j)+\left(1-\frac{1}{j}\right)m_K(A_j^{\mathrm{c}})=1-\frac{1}{j}+\frac{m_K(A_j)}{j},$$

which gives that

$$m_K(A_j)>1-\frac{j}{2^j}. \tag{4.14}$$

Combining (4.12), (4.13) and (4.14), we obtain that

$$\sum_{j\in\mathbb{N}}m_K(K\setminus W_j)=\sum_{j\in\mathbb{N}}m_K(K\setminus(K_j\cap A_j))<\sum_{j\in\mathbb{N}}\frac{1+j}{2^j}<\infty.$$

Let $W=\bigcup_{J\in\mathbb{N}}\bigcap_{j\ge J}W_j$. It follows by the Borel-Cantelli lemma, using the last equation above, that $m_K(W)=1$. Now let $L:=\{x\in X\colon x\in\mathrm{supp}(\eta_{\pi(x)})\}\cap\pi^{-1}(p(W))$. Observe that $p(W)=\bigcup_{J\in\mathbb{N}}p\big(\bigcap_{j\ge J}W_j\big)$. For each $J\in\mathbb{N}$, $\bigcap_{j\ge J}W_j$ is a closed subset of K and thus is compact, and since p is continuous, we get that $p\big(\bigcap_{j\ge J}W_j\big)$ is also compact; thus it is Borel measurable. As a result, we get that $p(W)$ is indeed a Borel subset of Z. In addition, $p^{-1}(p(W))\supset W$, and since $m_K(W)=1$, we have that $m_K(p^{-1}(p(W))=1$. Therefore, $m(p(W))=1$ and hence, $\mu(\pi^{-1}(p(W)))=1$. Then, in view of Lemma 4.12, it follows that L is a Borel subset of X and $\mu(L)=1$.

We now show that elements of L satisfy (4.11), and this will conclude the proof. Let $x\in L=\{x\in X\colon x\in\mathrm{supp}(\eta_{\pi(x)})\}\cap\pi^{-1}(p(W))$. Then $x\in\pi^{-1}(p(W))$ so there is $w\in W$ such that $\pi(x)=p(w)=wH$. Let U be an open neighborhood of x. Then we have that there exists $J\in\mathbb{N}$ such that for any $j\ge J$, $w\in W_j$ and $\mathtt{B}(x,\frac{1}{j})\subset U$.

We now claim that

$$\mathtt{B}_K(w,\delta_j)\cap K_j\subset\{k\in K\colon\eta_{kH}(U)>0\}.$$

To prove this, we let $w'\in\mathtt{B}_K(w,\delta_j)\cap K_j$. Then $d_K(w',w)<\delta_j$, and so, $\mathtt{D}(F(w'),F(w))<\frac{1}{j}$. Notice now that $F(w)=\mathrm{supp}(\eta_{wH})=\mathrm{supp}(\eta_{\pi(x)})$, and so, $x\in F(w)$, as $x\in L$. Then, by the definition of the Hausdorff metric, there exists $x'\in F(w')$ with $d_X(x,x')<\frac{1}{j}$, and so, $x'\in U$, which, combined with the fact that $x'\in F(w')$, yields $U\cap F(w')\ne\emptyset$. It follows that $\eta_{w'H}(U)>0$.

It follows from the above claim that

$$\frac{m_K(\{k\in K\colon\eta_{kH}(U)>0\}\cap\mathtt{B}_K(w,\delta_j))}{m_K(\mathtt{B}_K(w,\delta_j))}\ge\frac{m_K(\mathtt{B}_K(w,\delta_j)\cap K_j)}{m_K(\mathtt{B}_K(w,\delta_j))}=\frac{\int_K\mathbb{1}_{\mathtt{B}_K(w,\delta_j)}(u)\mathbb{1}_{K_j}(u)\,\mathrm{d}m_K(u)}{m_K(\mathtt{B}_K(e_K,\delta_j))}. \tag{4.15}$$

The denominator in the right-most term in (4.15) is smaller or equal to $\frac{2^j}{2^j-1}m_K(C_j)$, which then is smaller or equal to $\frac{2^j}{2^j-1}\int_K f_j(u)\,\mathrm{d}m_K(u)$, and therefore, the expression in (4.15) is greater or equal to

$$\frac{\int_K\mathbb{1}_{\mathtt{B}_K(w,\delta_j)}(u)\mathbb{1}_{K_j}(u)\,\mathrm{d}m_K(u)}{\int_K f_j(u)\,\mathrm{d}m_K(u)}\left(1-\frac{1}{2^j}\right). \tag{4.16}$$

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Observe that for all $u \in K$, $\mathbb{1}_{\mathrm{B}_K(w,\delta_j)}(uw) = \mathbb{1}_{\mathrm{B}_K(e_K,\delta_j)}(u)$, so we have that

$$\begin{aligned}\int_K \mathbb{1}_{\mathrm{B}_K(w,\delta_j)}(u)\mathbb{1}_{K_j}(u)\,\mathrm{d}m_K(u) &= \int_K \mathbb{1}_{\mathrm{B}_K(w,\delta_j)}(uw)\mathbb{1}_{K_j}(uw)\,\mathrm{d}m_K(u)\\ &= \int_K \mathbb{1}_{\mathrm{B}_K(e_K,\delta_j)}(u)\mathbb{1}_{K_j}(uw)\,\mathrm{d}m_K(u) \geq \int_K f_j(u)\mathbb{1}_{K_j}(uw)\,\mathrm{d}m_K(u)\\ &= \int_K f_j(uw^{-1})\mathbb{1}_{K_j}(u)\,\mathrm{d}m_K(u) = \int_{K_j} f_j(uw^{-1})\,\mathrm{d}m_K(u).\end{aligned}$$

Combining the last equation with (4.15) and (4.16), we get that

$$\begin{aligned}\frac{m_K(\{k \in K\colon \eta_{kH}(U) > 0\} \cap \mathrm{B}_K(w,\delta_j))}{m_K(\mathrm{B}_K(w,\delta_j))} &\geq \frac{\int_{K_j} f_j(uw^{-1})\,\mathrm{d}m_K(u)}{\int_K f_j(u)\,\mathrm{d}m_K(u)}\left(1-\frac{1}{2^j}\right)\\ &\geq \left(1-\frac{1}{j}\right)\left(1-\frac{1}{2^j}\right),\end{aligned}$$

where the least inequality is due to the fact that $w \in W_j$. Then, taking the limit as $j \to \infty$, we obtain that

$$\lim_{j\to\infty} \frac{m_K(\{k \in K\colon \eta_{kH}(U) > 0\} \cap \mathrm{B}_K(w,\delta_j))}{m_K(\mathrm{B}_K(w,\delta_j))} = 1,$$

and this concludes the proof. □

We are now ready to prove Theorem 4.9.

Proof of Theorem 4.9. Let $S = \{(x_1,x_2) \in X \times X\colon (x_1,x_2) \in \mathrm{supp}(\lambda_{(x_1,x_2)})\}$. By Lemma 4.11, S is a Borel subset of $X \times X$. Consider a sequence $\delta_j \to 0$ such that Proposition 4.13 is satisfied, and let $L \subset X$ be the set of $x \in X$ that satisfy (4.11). By Proposition 4.13, $\mu(L) = 1$. Following the argument in [15, Proposition 3.11], we will show that $\sigma(L \times L) = 1$ and $L \times L \subset S$. Consequently, we will have that $\sigma(S) = 1$, concluding the proof.

We start by showing that $\sigma(L \times L) = 1$. We write $L \times L = (L \times X) \cap (X \times L)$, and so it is enough to show that both sets in this intersection have full measure σ. By Proposition 4.5 (i), we have that $\sigma(L \times X) = \pi_1\sigma(L) = \mu(L) = 1$. By Proposition 4.5 (ii), the measure $\pi_2\sigma$ is absolutely continuous with respect to μ, and since $\mu(L) = 1$, it follows that $\sigma(X \times L) = \pi_2\sigma(L) = 1$.

To conclude the proof, we show that $L \times L \subset S$. Let $(x_1,x_2) \in L \times L$. To show that $(x_1,x_2) \in S$, it is enough to show that for all open neighborhoods U_1, U_2 of x_1, x_2, we have $\lambda_{(x_1,x_2)}(U_1 \times U_2) > 0$.

Let U_1, U_2 be open neighborhoods of x_1, x_2, respectively. By writing $\lambda_{(x_1,x_2)} = \int_K \eta_{kH} \times \eta_{kk_1^{-1}k_2H}\,\mathrm{d}m_K(k)$, where $k_1, k_2 \in K$ are such that $\pi(x_1) = k_1H$, $\pi(x_2) = k_2H$, we see that it suffices to show that the set $W = W(k_1,k_2) := \{k \in K : \eta_{kH}(U_1) > 0 \text{ and } \eta_{kk_1^{-1}k_2H}(U_2) > 0\}$ has positive measure m_K, for some choice of the k_1, k_2 as above. By Proposition 4.13, we can choose the elements $k_1, k_2 \in K$ such that

$$\frac{m_K(\{k \in K : \eta_{kH}(U_1) > 0\} \cap \mathrm{B}_K(k_1,\delta))}{m_K(\mathrm{B}_K(k_1,\delta))} \geq \frac{3}{4} \tag{4.17}$$

and

$$\frac{m_K(\{k \in K : \eta_{kH}(U_2) > 0\} \cap \mathrm{B}_K(k_2,\delta))}{m_K(\mathrm{B}_K(k_2,\delta))} \geq \frac{3}{4}, \tag{4.18}$$

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for some $\delta > 0$. Now using (4.18) along with the bi-invariance of both d_K and m_K, we have that

$$\begin{aligned}\frac{m_K(\{k \in K : \eta_{kk_1^{-1}k_2H}(U_2) > 0\} \cap \mathrm{B}_K(k_1,\delta))}{m_K(\mathrm{B}_K(k_1,\delta))} &= \frac{m_K(\{k \in K : \eta_{kH}(U_2) > 0\} \cdot k_2^{-1}k_1 \cap \mathrm{B}_K(k_2,\delta) \cdot k_2^{-1}k_1)}{m_K(\mathrm{B}_K(k_2,\delta) \cdot k_2^{-1}k_1)} \\ &= \frac{m_K(\{k \in K : \eta_{kH}(U_2) > 0\} \cap \mathrm{B}_K(k_2,\delta))}{m_K(\mathrm{B}_K(k_2,\delta))} \geq \frac{3}{4}. \end{aligned} \tag{4.19}$$

Combining (4.17) and (4.19) yields $\frac{m_K(W)}{m_K(\mathrm{B}_K(k_1,\delta))} \geq \frac{1}{2}$. This implies that $m_K(W) > 0$ and concludes the proof. □

It remains to show Theorem 4.10. To this end, we need the following lemma, which is the analog of [15, Lemma 3.7] in our setting.

Lemma 4.14. *For σ-almost every $(x_1, x_2) \in X \times X$, we have $\lambda_{(a,x_1)} = \lambda_{(x_1,x_2)}$.*

Proof. By the definition of σ and the fact that $z \mapsto \eta_z$ is a disintegration, it follows that for σ-almost every (x_1, x_2), we have $\pi(x_1) = w\pi(a)$ and $\pi(x_2) = w\pi(x_1)$, for some $w \in K$. For any such (x_1, x_2), using the right invariance of m_K, we have

$$\begin{aligned}\lambda_{(x_1,x_2)} &= \int_{X\times X} \eta_{k\pi(x_1)} \times \eta_{k\pi(x_2)}\, \mathrm{d}m_K(k) = \int_{X\times X} \eta_{kw\pi(a)} \times \eta_{kw\pi(x_1)}\, \mathrm{d}m_K(k) \\ &= \int_{X\times X} \eta_{k\pi(a)} \times \eta_{k\pi(x_1)}\, \mathrm{d}m_K(kw^{-1}) = \int_{X\times X} \eta_{k\pi(a)} \times \eta_{k\pi(x_1)}\, \mathrm{d}m_K(k) = \lambda_{(a,x_1)}.\end{aligned}$$

This concludes the proof. □

We are now ready to prove Theorem 4.10.

Proof of Theorem 4.10. Consider the measure $\nu_a := \delta_a \times \mu$, where δ_a denotes the Dirac mass at a. □

Claim. There exists some Følner sequence Ψ such that

$$\nu_a(\{(x_0, x_1) \in X \times X : (x_0, x_1) \in \mathsf{gen}(\lambda_{(x_0,x_1)}, \Psi)\}) = 1.$$

From the definition of ν_a, it is clear that for ν_a-almost every $(x_0, x_1) \in X \times X$, $x_0 = a$. Therefore, assuming the claim, we have that

$$\nu_a(\{(a, x_1) \in X \times X : (a, x_1) \in \mathsf{gen}(\lambda_{(a,x_1)}, \Psi)\}) = 1,$$

which, by the definition of ν_a implies that

$$\mu(\{x_1 \in X : (a, x_1) \in \mathsf{gen}(\lambda_{(a,x_1)}, \Psi)\}) = 1.$$

Then, by Proposition 4.5 (i), we have that $\pi_1\sigma = \mu$, and therefore,

$$\sigma(\{(x_1, x_2) \in X \times X : (a, x_1) \in \mathsf{gen}(\lambda_{(a,x_1)}, \Psi)\}) = 1.$$

From Lemma 4.14, we know that $\lambda_{(a,x_1)} = \lambda_{(x_1,x_2)}$ for σ-almost every (x_1, x_2), and consequently,

$$\sigma(\{(x_1, x_2) \in X \times X : (a, x_1) \in \mathsf{gen}(\lambda_{(x_1,x_2)}, \Psi)\}) = 1,$$

which was to be proved. Now, to finish the proof of Theorem 4.10, it only remains to prove the claim.

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Proof of Claim. In this proof, we follow the argument used in the proof of [14, Theorem 7.6]. Apply Lemma 2.8 for the ergodic decomposition $(x_0, x_1) \mapsto \lambda_{(x_0,x_1)}$ to obtain a Følner sequence Φ such that

$$(\mu \times \mu)(\{(x_0, x_1) \in X \times X : (x_0, x_1) \in \mathtt{gen}(\lambda_{(x_0,x_1)}, \Phi)\}) = 1. \tag{4.20}$$

Consider the map $X \to M(X)$, $s \mapsto \nu_s = \delta_s \times \mu$, where δ_s denotes the Dirac mass at s. It is quite straightforward to see that $s \mapsto \nu_s$ is a continuous disintegration of $\mu \times \mu$ and, moreover, satisfies

$$(T_g \times T_g)\nu_s = \nu_{T_g s} \tag{4.21}$$

for any $s \in X$. By (4.20) and the fact that $s \mapsto \nu_s$ is a disintegration of $\mu \times \mu$, it follows that for μ-almost every $s \in X$, ν_s-almost every $(x_0, x_1) \in X \times X$ is in $\mathtt{gen}(\lambda_{(x_0,x_1)}, \Phi)$. Fix $b \in \mathtt{supp}(\mu)$. Then

$$\nu_b\text{-almost every } (x_0, x_1) \text{ is in } \mathtt{gen}(\lambda_{(x_0,x_1)}, \Phi). \tag{4.22}$$

By Lemma 2.5, there exists some sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $T_{g_n} a \to b$, and now by continuity of the disintegration $s \mapsto \nu_s$, combined with (4.21), it follows that

$$(T_{g_n} \times T_{g_n})\nu_a \to \nu_b. \tag{4.23}$$

Now, let $\mathcal{F} = (F_k)_{k\in\mathbb{N}}$ be a dense subset of $C(X \times X)$ and for $k, N \in \mathbb{N}$, consider the sets

$$A_{k,N} = \left\{(x_0, x_1) \in X \times X : \max_{1\le j\le k} \left| \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} F_j((T_g \times T_g)(x_0, x_1)) - \int_{X\times X} F_j \, \mathrm{d}\lambda_{(x_0,x_1)} \right| \le \frac{1}{k} \right\}.$$

Using (4.22) and the monotone convergence theorem, it follows that for any $k \in \mathbb{N}$, there exists some $N(k) \in \mathbb{N}$, such that

$$\nu_b(A_{k,N(k)}) \ge 1 - 2^{-k}. \tag{4.24}$$

For $k, N \in \mathbb{N}$, we define

$$B_{k,N} = \left\{(x_0, x_1) \in X \times X : d_{X\times X}((x_0, x_1), A_{k,N}) < \frac{1}{k} \right\},$$

where $d_{X\times X}$ is the metric on $X \times X$. Then for k sufficiently large, we have

$$\max_{1\le j\le k} \left| \frac{1}{|\Phi_{N(k)}|} \sum_{g\in\Phi_{N(k)}} F_j((T_g \times T_g)(x_0, x_1)) - \int_{X\times X} F_j \, \mathrm{d}\lambda_{(x_0,x_1)} \right| \le \frac{2}{k}, \tag{4.25}$$

for any $(x_0, x_1) \in B_{k,N(k)}$. The sets $A_{k,N}$ are open, while the sets $B_{k,N}$ are closed subsets of $X \times X$, and also $A_{k,N(k)} \subset B_{k,N(k)}$, so by Urysohn's lemma, we can find, for all $k \in \mathbb{N}$, continuous functions $f_k : X \times X \to [0, 1]$ such that

$$f_k|_{A_{k,N(k)}} = 1 \quad \text{and} \quad f_k|_{(X\times X)\setminus B_{k,N(k)}} = 0.$$

By (4.22), for each $k \in \mathbb{N}$, there exists $n(k) \in \mathbb{N}$ such that

$$\left| \int_{X\times X} (T_{g_{n(k)}} \times T_{g_{n(k)}}) f_k \, \mathrm{d}\nu_a - \int_{X\times X} f_k \, \mathrm{d}\nu_b \right| \le 2^{-k}.$$

Let $(h_k)_{k\in\mathbb{N}}$ be the subsequence of $(g_n)_{n\in\mathbb{N}}$ defined by $h_k = g_{n(k)}$, for $k \in \mathbb{N}$. Then, by the equation above, we have

$$\begin{aligned}\nu_a\big((T_{h_k}\times T_{h_k})^{-1}B_{k,N(k)}\big) &\geq \int_{X\times X}(T_{h_k}\times T_{h_k})f_k \,\mathrm{d}\nu_a \geq \int_{X\times X} f_k \,\mathrm{d}\nu_b - 2^{-k}\\ &\geq \nu_b(A_{k,N(k)}) - 2^{-k} \geq 1-2^{-k+1} \quad \text{(by (4.24))},\end{aligned}$$

for any $k \in \mathbb{N}$. Therefore, it holds that

$$\sum_{k\geq 1}\nu_a\big((T_{h_k}\times T_{h_k})^{-1}B_{k,N(k)}\big) = \infty,$$

and then, by the Borel-Cantelli lemma, it follows that ν_a-almost every $(x_0,x_1) \in \mathtt{supp}(\nu_a)$ belong to all, but finitely many, sets $(T_{h_k}\times T_{h_k})^{-1}B_{k,N(k)}$. Then by (4.25), it follows that for ν_a-almost every $(x_0,x_1) \in X\times X$ and k sufficiently large, we have

$$\max_{1\leq j\leq k}\left|\frac{1}{|\Psi_k|}\sum_{g\in\Psi_k}F_j((T_g\times T_g)(x_0,x_1)) - \int_{X\times X}F_j \,\mathrm{d}\lambda_{(T_{h_k}\times T_{h_k})(x_0,x_1)}\right| \leq \frac{2}{k},$$

where Ψ is the Følner sequence defined by $\Psi_k = \Phi_{N(k)}h_k$, for $k \in \mathbb{N}$. Using (4.6), the above equation becomes

$$\max_{1\leq j\leq k}\left|\frac{1}{|\Psi_k|}\sum_{g\in\Psi_k}F_j((T_g\times T_g)(x_0,x_1)) - \int_{X\times X}F_j \,\mathrm{d}\lambda_{(x_0,x_1)}\right| \leq \frac{2}{k}.$$

Sending $k \longrightarrow \infty$, we have shown that

$$\lim_{k\to\infty}\frac{1}{|\Psi_k|}\sum_{g\in\Psi_k}F((T_g\times T_g)(x_0,x_1)) = \int_{X\times X}F \,\mathrm{d}\lambda_{(x_0,x_1)}$$

holds for ν_a-almost every $(x_0,x_1) \in X\times X$ and for any $F \in \mathcal{F}$. An approximation argument concludes the proof of the Claim. □

The proof of the theorem is complete.

4.3. The proof of Theorem 3.8

We are now ready to prove Theorem 3.8.

Let G be a square absolutely continuous group, let (X,μ,T) be an ergodic G-system admitting a continuous factor map to its Kronecker factor, and let $a \in \mathtt{gen}(\mu,\Phi)$ for some Følner sequence Φ. Let also E be a clopen subset of X with $\mu(E) > 0$. Consider the measure σ given in (4.1). By Theorems 4.9 and 4.10, we have that there is a Følner sequence Ψ such that

$$\sigma(\{(x_1,x_2) : (a,x_1) \in \mathtt{gen}(\lambda_{(x_1,x_2)},\Psi) \text{ and } (x_1,x_2) \in \mathtt{supp}(\lambda_{(x_1,x_2)})\}) = 1. \tag{4.26}$$

However, by Theorem 4.6, we have that

$$\sigma\left(E\times\bigcup_{t\in G}T_t^{-1}E\right) > 0. \tag{4.27}$$

Combining (4.26) and (4.27), we get that there exists $(x_1,x_2) \in X\times X$ such that for the $T\times T$-invariant measure $\lambda := \lambda_{(x_1,x_2)}$, we have that $(a,x_1) \in \mathtt{gen}(\lambda,\Psi)$, $(x_1,x_2) \in \mathtt{supp}(\lambda)$ and also $(x_1,x_2) \in E\times\left(\bigcup_{t\in G}T_t^{-1}E\right)$. Hence, there is $t \in G$ such that $x_1 \in E$ and $x_2 \in T_t^{-1}E$. Finally, applying

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Lemma 2.5 for $Y = X\times X$, $S = T\times T$, $y = (a, x_1)$ and $w = (x_1, x_2)$, $\nu = \lambda$ and for the Følner sequence Ψ, we get that there is an infinite sequence $(g_n)_{n\in N}$ in G such that $(T_{g_n}\times T_{g_n})(a, x_1) \to (x_1, x_2)$ in $X\times X$. Therefore, $(a, x_1, x_2) \in X^3$ forms an Erdős progression, and this concludes the proof of Theorem 3.8.

5. Proof of the corollaries of Theorem 1.6

We start by showing Theorem 1.8, and then we will prove Corollaries 1.9 and 1.10. We split the proof of Theorem 1.8 into the following three lemmas:

Lemma 5.1. *Let G be an amenable group, let $M > 0$, and let $\phi : G \to G$ be a map such that for every $g \in G$, $|\phi^{-1}(\{g\})| \le M$. Suppose that there exist two Følner sequences Φ and Ψ in G and some $\eta > 0$ such that for any $N \in \mathbb{N}$, we have that $\phi(\Psi_N) \subset \Phi_N$ and*

$$\frac{|\phi(\Psi_N)|}{|\Phi_N|} \ge \eta.$$

Then G is ϕ-absolutely continuous.

Lemma 5.2. *Let G be a torsion-free finitely generated nilpotent group. Then the squaring map s_G on G is injective.*

Lemma 5.3. *Let G be a torsion-free finitely generated nilpotent group. Then there exists some Følner sequence Ψ in G and some $\eta > 0$ such that for any $N \in \mathbb{N}$, we have that $s_G(\Psi_N) \subset \Psi_{N+1}$ and*

$$\frac{|s_G(\Psi_N)|}{|\Psi_{N+1}|} = \eta. \tag{5.1}$$

It is clear that Theorem 1.8 follows immediately from Lemmas 5.1, 5.2 and 5.3, where Lemma 5.1 is applied for $\phi = s_G$, Ψ the Følner guaranteed by Lemma 5.3 and Φ the Følner given by $\Phi_N = \Psi_{N+1}$.

Proof of Lemma 5.1. Let G, M, ϕ, Φ, Ψ and η be as in the assumptions of Lemma 5.1. We will prove something stronger than we require, namely that for any $u : G \to [0, 1]$ we have

$$\limsup_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} u(\phi(g)) \le \frac{M}{\eta} \limsup_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} u(g),$$

and then for any $\varepsilon > 0$, taking $\delta = \eta\varepsilon/M > 0$ yields that G is ϕ-absolutely continuous. Let $u : G \to [0, 1]$, and let $(\Psi_{N_k})_{k\in\mathbb{N}}$ satisfy

$$\lim_{k\to\infty} \frac{1}{|\Psi_{N_k}|} \sum_{g\in\Psi_{N_k}} u(\phi(g)) = \limsup_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} u(\phi(g)).$$

For each $k \in \mathbb{N}$, using the assumption on Φ and Ψ, we have that $\phi(\Psi_{N_k}) \subset \Phi_{N_k}$ and $\frac{|\phi(\Psi_{N_k})|}{|\Phi_{N_k}|} \ge \eta$. Then for every $k \in \mathbb{N}$, we have

$$\frac{1}{|\Psi_{N_k}|} \sum_{g\in\Psi_{N_k}} u(\phi(g)) \le \frac{M}{|\phi(\Psi_{N_k})|} \sum_{g\in\phi(\Psi_{N_k})} u(g) \le \frac{M}{\eta} \frac{1}{|\Phi_{N_k}|} \sum_{g\in\Phi_{N_k}} u(g),$$

so letting $k \to \infty$, we obtain that

$$\lim_{k\to\infty} \frac{1}{|\Psi_{N_k}|} \sum_{g\in\Psi_{N_k}} u(\phi(g)) \le \frac{M}{\eta} \limsup_{k\to\infty} \frac{1}{|\Phi_{N_k}|} \sum_{g\in\Phi_{N_k}} u(g) \le \frac{M}{\eta} \limsup_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} u(g),$$

which concludes the proof. □

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Let G be a torsion-free finitely generated nilpotent group and fix a Mal'cev coordinate system $(t_1, \dots, t_s)$. By [11, Theorem 17.2.5], we have that for any $1 \leq i \leq s$, there are polynomials p_i in $i-1$ variables, with rational coefficients, satisfying $p_i(\mathbb{Z}^{i-1}) \subset \mathbb{Z}$ such that for any $x \in G$, we have

$$t_i(x^2) = 2t_i(x) + p_i(t_1(x), \dots, t_{i-1}(x)). \tag{5.2}$$

In the previous, p_1 is a polynomial in 0 variables, which means that it is a constant polynomial.

Proof of Lemma 5.2. The proof becomes obvious by using (5.2) and the injectivity of the coordinate map $G \to \mathbb{Z}^s$, $x \mapsto (t_1(x), \dots, t_s(x))$, which follows from the fact that G is torsion-free. □

Proof of Lemma 5.3. Let G be a torsion-free finitely generated nilpotent group and fix a Mal'cev coordinate system $(t_1, \dots, t_s)$. We identify the group G with $\mathbb{Z}^s$ as implied by the Mal'cev coordinate system. Let $(p_i)_{1\leq i\leq s}$ be the sequence of polynomials satisfying (5.2), and for each i, we denote the number of terms of p_i by γ_i. Now we will define three integer-valued sequences $(b_i)_{1\leq i\leq s}$, $(c_i)_{1\leq i\leq s}$ and $(d_i)_{1\leq i\leq s}$ that will help us to construct the Følner sequence Ψ with the required properties. We start with the latter one, and we define it recursively as follows: Let $d_1 = 1$. Now let $1 < i \leq s$ and suppose that d_j has been defined for all $1 \leq j < i$. Given a monomial $m(x_1, \dots, x_{i-1}) = x_1^{e_1} \dots x_{i-1}^{e_{i-1}}$, with $e_j \geq 0$ for any $1 \leq j < i$, we let $d(m) = \sum_{1\leq j<i} e_j d_j$, and then we also let $d(p_i) = \max\{d(m)\colon \text{the monomial } m \text{ appears in } p_i\}$. Then we define $d_i = \max\{d(p_i), 1\}$. The other two sequences are defined as follows: Let $b_1 = 0$. Now we fix $1 < i \leq s$. We denote by m_i the monomial m appearing in p_i with maximal coefficient in absolute value such that $d(p_i) = d(m)$. Then we define b_i to be the ceiling of the absolute value of the coefficient of m_i. Finally, for any $1 \leq i \leq s$, we let $c_i = \gamma_i b_i + 2$. We also note that $d_i = d(m_i)$ for each $1 \leq i \leq s$.

Let us now define the Følner sequence Ψ. By the definition of $(b_i)_{1\leq i\leq s}$ and $(d_i)_{1\leq i\leq s}$, we have that for any $1 \leq i \leq s$, if $|x_j| \leq M^{d_j}$ for every $1 \leq j < i$, for some $M > 0$, then $|m_i(x_1, \dots, x_{i-1})| \leq b_i M^{d_i}$, and then, $|p_i(x_1, \dots, x_{i-1})| \leq \gamma_i b_i M^{d_i}$. It follows that for any $1 \leq i \leq s$, the following implication holds:

$$|x_j| \leq c_j^{N d_j} \quad \forall\, 1 \leq j < i \Longrightarrow |p_i(x_1, \dots, x_{i-1})| \leq \gamma_i b_i c_i^{N d_i}. \tag{5.3}$$

Given $M > 0$, we use the notation $[M] := (-M, M] \cap \mathbb{Z}$. Now, for any $N \in \mathbb{N}$, we define

$$\Psi_N = [c_1^{(N-1)d_1}] \times [c_2^{(N-1)d_2}] \times \cdots \times [c_s^{(N-1)d_s}].$$

It is not hard to check that $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ is a Følner sequence in G. We show that $s_G(\Psi_N) \subset \Psi_{N+1}$ for every $N \in \mathbb{N}$. Let $N \in \mathbb{N}$ and then

$$s_G(\Psi_N) = \left\{(2t_i(x) + p_i((t_j(x))_{1\leq j<i}))_{1\leq i\leq s}\colon t_i(x) \in [c_i^{(N-1)d_i}] \quad \forall\, 1 \leq i \leq s\right\}.$$

Let $x^2 = (t_1(x^2), \dots, t_s(x^2)) \in s_G(\Psi_N)$. We claim that for each $1 \leq i \leq s$, we have that

$$t_i(x^2) \in [c_i^{N d_i}]. \tag{5.4}$$

Let $1 \leq i \leq s$. For any $1 \leq j < i$, we have that $t_j(x) \in [c_j^{(N-1)d_j}]$, and then we have that

$$\begin{aligned} t_i(x^2) &= 2t_i(x) + p_i((t_j(x)\colon j < i)) \\ &\in (p_i((t_j(x))_{1\leq j<i}) - 2c_i^{(N-1)d_i}, p_i((t_j(x))_{1\leq j<i}) + 2c_i^{(N-1)d_i}] \cap (2\mathbb{Z} + p_i((t_j(x))_{1\leq j<i})) \\ &\subset [c_i^{N d_i}], \end{aligned}$$

where the last one follows from $|p_i((t_j(x))_{1\leq j<i})| \leq \gamma_i b_i c_i^{(N-1)d_i}$, by (5.3), and from the definition of c_i along with that $d_i \geq 1$. This shows (5.4) and hence that $s_G(\Psi_N) \subset \Psi_{N+1}$.

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It remains to prove (5.1), and we show it for the Følner sequence Ψ and with $\eta := \prod_{1\le i\le s} c_i^{-d_i} > 0$. In other words, we prove that for any $N \in \mathbb{N}$, we have

$$\frac{|s_G(\Psi_N)|}{|\Psi_{N+1}|} = \prod_{1\le i\le s} c_i^{-d_i}. \tag{5.5}$$

Let $N \in \mathbb{N}$. By Lemma 5.2, s_G is injective, and since Ψ_N is finite, we have that

$$|s_G(\Psi_N)| = |\Psi_N| = 2^s\left(\prod_{1\le i\le s} c_i^{d_i}\right)^{N-1}.$$

Moreover, we clearly have

$$|\Psi_{N+1}| = 2^s\left(\prod_{1\le i\le s} c_i^{d_i}\right)^{N},$$

and so (5.5) follows from comparing the last two equations. This concludes the proof. □

Having established Theorem 1.8, we are ready to prove Corollary 1.9. Before we prove it, let us make some remarks.

Let $s \in \mathbb{N}$ and $1 \le i \le s$. We say that a nonempty set $L \subset \mathbb{Z}^s$ is a *line* in the i-th coordinate if there is a nonempty interval I in $\mathbb{Z}$ (i.e., a set of the form $\{m, \dots, m+r\}$ for some $m, r \in \mathbb{Z}$ with $r \ge 0$) and some integers x_j, $j \in \{1, \dots, s\} \setminus \{i\}$ such that

$$L = \{x_1\} \times \cdots \times \{x_{i-1}\} \times I \times \{x_{i+1}\} \times \cdots \times \{x_s\}.$$

We refer to the cardinality of the interval I as the length of the line L.

Let G be a torsion-free finitely generated nilpotent group and $(t_1, \dots, t_s)$ a Mal'cev coordinate system. For each $1 \le i \le s$, let e_i be the element with coordinates $(e_1^{(i)}, \dots, e_s^{(i)})$, where $e_i^{(i)} = 1$ and $e_j^{(i)} = 0$ for $j \ne i$. Every $g \in G$ has a unique representation as $(t_1(g), \dots, t_s(g)) \in \mathbb{Z}^s$, and this defines a bijective map from G to $\mathbb{Z}^s$. From now on, we identify each $g \in G$ with its coordinates $(t_1(g), \dots, t_s(g)) \in \mathbb{Z}^s$, and every set $E \subset G$ with the corresponding set of coordinates in $\mathbb{Z}^s$. We freely pass from viewing a set $E \subset G$ as a subset of $\mathbb{Z}^s$ and vice versa, without stating it, as it will be clear from the context.

If E is a finite subset of G and $1 \le i \le s$, then E can be written as a finite disjoint union of lines in the i-th coordinate, and this can be done in many ways. We want to write E as a union of lines which is going to be maximal in some sense that is going to be useful for us in our proof of Corollary 1.9.

More precisely, if E is a finite subset of G, and $1 \le i \le s$, then we can always write it as a disjoint union $E = \bigsqcup_{j=1}^{\ell} L^{(j)}$ such that the sets $L^{(j)}$ are lines in the i-th coordinate, and for each j, the line $L^{(j)}$ is maximal within E, meaning that for each $1 \le j \le s$, $e_i L^{(j)}$ is not a subset of E. Note that although there always exists such a choice of lines, it may not be unique, but uniqueness is not necessary for our purposes. We are now ready to prove Corollary 1.9.

Proof of Corollary 1.9. The first part of Corollary 1.9 follows immediately by combining Theorems 1.6 and 1.8. It remains to prove that if G is a torsion-free finitely generated nilpotent group, then given a Mal'cev coordinate system $(t_1, \dots, t_s)$ on G, we can choose B so that for any finite set $C \subset \mathbb{Z}$ and any $1 \le i \le s$, the set $\{b \in B : t_i(b) \in C\}$ is finite.

Let G be a torsion-free finitely generated nilpotent group, let A have positive left upper Banach density, and $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ a left Følner sequence such that $d_\Psi(A) > 0$, where the previous density exists. In addition, let $(t_1, \dots, t_s)$ be any Mal'cev coordinate system on G.

Claim. *There is a Følner sequence Ψ' such that $d_{\Psi'}(A) > 0$, and for any finite set $C \subset \mathbb{Z}$ and any $1 \le i \le s$, the set $\{N \in \mathbb{N} : t_i(\Psi'_N) \cap C \ne \emptyset\}$ is finite.*

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Assume that we have proved the Claim. Then we can consider the set $A' := \bigcup_{N\in\mathbb{N}} A \cap \Psi'_N$. Then we have that $d_{\Psi'}(A) > 0$, so by Corollary 1.9, there is an infinite sequence $B \subset A' \subset A$ and some $t \in G$ such that $t \cdot B \blacktriangleleft B \subset A' \subset A$. Let $C \subset \mathbb{Z}$ be finite and $1 \leq i \leq s$. For each $N \in \mathbb{N}$, Ψ'_N is finite and $\{N \in \mathbb{N} : t_i(\Psi'_N) \cap C \neq \emptyset\}$ is also finite, so since $B \subset \bigcup_{N\in\mathbb{N}} \Psi'_N$ is infinite, one easily sees that $\{b \in B : t_i(b) \in C\}$ is finite. So it remains to prove the Claim.

Proof of Claim. We know that $d_\Psi(A) = \lim_{N\to\infty} \frac{|A\cap\Psi_N|}{|\Psi_N|} > 0$. For each $1 \leq i \leq s$ and $N \in \mathbb{N}$, let

$$\delta_{i,N} = \frac{|\Psi_N \triangle (e_i \Psi_N)|}{|\Psi_N|}.$$

Since Ψ is a Følner sequence, for all i, $\delta_{i,N} \to 0$ as $N \to \infty$. Then we can choose a sequence Q_N of natural numbers such that $Q_N \to \infty$ as $N \to \infty$ and for all i, $Q_N \delta_{i,N} \to 0$ as $N \to \infty$.

Step 1: For each $N \in \mathbb{N}$, Ψ_N is a finite subset of G, so we can write it as a disjoint union of lines in the 1st coordinate, which are maximal within Ψ_N. Let $\Psi_N^{(1)}$ be the union of those lines whose length is greater that Q_N, and m_N be the number of those lines whose length is less than or equal to Q_N. Then

$$Q_N \delta_{1,N} = Q_N \frac{|\Psi_N \triangle (e_1 \Psi_N)|}{|\Psi_N|} \geq \frac{Q_N m_N}{|\Psi_N|} \geq \frac{|\Psi_N| - |\Psi_N^{(1)}|}{|\Psi_N|}.$$

Therefore, $\frac{|\Psi_N^{(1)}|}{|\Psi_N|} \to 1$ as $N \to \infty$, from which one gets that $\Psi^{(1)} = (\Psi_N^{(1)})_{N\in\mathbb{N}}$ is a left Følner sequence in G. In addition, it is not difficult to see that the density $d_{\Psi^{(1)}}(A)$ exists and $d_{\Psi^{(1)}}(A) = d_\Psi(A)$.

Recall that for each $N \in \mathbb{N}$, $\Psi_N^{(1)}$ is a disjoint union of some lines in the 1-st coordinate $L^{(1,N)}, \ldots, L^{(\ell_N,N)}$ whose length is greater than Q_N. Let $N_1 = 1$ and set $\widetilde{\Psi}_1^{(1)} = \Psi_1^{(1)}$. Since $\Psi_1^{(1)}$ is finite, the projection P_1 of $\Psi_1^{(1)}$ in the first coordinate is also finite. For each $N \in \mathbb{N}$, let

$$\Psi_N^{(1,2)} = \{g \in \Psi_N^{(1)} : t_1(g) \notin P_1\} = \bigsqcup_{j=1}^{\ell_N} \{g \in L^{(j,N)} : t_1(g) \notin P_1\}.$$

Then $\Psi_N^{(1,2)} \subset \Psi_N^{(1)}$ and

$$\begin{aligned}\frac{|\Psi_N^{(1,2)}|}{|\Psi_N^{(1)}|} &= \frac{\sum_{j=1}^{\ell_N} |\{g \in L^{(j,N)} : t_1(g) \notin P_1\}|}{\sum_{j=1}^{\ell_N} |L^{(j,N)}|} \geq \frac{\sum_{j=1}^{\ell_N} (|L^{(j,N)}| - |P_1|)}{\sum_{j=1}^{\ell_N} |L^{(j,N)}|} \\ &= 1 - \frac{\sum_{j=1}^{\ell_N} |P_1|}{\sum_{j=1}^{\ell_N} |L^{(j,N)}|} \geq 1 - \frac{\ell_N |P_1|}{\ell_N Q_N} = 1 - \frac{|P_1|}{Q_N}.\end{aligned}$$

Since $Q_N \to \infty$ as $N \to \infty$, we have that $\frac{|P_1|}{Q_N} \to 0$ as $N \to \infty$, so $\frac{|\Psi_N^{(1,2)}|}{|\Psi_N^{(1)}|} \to 1$ as $N \to \infty$. Hence, we can pick $N_2 \in N, N_2 > N_1$ such that $\frac{|\Psi_{N_2}^{(1,2)}|}{|\Psi_{N_2}^{(1)}|} > \frac{1}{2}$. Set $\widetilde{\Psi}_2^{(1)} = \Psi_{N_2}^{(1,2)}$.

Now since $\widetilde{\Psi}_2^{(1)}$ is finite, the projection P_2 of $\widetilde{\Psi}_2^{(1)}$ in the first coordinate is also finite. For each $N \in \mathbb{N}$, let

$$\Psi_N^{(1,3)} = \{g \in \Psi_N^{(1)} : t_1(g) \notin P_1 \cup P_2\} = \bigsqcup_{j=1}^{\ell_N} \{g \in L^{(j,N)} : t_1(g) \notin P_1 \cup P_2\}.$$

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Then $\Psi_N^{(1,3)} \subset \Psi_N^{(1)}$ and as before, we have that

$$\frac{|\Psi_N^{(1,3)}|}{|\Psi_N^{(1)}|} \geq 1 - \frac{|P_1| + |P_2|}{Q_N}.$$

Again, since $Q_N \to \infty$ as $N \to \infty$, we have that $\frac{|P_1|+|P_2|}{Q_N} \to 0$ as $N \to \infty$, so $\frac{|\Psi_N^{(1,3)}|}{|\Psi_N^{(1)}|} \to 1$ as $N \to \infty$. Hence, we can pick $N_3 \in N, N_3 > N_2$ such that $\frac{|\Psi_{N_3}^{(1,3)}|}{|\Psi_{N_3}^{(1)}|} > \frac{2}{3}$. Set $\widetilde{\Psi}_3^{(1)} = \Psi_{N_3}^{(1,3)}$.

Continuing inductively, we find a strictly increasing sequence of natural numbers $(N_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, $\widetilde{\Psi}_k^{(1)} \subset \Psi_{N_k}^{(1)} \subset \Psi_{N_k}$ and $\frac{|\widetilde{\Psi}_k^{(1)}|}{|\Psi_{N_k}^{(1)}|} > 1 - \frac{1}{k}$. Then $\frac{|\widetilde{\Psi}_k^{(1)}|}{|\Psi_{N_k}^{(1)}|} \to 1$ as $k \to \infty$, from which one gets that $\widetilde{\Psi}^{(1)} = (\widetilde{\Psi}_k^{(1)})_{k\in\mathbb{N}}$ is a left Følner sequence in G. In addition, it is not difficult to see that the density $d_{\widetilde{\Psi}^{(1)}}(A)$ exists and $d_{\widetilde{\Psi}^{(1)}}(A) = d_{(\Psi_{N_k}^{(1)})_{k\in\mathbb{N}}}(A) = d_{\Psi^{(1)}}(A) = d_\Psi(A)$. In addition, from the construction of $\widetilde{\Psi}^{(1)}$, we have that for any finite set $C \subset \mathbb{Z}$, the set $\{k \in \mathbb{N} : t_1(\widetilde{\Psi}_k^{(1)}) \cap C \neq \emptyset\}$ is finite.

Step 2: Repeat Step 1 with $\widetilde{\Psi}^{(1)}$ in place of Ψ, which we write as a disjoint union of lines in the 2nd coordinate that are maximal within $\widetilde{\Psi}_N^{(1)}$, to obtain a strictly increasing sequence of natural numbers $(N_k)_{k\in\mathbb{N}}$ and a left Følner sequence $\widetilde{\Psi}^{(2)}$ such that for all $k \in \mathbb{N}$, $\widetilde{\Psi}_k^{(2)} \subset \widetilde{\Psi}_{N_k}^{(1)}$, $\frac{|\widetilde{\Psi}_k^{(2)}|}{|\widetilde{\Psi}_{N_k}^{(1)}|} \to 1$ as $k \to \infty$, and such that for any finite set $C \subset \mathbb{Z}$, the set $\{k \in \mathbb{N} : t_2(\widetilde{\Psi}_k^{(2)}) \cap C \neq \emptyset\}$ is finite. Then we will also have that the density $d_{\widetilde{\Psi}^{(2)}}(A)$ exists and $d_{\widetilde{\Psi}^{(2)}}(A) = d_{\widetilde{\Psi}^{(1)}}(A) = d_\Psi(A)$.

Recall that $\widetilde{\Psi}^{(1)}$ has the property that for any finite set $C \subset \mathbb{Z}$, the set $\{N \in \mathbb{N} : t_1(\widetilde{\Psi}_N^{(1)}) \cap C \neq \emptyset\}$ is finite. As $\widetilde{\Psi}_k^{(2)} \subset \widetilde{\Psi}_{N_k}^{(1)}$ for all $k \in \mathbb{N}$, we get that for any finite set $C \subset \mathbb{Z}$, the set $\{k \in \mathbb{N} : t_1(\widetilde{\Psi}_k^{(2)}) \cap C \neq \emptyset\}$ is finite. Hence, after all, for any finite set $C \subset \mathbb{Z}$ and any $i \in \{1, 2\}$, the set $\{k \in \mathbb{N} : t_i(\widetilde{\Psi}_k^{(2)}) \cap C \neq \emptyset\}$ is finite.

Repeating the same procedure, after s steps, we find a left Følner sequence $\widetilde{\Psi}^{(s)}$ such that the density $d_{\widetilde{\Psi}^{(s)}}(A)$ exists, $d_{\widetilde{\Psi}^{(s)}}(A) = d_\Psi(A) > 0$ and for any finite set $C \subset \mathbb{Z}$ and any $i \in \{1, \ldots, s\}$, the set $\{N \in \mathbb{N} : t_i(\widetilde{\Psi}_N^{(s)}) \cap C \neq \emptyset\}$ is finite. Taking $\Psi' := \widetilde{\Psi}^{(s)}$, we see that Ψ' satisfies the claim, thus concluding its proof. □

Since the claim is established, the proof of the corollary is complete. □

Now, we have to show Corollary 1.10, but this is not hard using the fact that finitely generated nilpotent groups are virtually torsion-free. Let us first show the following simple lemma:

Lemma 5.4. *Let G be an amenable group, H be a subgroup of G with $[G : H] = r < \infty$ and Φ be a Følner sequence in G. Then the following hold:*

(i) $d_\Phi(H) = \frac{1}{r}$.
(ii) *Letting $\Psi_N = \Phi_N \cap H$ for each $N \in \mathbb{N}$, the sequence $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ is a Følner sequence in H.*

In the following proof and onwards, we use the symbol $\sqcup$ to denote the disjoint union.

Proof. (i) This is quite easy to check.

(ii) Let $h \in H$ and suppose for sake of contradiction that

$$\ell = \liminf_{N\to\infty} \frac{|h\Psi_N \cap \Psi_N|}{|\Psi_N|} < 1.$$

For any $N \in \mathbb{N}$, we have that

$$h\Phi_N \cap \Phi_N = (h\Psi_N \cap \Psi_N) \sqcup ((h\Phi_N \cap \Phi_N) \setminus H);$$

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hence,

$$\frac{|h\Phi_N \cap \Phi_N|}{|\Phi_N|} = \frac{|h\Psi_N \cap \Psi_N|}{|\Phi_N|} + \frac{|(h\Phi_N \cap \Phi_N) \setminus H|}{|\Phi_N|} \leq \frac{|h\Psi_N \cap \Psi_N|}{|\Psi_N|} \cdot \frac{|\Psi_N|}{|\Phi_N|} + \frac{|\Phi_N \setminus H|}{|\Phi_N|}.$$

Using (i), it follows that

$$\lim_{N\to\infty} \frac{|h\Phi_N \cap \Phi_N|}{|\Phi_N|} \leq \frac{\ell}{r} + 1 - \frac{1}{r} < 1,$$

which contradicts the fact that Φ is a Følner sequence in G. Therefore, Ψ is indeed a Følner sequence in H, and the proof of the lemma is complete. □

Proof of Corollary 1.10. Let G be a finitely generated virtually nilpotent group, $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ be a Følner sequence in G and A be a subset of G with $\overline{d}_\Phi(A) > 0$. By passing to a subsequence for which the limit exists, we may assume that $d_\Phi(A) > 0$. Let G' be a nilpotent finite-index subgroup of G. Since G is finitely generated, by writing it as a disjoint union of finitely many left cosets of G', it is easy to see that G' is also finitely generated, and it is also nilpotent. By [11, Theorem 17.2.2], there exists a normal subgroup H of G' with finite index, which is torsion-free. Hence, H is a torsion-free finitely generated nilpotent group, which has finite index in G. By writing G as finite disjoint union of left cosets of H, we can see that there exists some $g \in G$ such that $\overline{d}_\Phi(g^{-1}A \cap H) = \overline{d}_\Phi(A \cap gH) > 0$. Again, by passing to a subsequence, we may assume that the limit exists, so $d_\Phi(g^{-1}A \cap H) = d_\Phi(A \cap gH) > 0$. We let $\Psi_N = \Phi_N \cap H$ for each $N \in \mathbb{N}$, and by Lemma 5.4 (ii), $\Psi = (\Psi_N)_{N\in\mathbb{N}}$ is a Følner sequence on H. Hence, for every $N \in \mathbb{N}$, we have that

$$\frac{|g^{-1}A \cap \Psi_N|}{|\Psi_N|} = \frac{|g^{-1}A \cap \Phi_N \cap H|}{|\Phi_N|} \cdot \frac{|\Phi_N|}{|\Psi_N|},$$

and thus,

$$d_\Psi(g^{-1}A \cap H) = \frac{d_\Phi(g^{-1}A \cap H)}{d_\Phi(H)} > 0,$$

using Lemma 5.4 (i). Therefore, recalling that H is torsion-free finitely generated nilpotent group, Corollary 1.9 yields an infinite sequence $B \subset g^{-1}A \cap H \subset g^{-1}A$ and some $t_0 \in H$ such that $B \blacktriangleleft B \subset t_0^{-1}g^{-1}A \cap H \subset t^{-1}A$, where we have set $t = gt_0 \in G$. This concludes the proof. □

Having established the corollaries concerning finitely generated nilpotent groups, we move on to showing the results corresponding to abelian groups – namely, Theorem 1.12 and Corollary 1.13. Obviously, the latter is an immediate consequence of the former and of Theorem 1.6, so it suffices to show Theorem 1.12.

Proof of Theorem 1.12. Let $(G, +)$ be an abelian group such that $2G$ has finite index in G, let $r = [G : 2G]$, and consider a collection $\beta_i, 1 \leq i \leq r$ such that $\beta_1 = e_G$ (= the identity element in G) and $G = \bigsqcup_{i=1}^r \beta_i + 2G$. Let s denote the doubling map $s : G \to 2G, g \mapsto 2g$. From the first isomorphism theorem for groups, we know that the map $\widetilde{s} : {}^{G}\!/_{ker(s)} \to 2G, \widetilde{s}(g + ker(s)) = 2g$ is an isomorphism. Since $2G$ has finite index in G and G is infinite, we know that $2G$ is also infinite so the quotient ${}^{G}\!/_{ker(s)}$ is infinite. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in G such that ${}^{G}\!/_{ker(s)} = \{g_n + ker(s) : n \in \mathbb{N}\}$ and such that for any $n \neq m, g_n + ker(s) \neq g_m + ker(s)$.

Consider a Følner sequence $F = (F_N)_{N\in\mathbb{N}}$ in ${}^{G}\!/_{ker(s)}$, and for each N, take $x_{N,1}, \ldots, x_{N,\ell(N)}$ from the sequence $(g_n)_{n\in\mathbb{N}}$ such that $F_N = \{x_{N,1} + ker(s), \ldots, x_{N,\ell(N)} + ker(s)\}$ and such that $x_{N,i} + ker(s) \neq x_{N,j} + ker(s)$ whenever $i \neq j$. For each N, let $\widetilde{\Phi}_N = \widetilde{s}(F_N) = \{2x_{N,1}, \ldots, 2x_{N,\ell(N)}\}$. Since $\widetilde{s}_G$ is an isomorphism, we have that $\widetilde{\Phi} = (\widetilde{\Phi}_N)_{N\in\mathbb{N}}$ is a Følner in $2G$ and $|\widetilde{\Phi}_N| = |F_N| = \ell(N)$.

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For each N, we define $\Phi_N := \bigsqcup_{i=1}^r \beta_i + \widetilde{\Phi}_N$, and we observe that $|\Phi_N| = r|\widetilde{\Phi}_N|$.

Claim 1. *$\Phi = (\Phi_N)_{N\in\mathbb{N}}$ is a Følner in G.*

Proof of Claim 1. Let $y \in G$ and $\varepsilon > 0$. Then there exist $1 \le i_0 \le r$ and $h \in G$ such that $y = \beta_{i_0} + 2h$. Then

$$(y + \Phi_N) \cap \Phi_N = \left(\bigsqcup_{i=1}^r (\beta_{i_0} + \beta_i + 2h + \widetilde{\Phi}_N))\right) \cap \left(\bigsqcup_{j=1}^r (\beta_j + \widetilde{\Phi}_N)\right). \tag{5.6}$$

Since the cosets $\beta_j + 2G$ are disjoint, it follows that for each $1 \le i \le r$, there exists a unique $1 \le j(i) \le r$ such that $\beta_{i_0} + \beta_i \in \beta_{j(i)} + 2G$, so there is $h_i \in G$ such that $\beta_{i_0} + \beta_i = \beta_{j(i)} + 2h_i$. In addition, if we assume that for $i_1 \neq i_2$ we have $j(i_1) = j(i_2)$, then we have that $\beta_{i_1} - \beta_{i_2} = 2h_{i_1} - 2h_{i_2} \in 2G$, so $\beta_{i_1} + 2G = \beta_{i_2} + 2G$, which is a contradiction. Therefore, the map $j : \{1, \dots, r\} \to \{1, \dots, r\}, i \mapsto j(i)$ is a bijection, and then (5.6) becomes

$$\begin{aligned}(y + \Phi_N) \cap \Phi_N &= \bigsqcup_{i=1}^r \left((\beta_{j(i)} + 2h_i + 2h + \widetilde{\Phi}_N)) \cap (\beta_{j(i)} + \widetilde{\Phi}_N)\right) \\ &= \bigsqcup_{i=1}^r (\beta_{j(i)} + ((2h_i + 2h + \widetilde{\Phi}_N) \cap \widetilde{\Phi}_N)),\end{aligned}$$

and thus, we have that

$$|(y + \Phi_N) \cap \Phi_N| = \sum_{i=1}^r |(2h_i + 2h + \widetilde{\Phi}_N) \cap \widetilde{\Phi}_N)|. \tag{5.7}$$

Now since $\widetilde{\Phi}$ is a Følner sequence in $2G$, for N sufficiently large, we have that for every $1 \le i \le r$, $|(2h_i + 2h + \widetilde{\Phi}_N)\triangle\widetilde{\Phi}_N| \le \varepsilon|\widetilde{\Phi}_N|$, and then we have that

$$\frac{|(2h_i + 2h + \widetilde{\Phi}_N) \cap \widetilde{\Phi}_N)|}{|\widetilde{\Phi}_N|} \ge 1 - \frac{|(2h_i + 2h + \widetilde{\Phi}_N)\triangle\widetilde{\Phi}_N|}{|\widetilde{\Phi}_N|} \ge 1 - \varepsilon. \tag{5.8}$$

Then, combining (5.7) and (5.8), we get that for N sufficiently large,

$$\frac{|(y + \Phi_N) \cap \Phi_N|}{|\Phi_N|} = \frac{\sum_{i=1}^r |(2h_i + 2h + \widetilde{\Phi}_N) \cap \widetilde{\Phi}_N)|}{r|\widetilde{\Phi}_N|} \ge \frac{\sum_{i=1}^r (1 - \varepsilon)|\widetilde{\Phi}_N|}{r|\widetilde{\Phi}_N|} = 1 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\lim_{N\to\infty} \frac{|(y+\Phi_N)\cap\Phi_N|}{|\Phi_N|} = 1$. Thus, Φ is a Følner sequence in G, and the proof of the claim is complete. □

Assume that $ker(s)$ is infinite, and let $(h_n)_{n\in\mathbb{N}}$ be an enumeration of $ker(s)$ (the case when $ker(s)$ is finite is easier and can be treated with a similar argument). Let also $(E_k)_{k\in\mathbb{N}}$ be a Følner sequence in $ker(s)$ (in case $ker(s)$ is finite, one can take $E_k = ker(s)$ for all $k \in \mathbb{N}$). Fix $N \in \mathbb{N}$. Then for any n, m with $1 \le n, m \le N$ and any $i, j \in \{1, \dots, \ell(N)\}$, if $g_n + x_{N,i} - x_{N,j} + z_m \in ker(s)$, then

$$\lim_{k\to\infty} \frac{|(g_n + x_{N,i} - x_{N,j} + z_m + E_k)\triangle E_k|}{|E_k|} = 0.$$

We can then pick k_N large enough such that for all n, m with $1 \leq n, m \leq N$ and all $i, j \in \{1, \ldots, \ell(N)\}$, if $g_n + x_{N,i} - x_{N,j} + z_m \in ker(s)$, then

$$\frac{|(g_n + x_{N,i} - x_{N,j} + z_m + H_N)\triangle H_N|}{|H_N|} < \frac{1}{N}, \tag{5.9}$$

where we have set $H_N := E_{k_N}$. For every $N \in \mathbb{N}$, let $\Psi_N = \bigsqcup_{i=1}^{\ell(N)} x_{N,i} + H_N$.

Claim 2. *$\Psi = (\Psi_N)_{N\in\mathbb{N}}$ is a Følner in G.*

Proof of Claim 2. Let $y \in G$, and take $n_y, m_y \in \mathbb{N}$ such that $y = g_{n_y} + z_{m_y}$. Let $\varepsilon > 0$. Since F is a Følner in G, there is $N_0 \in \mathbb{N}$, $N_0 > n_y, m_y$, such that for every $N \geq N_0$,

$$\frac{|(g_{n_y} + ker(s) + F_N)\triangle F_N|}{|F_N|} < \frac{\varepsilon}{2}$$

and $\frac{1}{N} < \frac{\varepsilon}{2}$. Let $N \geq N_0$. For each $i \in \{1, \ldots, \ell(N)\}$, if there is $j \in \{1, \ldots, \ell(N)\}$ such that $g_{n_y} + x_{N,i} \in x_{N,j} + ker(s)$, then this j is unique by the choice of $(x_{N,i})_{i\in\{1,\ldots,\ell(N)\}}$. Let

$$A_N = \{i \in \{1, \ldots, \ell(N)\} : g_{n_y} + x_{N,i} \in x_{N,j} + ker(s) \text{ for some } j \in \{1, \ldots, \ell(N)\}\},$$

and for each $i \in A_N$, denote the corresponding unique j by $j(i)$. Let also

$$B_N := \{i \in \{1, \ldots, \ell(N)\} : g_{n_y} + x_{N,i} \notin x_{N,j} + ker(s) \text{ for any } j \in \{1, \ldots, \ell(N)\}\}$$

and

$$C_N := \{j \in \{1, \ldots, \ell(N)\} : g_{n_y} + x_{N,i} \notin x_{N,j} + ker(s) \text{ for any } i \in \{1, \ldots, \ell(N)\}\},$$

and observe that $\{g_{n_y} + x_{N,i} : i \in B_N\} = \{a \in G : a + ker(s) \in (g_{n_y} + ker(s) + F_N) \setminus F_N\}$, $\{x_{N,j} : j \in C_N\} = \{a \in G : a + ker(s) \in F_N \setminus (g_{n_y} + ker(s) + F_N)\}$, $|B_N| = |(g_{n_y} + ker(s) + F_N) \setminus F_N|$ and $|C_N| = |F_N \setminus (g_{n_y} + ker(s) + F_N)|$. We then obtain that

$$\begin{aligned}(y + \Psi_N)\triangle\Psi_N &= \Big(\bigsqcup_{i=1}^{\ell(N)} g_{n_y} + x_{N,i} + z_{m_y} + H_N\Big)\triangle\Big(\bigsqcup_{i=1}^{\ell(N)} x_{N,i} + H_N\Big)\\ &\subset \Big(\bigcup_{i\in B_N} g_{n_y} + x_{N,i} + z_{m_y} + H_N\Big) \cup \Big(\bigcup_{i\in C_N} x_{N,i} + H_N\Big)\cup\\ &\quad \cup \Big(\bigcup_{i\in A_N} (g_{n_y} + x_{N,i} + z_{m_y} + H_N)\triangle(x_{N,j(i)} + H_N)\Big).\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&|(y + \Psi_N)\triangle\Psi_N| \leq |B_N||H_N| + |C_N||H_N| + \sum_{i\in A_N} |(g_{n_y} + x_{N,i} + z_{m_y} + H_N)\triangle(x_{N,j(i)} + H_N)|\\ &= |(g_{n_y} + ker(s) + F_N)\triangle F_N||H_N| + \sum_{i\in A_N} |(g_{n_y} + x_{N,i} + z_{m_y} + H_N)\triangle(x_{N,j(i)} + H_N)|.\end{aligned}$$

Now for each $i \in A_N$, we have that $g_{n_y} + x_{N,i} \in x_{N,j(i)} + ker(s)$, so also $g_{n_y} + x_{N,i} + z_{m_y} \in x_{N,j(i)} + ker(s)$, and hence, there is $w_y \in ker(s)$ such that $g_{n_y} + x_{N,i} + z_{m_y} = x_{N,j(i)} + w_y$. This implies $|(g_{n_y} + x_{N,i} + z_{m_y} + H_N)\triangle(x_{N,j(i)} + H_N)| = |x_{N,j(i)} + w_y + H_N)\triangle(x_{N,j(i)} + H_N)| = |(w_y + H_N)\triangle H_N|$. Since $w_y = g_{n_y} + x_{N,i} - x_{N,j(i)} + z_{m_y}$, using (5.9) and combining with the fact that $N > n_y, m_y$, we get that $\frac{|(w_y + H_N)\triangle H_N|}{|H_N|} \leq \frac{1}{N}$.

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Recall that $|\Psi_N| = \ell(N)|H_N| = |F_N||H_N|$ and $|A_N| \le \ell(N) = |F_N|$, so after all, we have

$$\frac{|(y+\Psi_N)\triangle\Psi_N|}{|\Psi_N|} \le \frac{|(g_{n_y}+ker(s)+F_N)\triangle F_N||H_N| + \sum_{i\in A_N}\frac{|H_N|}{N}}{|F_N||H_N|} \le \frac{|(g_{n_y}+ker(s)+F_N)\triangle F_N|}{|F_N|} + \frac{1}{N} < \varepsilon.$$

Since $\varepsilon > 0$ and $y \in G$ were arbitrary, this concludes the proof of the claim. □

We will now prove that Φ, Ψ satisfy Definition 1.4 for the map s and therefore G is *square absolutely continuous*. Observe that $2\Psi_N = \{2g : g \in \Psi_N\} = \{2x_{N,i} : i \in \{1,\dots,\ell(N)\}\} = \widetilde{\Phi}_N$, and if $u : G \to [0,1]$, then for each $N \in \mathbb{N}$, we have $\sum_{g\in\Psi_N} u(2g) = \sum_{g\in\bigsqcup_{i=1}^{\ell(N)} x_{N,i}+H_N} u(2g) = \sum_{i=1}^{\ell(N)} |H_N|u(2x_{N,i}) = |H_N|\sum_{g\in\widetilde{\Phi}_N} u(g)$. Therefore, for each $N \in \mathbb{N}$, we have

$$\frac{1}{|\Psi_N|}\sum_{g\in\Psi_N} u(2g) = \frac{1}{|\widetilde{\Phi}_N|}\sum_{g\in\widetilde{\Phi}_N} u(g) = \frac{r}{|\Phi_N|}\sum_{g\in\widetilde{\Phi}_N} u(g) \le \frac{r}{|\Phi_N|}\sum_{g\in\Phi_N} u(g).$$

Therefore, given $\varepsilon > 0$, we can take $\delta = \frac{\varepsilon}{r}$, and then for any $u : G \to [0,1]$, if

$$\limsup_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N} u(g) < \delta = \frac{\varepsilon}{r},$$

then

$$\limsup_{N\to\infty}\frac{1}{|\Psi_N|}\sum_{g\in\Psi_N} u(2g) < \varepsilon.$$

This concludes the proof of Theorem 1.12. □

6. Counterexamples on product sets

To construct the counterexamples, we introduce some convenient notation. We denote by H_3 the 3×3 discrete Heisenberg group – that is, the group of 3×3 upper triangular matrices with 1 in the diagonal and integer entries. Using the obvious Mal'cev coordinate system in this group, we identify H_3 with $\mathbb{Z}^3$ by writing elements of H_3 as $a = (a_1, a_2, a_3)$, where the group operation is given by $ab = (a_1+b_1, a_2+b_2, a_3+b_3+a_1b_2)$. For $N \in \mathbb{N}$, we denote $[N] := [1, N]$, where all intervals are considered in $\mathbb{Z}$, and $[N]' := [1,N] \cap (2\mathbb{Z}+1)$. All the counterexamples below are constructed on the group $G = H_3$. We may assume that any infinite sequence $B \subset H_3$ arising from Corollary 1.9 considered below is infinite an all coordinates.

Example 6.1. We construct a set $A \subset H_3$ and a Følner sequence Φ with $\overline{d}_\Phi(A) > 0$ such that there is no infinite sequence $B = (b(n))_{n\in\mathbb{N}} \subset A$ satisfying $B \blacktriangleleft B \subset At^{-1}$ for some $t \in G$.

Consider the Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ with $\Phi_N = (2^N + [N]) \times [N] \times [N^2]$. It is not hard to see that Φ is a left Følner sequence, but not a right one. Let $\Phi'_N = (2^N + [N]') \times [N]' \times [N^2]'$ and consider the set $A = \bigcup_{N\in\mathbb{N}} \Phi'_N$. Clearly, $\overline{d}_\Phi(A) > 0$. Suppose, for sake of contradiction, that there exist an infinite sequence $B = (b(n))_{n\in\mathbb{N}} \subset A$ and some $t \in H_3$ such that $\{b(i)b(j) : i < j\} \subset At^{-1}$. We denote $t^{-1} = (t_1, t_2, t_3)$. We observe that $B \cap \Phi'_N \neq \emptyset$ for infinitely many $N \in \mathbb{N}$. Let $b = b(i) = (b_1, b_2, b_3)$, for some $i \in \mathbb{N}$, such that $b \in \Phi'_N$ for some N large (compared to the t_i's). Then we can find some $j > i$ such that the element $c = b(j) = (c_1, c_2, c_3)$ belongs in some Φ'_M for some M much larger than N, and

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then $bc \in At^{-1}$. It follows that $bc \in \Phi'_Q t^{-1}$, for some $Q \in \mathbb{N}$, where

$$\Phi'_Q t^{-1} = (2^Q + [Q]' + t_1) \times ([Q]' + t_2) \times ([Q^2]' + t_3 + t_2(2^Q + [Q]')).$$

Then $2^Q - Q + t_1 \leq (bc)_1 \leq 2^Q + Q + t_1$, and on the other hand, $(bc)_1 = b_1 + c_1 \in 2^N + 2^M + [N+M]'$, so $2^N + 2^M - N - M \leq (bc)_1 \leq 2^N + 2^M + N + M$. Combining those with the fact that M is assumed to be sufficiently large (and much larger than N), we obtain that $Q = M$. We now want to show that $t_2 \neq 0$. By the fact that $bc \in \Phi'_Q t^{-1}$, we have that

$$bc \in (2\mathbb{Z} + 1 + t_1) \times (2\mathbb{Z} + 1 + t_2) \times (2\mathbb{Z} + 1 + t_3 + t_2).$$

However, multiplying b and c, and using that b_i and c_i are odd for all i, gives that

$$bc = (b_1 + c_1, b_2 + c_2, b_3 + c_3 + b_1 c_2) \in 2\mathbb{Z} \times 2\mathbb{Z} \times (2\mathbb{Z} + 1).$$

It follows that all the t_i's are odd, and in particular, $t_2 \neq 0$.

Now, since $bc \in \Phi'_Q t^{-1}$ and $t_2 \neq 0$, we have that

$$(bc)_3 \gg 2^Q = 2^M.$$

However, for M sufficiently large, we have that

$$(bc)_3 = b_3 + c_3 + b_1 c_2 \leq N^2 + M^2 + (2^N + N)M \ll M^2,$$

which yields a contradiction.

Example 6.2. We construct a set $A \subset H_3$ and a Følner sequence Φ with $\overline{d}_\Phi(A) = 1$ such that there is no infinite sequence $B = (b(n))_{n\in\mathbb{N}} \subset G$ satisfying $B \blacktriangleright B \subset t^{-1}A$ for some $t \in G$.

Consider the same Følner sequence Φ as in Example 6.1. We observe that $\Phi_N \cap \Phi_M = \emptyset$ for any $N \neq M$, and in particular, the projections of any two such sets in the first coordinate are disjoint subsets of $\mathbb{Z}$. We define the set $A = \bigcup_{N\in\mathbb{N}} \Phi_N$, and clearly, we have $\overline{d}_\Phi(A) = 1$. Suppose, for sake of contradiction, that there exist an infinite sequence $B = (b(n))_{n\in\mathbb{N}} \subset H_3$ and some $t \in H_3$ such that $\{b(i)b(j) \colon i > j\} \subset t^{-1}A$. We denote $t = (t_1, t_2, t_3)$ and $b(1) = b = (b_1, b_2, b_3)$. We may assume without loss of generality that $b_2 \neq 0$. We let $B' = (b(n))_{n\geq 2}$. Moreover, we denote $b^{-1} = y = (y_1, y_2, y_3)$, and then we have that $B' \subset t^{-1}Ay$. It follows that $B' \cap t^{-1}\Phi_N y \neq \emptyset$ for infinitely many $N \in \mathbb{N}$. Fix $c = b(i)$ for some $i > 1$ such that $c \in B' \cap t^{-1}\Phi_M y$ for some large $M \in \mathbb{N}$. Then $cb \in t^{-1}A$, which implies that cb belongs in exactly one set of the form $t^{-1}\Phi_N$. We have that

$$(cb)_1 = c_1 + b_1 \in 2^M + [M] + t_1 + y_1 + b_1 = 2^M + [M] + t_1,$$

and hence, $cb \in t^{-1}\Phi_M$. Then we have that $(cb)_3 \in [M^2] + t_1[M] + t_3$, which imples that $(cb)_3 \ll M^2$, for M sufficiently large. However, multiplying c and b gives that

$$(cb)_3 = c_3 + b_3 + c_1 b_2 \gg c_1 \in 2^M + [M] + t_1 + y_1,$$

which implies that $(cb)_3 \gg 2^M = 2^M$, for M sufficiently large, where the implied constant is again absolute. This yields a contradiction.

Example 6.3. Consider the same Φ and $A \subset H_3$ as in Example 6.2. Then we show that there is no infinite sequence $B \subset H_3$ satisfying $B \blacktriangleright B \subset At^{-1}$.

To see why, suppose, for sake of contradiction, that there exists such a sequence B and, as we did in Remark 1.14, consider the sequence $B' = t^{-1}Bt$. Then we have that $B' \blacktriangleright B' \subset t^{-1}A$, which cannot hold for this particular set A as we saw in Example 6.2. This yields a contradiction.

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A. A result of Host and Kra for amenable groups

The purpose of this appendix is to prove Lemma 3.5. The proof follows the ideas in the proof of [9, Proof of Proposition 6.1], adapted in our setting. We state the following classical result (see, for example, [19, Example 11.13 (a)]), which we will need in the proof of Lemma 3.5:

Lemma A.1. *Let X be a compact metric space. Then the only linear multiplicative functionals on the algebra $C(X)$ are the point evaluations, (i.e., $\mathrm{ev}_x(f) = f(x)$, for $x \in X$).*

For convenience, we restate the lemma we want to prove:

Lemma 3.5. [9, Proposition 6.1 for group actions] *Let G be an amenable group, let (X, μ, T) be an ergodic G-system, and (Z, m, R) be its Kronecker factor and $\rho : (X, \mu, T) \to (Z, m, R)$ be a factor map. If $a \in X$ is a transitive point, then there exists a point $z \in Z$ and a Følner sequence Ψ such that*

$$\lim_{N\to\infty} \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} f_1(T_g a) \cdot f_2(R_g z) = \int_X f_1 \cdot (f_2 \circ \rho)\, \mathrm{d}\mu \tag{A.1}$$

holds for any $f_1 \in C(X)$ and $f_2 \in C(Z)$.

We remark that the result still holds if we replace (Z, m, R) by any factor of (X, μ, T) that is distal as a topological system.

Proof. As in [9], we split the proof into two parts.

Construction of a common extension. Let $\mathcal{A} \subset \{f : X \to \mathbb{C} \colon f \text{ is measurable and bounded}\}$ be the closed (in norm) subalgebra that is spanned by $C(X)$ and $\{f \circ \rho : f \in C(Z)\}$. This is a unital commutative separable algebra, which contains the constants and is invariant under both complex conjugation and T. Consider the Gelfand spectrum of $\mathcal{A}$, which is defined as

$$W = \{\chi : \mathcal{A} \to \mathbb{C} \colon \chi \text{ is linear and multiplicative}\}.$$

Note that W is compact and metrizable since $\mathcal{A}$ is separable. By Gelfand's theorem, there exists an isometric isomorphism $F : C(W) \to \mathcal{A}$, satisfying $F^{-1}(f)(\chi) = \chi(f)$ for all $f \in \mathcal{A}$ and all $\chi \in W$. Hence, for all $\widetilde{f} \in C(W)$ and all $\chi \in W$, we have $\widetilde{f}(\chi) = \chi(F(\widetilde{f}))$. For $g \in G$ and $\chi \in W$, we define $S_g(\chi) : A \to \mathbb{C}$, $S_g(\chi)(f) = \chi(f \circ T_g)$. Then it is not too difficult to see that for each $g \in G$, $S_g : W \to W$ is a homeomorphism, and we also let $S = (S_g)_{g\in G}$. Then for every $\chi \in W$, we have that

$$\chi(F(\widetilde{f} \circ S_g)) = \widetilde{f}(S_g(\chi)) = S_g(\chi)(F(\widetilde{f})) = \chi(F(\widetilde{f}) \circ T_g)$$

for $\widetilde{f} \in C(W)$ and any $g \in G$. In particular, for every $x \in X$, by considering the evaluation functional $\mathrm{ev}_x \in W$, it follows that

$$F(\widetilde{f} \circ S_g)(x) = \mathrm{ev}_x(F(\widetilde{f} \circ S_g)) = \mathrm{ev}_x(F(\widetilde{f}) \circ T_g) = (F(\widetilde{f}) \circ T_g)(x)$$

holds for any $\widetilde{f} \in C(W)$ and any $g \in G$. Thus, we have that

$$F(\widetilde{f} \circ S_g) = F(\widetilde{f}) \circ T_g \tag{A.2}$$

for every $\widetilde{f} \in C(W)$ and for every $g \in G$.

Now, we consider the embedding $F^{-1}|_{C(X)} : C(X) \hookrightarrow C(W)$. Given $w \in W$, $\mathrm{ev}_w \circ F^{-1}|_{C(X)}$ is a linear multiplicative functional on $C(X)$, and by Lemma A.1, there exists a unique $x \in X$ such that $\mathrm{ev}_w \circ F^{-1}|_{C(X)} = \mathrm{ev}_x$. Thus, we can define $\pi_X : W \to X$ by $\pi_X(w) = x$ if and only if $\mathrm{ev}_w \circ F^{-1}|_{C(X)} = \mathrm{ev}_x$. The last equation is equivalent to that for any $f \in C(X)$ and $w \in W$,

$$f \circ \pi_X(w) = \mathrm{ev}_{\pi_X(w)}(f) = \mathrm{ev}_w \circ F^{-1}|_{C(X)}(f) = F^{-1}(f)(w).$$

Hence, π_X is the unique map from W to X satisfying

$$f \circ \pi_X = F^{-1}(f) \tag{A.3}$$

for any $f \in C(X)$. We claim that π_X is continuous and surjective.

To show continuity, we let $(w_n)_{n\in\mathbb{N}}$ in W such that $w_n \to w \in W$. Then for any $f \in C(W)$, we have $\mathrm{ev}_{w_n}(f) \to \mathrm{ev}_w(f)$. Hence, for any $f \in C(X)$, we have $\mathrm{ev}_{\pi_X(w_n)}(f) \to \mathrm{ev}_{\pi_X(w)}(f)$; that is, $f(\pi_X(w_n)) \to f(\pi_X(w))$. Since X is compact, $\pi_X(w_n)$ has a convergent subsequence, which by abuse of notation we denote by $\pi_X(w_n)$. Suppose for sake of contradiction that $\pi_X(w_n) \to y \neq \pi_X(w)$. Then by Urysohn's lemma, we can find some $f \in C(X)$ and some disjoint open neighborhoods $U_1 \ni \pi_X(w), U_2 \ni y$, such that $f|_{U_1} = 1$ and $f|_{U_2} = 0$. It follows that $\pi_X(w_n) \in U_2$ for large n, hence $f(\pi_X(w_n)) = 0$ for large n, while $f(\pi_X(w)) = 1$, but this contradicts the fact that $f(\pi_X(w_n)) \to f(\pi_X(w))$. This shows that every convergent subsequence of $\pi_X(w_n)$ converges to $\pi_X(w)$, and since X is compact, it follows that $\pi_X(w_n) \to \pi_X(w)$, showing the continuity of π_X.

To show that π_X is surjective, let $x \in X$ and consider the linear multiplicative functional ev_x on $C(X)$. Since $F^{-1}|_{C(X)}$ is an embedding, it follows that $F^{-1}|_{C(X)} \circ \mathrm{ev}_x$ is a linear multiplicative functional on $C(W)$. Then, by Lemma A.1, there exists some $w \in W$ such that $F^{-1}|_{C(X)} \circ \mathrm{ev}_x = \mathrm{ev}_w$. Hence, $\pi_X(w) = x$, showing that π_X is surjective.

Moreover, for any $g \in G$ and any $f \in C(X)$, let $\widetilde{f} = F^{-1}(f) \in C(W)$, and then $F(\widetilde{f}) = f \in C(X)$, and by (A.2), $F(\widetilde{f} \circ S_g) = f \circ T_g \in C(X)$. Then, using (A.2) and (A.3), we have that

$$f \circ T_g \circ \pi_X = F(\widetilde{f} \circ S_g) \circ \pi_X = \widetilde{f} \circ S_g = F(\widetilde{f}) \circ \pi_X \circ S_g = f \circ \pi_X \circ S_g$$

for any $g \in G$ and any $f \in C(X)$. It follows by Urysohn's lemma that

$$T_g \circ \pi_X = \pi_X \circ S_g \tag{A.4}$$

for any $g \in G$. Therefore, we have proved that W is an extension of X with π_X being a continuous topological factor map.

Similarly, by considering the embedding $F^{-1}|_{C(Z)\circ\rho} : C(Z) \circ \rho \hookrightarrow C(W)$, there exists a unique surjective continuous map $\pi_Z : W \to Z$ such that

$$f \circ \pi_Z = F^{-1}(f \circ \rho) \tag{A.5}$$

for any $f \in C(Z)$, and

$$R_g \circ \pi_Z = \pi_Z \circ S_g \tag{A.6}$$

for any $g \in G$. Hence, W is also an extension of Z with π_Z being a continuous topological factor map.

Now we will find a measure on W, with which W will become a measurable extension of X and Z. Since $f \mapsto \int f \,\mathrm{d}\mu$ is a positive linear functional on $\mathcal{A}$, there exists a unique probability measure ν on W such that

$$\int_X f \,\mathrm{d}\mu = \int_W F^{-1}(f) \,\mathrm{d}\nu$$

for any $f \in \mathcal{A}$. By (A.2), we have that ν is S-invariant; by (A.3), we have $\pi_X \nu = \mu$; and by (A.5), we have $\pi_Z \nu = m$. Consequently, π_X and π_Z are factor maps.

The last thing in this first step is to show that π_X is actually a measurable isomorphism between W and X and thus that the measure ν is ergodic. First, we want to extend (A.3) in $C(W) \simeq \mathcal{A}$. For $f \in \mathcal{A}$, it holds $\int_W |F^{-1}(f)|^2 \,\mathrm{d}\nu = \int_X |f|^2 \,\mathrm{d}\mu$, and F^{-1} is an isometry from $\mathcal{A}$ (with the $L^2(X,\mu)$ norm) into $L^2(W,\nu)$. Combining the facts that $C(X)$ is dense in $\mathcal{A}$ (with respect to the $L^2(X,\mu)$ norm) and that (A.3) holds for all $f \in C(X)$, we obtain that (A.3) holds for all $f \in \mathcal{A}$, ν-almost always. Then consider

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the map $H : L^2(X,\mu) \to L^2(W,\nu)$, such that $f \mapsto f \circ \pi_X$. Then $H(L^2(\mu))$ is closed in $L^2(W,\nu)$ since the map is an isometry, and notice that it contains $F^{-1}(\mathcal{A}) = C(W)$. Thus, $H(L^2(X,\mu)) = L^2(W,\nu)$, showing that π_X is a measurable isomorphism and, consequently, that (W,ν,S) is ergodic. Finally, for any $f \in C(Z)$, using (A.3) and (A.5), we see that $f \circ \pi_Z = F^{-1}(f \circ \rho) = f \circ \rho \circ \pi_X$ holds ν-almost always, and so, $\pi_Z = \rho \circ \pi_X$.

Construction of the Følner sequence. Since (W,ν,S) is ergodic, it follows that there exists $w_1 \in \mathsf{gen}(\nu,\Phi)$ for some Følner sequence Φ.

Set $x_1 = \pi_X(w_1)$. Transitivity of a implies that there exists a sequence $(h_N)_{N\in\mathbb{N}} \subset G$ such that

$$\lim_{N\to\infty} \sup_{g\in\Phi_N} d_X(T_g x_1, T_{gh_N} a) = 0, \tag{A.7}$$

where d_X is the metric on the space X. Now set $z_1 = \pi_Z(w_1)$. Let $E(Z,R)$ be the Ellis semigroup of (Z,R), that is, the closure of R as an element of Z^Z, where this space is equipped with the pointwise convergence topology. Let $R_0 \in \overline{(R_{h_N})}_{N\in\mathbb{N}} \subset E(Z,R)$. By Proposition 2.12, R is a rotation, which implies that is a bijection from Z to itself. In case that (Z,m,R) is any distal system (and not necessarily the Kronecker factor), then we also have that R is a bijection (see by [3, Chapter 5]). Therefore, there exists $z_0 \in Z$ such that $R_0(z_0) = z_1$. Then there exists a subsequence of $(h_N)_{N\in\mathbb{N}}$, which, by abuse of notation, we denote as $(h_N)_{N\in\mathbb{N}}$, such that $\lim_{N\to\infty} R_{h_N} z_0 = z_1$. Therefore, there exists a further subsequence, which once again we denote in the same way, for which it holds that

$$\lim_{N\to\infty} \sup_{g\in\Phi_N} d_Z(R_g z_1, R_{gh_N} z_0) = 0, \tag{A.8}$$

where d_Z is the metric on the space Z.

Let $f_1 \in C(X)$, $f_2 \in C(Z)$. By (A.7) and (A.8), we have that

$$\lim_{N\to\infty} \sup_{g\in\Phi_N} |f_1(T_g x_1) - f_1(T_{gh_N} a)| = 0 \quad \text{and} \quad \lim_{N\to\infty} \sup_{g\in\Phi_N} |f_2(R_g z_1) - f_2(R_{gh_N} z_0)| = 0.$$

We define the Følner sequence $\Psi = (\Psi_N)_{N\in\mathbb{N}}$, by $\Psi_N = \Phi_N h_N$ for any $N \in \mathbb{N}$. It is easy to check that since Φ is a left Følner sequence, then Ψ is also a left Følner sequence. It follows from the above equations that

$$\begin{aligned}
&\lim_{N\to\infty} \left| \frac{1}{|\Psi_N|} \sum_{g\in\Psi_N} f_1(T_g a) f_2(R_g z_0) - \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} f_1(T_g x_1) f_2(R_g z_1) \right| \\
&= \lim_{N\to\infty} \left| \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} \big(f_1(T_{gh_N} a) f_2(R_{gh_N} z_0) - f_1(T_g x_1) f_2(R_g z_1)\big) \right| \\
&\le \lim_{N\to\infty} \sup_{g\in\Phi_N} |f_1(T_{gh_N} a) f_2(R_{gh_N} z_0) - f_1(T_g x_1) f_2(R_g z_1)| \\
&\le \lim_{N\to\infty} \sup_{g\in\Phi_N} \big(|f_1(T_{gh_N} a) f_2(R_{gh_N} z_0) - f_1(T_{gh_N} a) f_2(R_g z_1)| \\
&\qquad\qquad + |f_1(T_{gh_N} a) f_2(R_g z_1) - f_1(T_g x_1) f_2(R_g z_1)|\big) \\
&\le \lim_{N\to\infty} \big(\|f_1\|_\infty \sup_{g\in\Phi_N} |f_2(R_{gh_N} z_0) - f_2(R_g z_1)| + \|f_2\|_\infty \sup_{g\in\Phi_N} |f_1(T_{gh_N} a) - f_1(T_g x_1)|\big) \\
&= 0.
\end{aligned} \tag{A.9}$$

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Moreover, recalling that $w_1 \in \mathtt{gen}(\nu, \Phi)$ and observing that $f_1 \circ \pi_X \in C(W)$ and $f_2 \circ \pi_Z \in C(W)$, we have that

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} f_1(T_g x_1) f_2(R_g z_1) &= \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} f_1(T_g(\pi_X(w_1))) f_2(R_g(\pi_Z(w_1))) \\ &= \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} f_1(\pi_X(S_g w_1)) f_2(\pi_Z(S_g w_1)) \quad \text{(by (A.4),(A.6))} \\ &= \int_W (f_1 \circ \pi_X)(f_2 \circ \pi_Z)\, \mathrm{d}\nu \\ &= \int_W (f_1 \circ \pi_X)(f_2 \circ \rho \circ \pi_X)\, \mathrm{d}\nu \qquad \text{(since } \pi_Z = \rho \circ \pi_X) \\ &= \int_W f_1 \cdot (f_2 \circ \rho)\, \mathrm{d}\mu \qquad \text{(since } \pi_X \nu = \mu). \end{aligned} \tag{A.10}$$

Combining (A.9) and (A.10) yields the desired result. The proof is complete. □

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