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PINNING SYNCHRONIZATION OF FRACTIONAL-ORDER COMPLEX NETWORKS BY A SINGLE CONTROLLER

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Abstract

We investigate the state feedback pinning synchronization of fractional-order complex networks. Based on the stability theory of fractional-order differential systems and state feedback control by a single controller, synchronization conditions for fractional-order complex networks are given. We assume that the coupling matrix is irreducible, and provide a numerical example to illustrate the validity of the proposed conclusions.

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1. Introduction

Recently, human society has entered the Internet age, and people live in a world filled with all kinds of complex networks due to the development of information technology. The synchronization phenomenon occurs in complex communication networks [16]. Network synchronization plays a very important role in the fields of signal generators, nuclear magnetic resonance spectrometers, superconducting materials, granule crumblers, laser gears, communication systems and so on. In the past few decades, integer-order complex network synchronization has been widely studied and many control methods have been employed to deal with complex network synchronization. However, there is not much work on fractional-order complex network synchronization, although it has strong practicability. There exist many fractional-order systems in the real world, such as the fractional-order Lorenz system [6], fractional-order Chua system [7] and fractional-order Chen system [11]. Synchronization of fractional-order chaos systems is becoming a challenging and interesting research topic due to its potential application in secure communication [2, 6, 22]. Many control schemes have been employed to achieve chaotic synchronization,

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such as adaptive control [1], the time delay feedback approach [5, 17], nonlinear feedback control [4] and active control [8, 19].

State feedback control is one of the characteristics of modern control theory. After recent decades of development, state feedback control is being more widely used in real control systems. In recent years, the state feedback control method has also been used in synchronization of fractional-order complex networks. Tang et al. [20] investigated pinning synchronization of fractional-order complex networks based on state feedback and established a local synchronization criterion by using the stability theory of fractional-order systems and a numerical algorithm. Wang et al. studied the synchronization of complex dynamical networks with fractional-order chaotic nodes, and proposed some sufficient synchronization criteria based on the Lyapunov stability theory and the LaSalle invariance principle [21].

When the network node number or system dimension is very large, state feedback may lead to a massive increase in the amount of calculation due to the involvement of a large number of system states. Chen et al. [3] investigated pinning synchronization of integer-order complex networks and proved that complex networks can be synchronized when only one node is controlled using state feedback. In this paper we study pinning synchronization of fractional-order complex networks using the idea of Chen et al. [3]. We establish synchronization conditions for fractional-order complex networks, and investigate the range of feedback gain.

The rest of the paper is organized as follows. In Section 2 the network model is described and some definitions and lemmas are given. Asymptotical synchronization conditions for fractional-order complex networks are discussed in Section 3. Section 4 shows the validity of the proposed synchronization schemes through numerical simulations. We conclude with some remarks in Section 5.

2. Model description and preliminaries

The fractional-order integro-differential operator is a generalized concept of an integer-order integro-differential operator. The commonly used definitions of fractional-order derivatives are the Grunwald–Letnikov, Riemann–Liouville and Caputo definitions [18]. We employ the Caputo fractional derivative operator in this paper, since it has well-understood physical meanings.

The Caputo fractional derivative is defined as follows:

$$ {}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{(n-\alpha-1)} f^{(n)}(\tau)\, d\tau, \quad n-1 < \alpha < n, $$

where $n \in Z_+$ and $\Gamma(\cdot)$ is the gamma function, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$. In the following, we denote ${}_{0}^{C}D_t^{\alpha}$ as D_t^{α}.

Consider a generic controlled complex network consisting of N coupled identical nodes, with each node being an n-dimensional fractional-order dynamical system, which is in the form:

$$ D_t^{\alpha} x_i(t) = f(x_i(t), t) + c \sum_{j=1}^{N} b_{ij} \Gamma x_j(t) + u_i(t), x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, N, \tag{2.1} $$

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where $0 < \alpha < 2$, $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in R^n$ is the state variable of the ith node, x_{i0} is an n-dimensional constant vector, $i = 1, 2, \ldots, N$, and $f : R^n \times R \to R^n$ is a smooth time-varying nonlinear function. The scalar $c > 0$ is the coupling strength. The coupling matrix $B = (b_{ij}) \in R^{N\times N}$ is determined by the topological structure of the network, where $b_{ij} \geq 0$ denotes the coupling coefficient from node j to node i, for $i, j = 1, \ldots, N, i \neq j$, and $b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij}, i = 1, 2, \ldots, N$. The matrix $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ denotes the inner connection at each node with $\gamma_i \geq 0$, $i = 1, 2, \ldots, n$, which means that two nodes are connected by their ith component, if $\gamma_i > 0$. The matrix $u(t) = [u_1(t), u_2(t), \ldots, u_N(t)] \in R^{n\times N}$ is the controller to be designed.

In this paper we make the following assumptions for the function $f(x, t)$.

Assumption 2.1. There exist a bounded function $\omega(x) \in R^{n\times n}$ and a diagonal matrix Σ =diag$(\sigma_1, \sigma_2, \ldots, \sigma_n)$ such that

$$f(x, t) - f(y, t) = \omega(x - y)(x - y), \tag{2.2}$$

$$(x - y)^T (f(x, t) - f(y, t)) \leq (x - y)^T \Sigma (x - y) \tag{2.3}$$

hold for all $x, y \in R^n$ and $t > 0$.

Remark. The most typical fractional-order chaotic systems, such as fractional-order Lorenz systems, Chen systems, Rössler systems [10], the Lü system [12], and so on, satisfy (2.2) and (2.3).

Definition 2.2. Assume $x_i(t, t_0, X_0)$, for $i = 1, 2, \ldots, N$, to be a solution of the controlled network (2.1), where $X_0 = (x_{10}, x_{20}, \ldots, x_{N0}) \in R^{n\times N}$, $f : \Omega \times [0, +\infty) \to R^n$ and $u_i : \Omega \times \cdots \times \Omega \to R^n$, $1 \leq i \leq N$, are continuous with $\Omega \subseteq R^n$. If there exist a controller $u(t)$ and a nonempty subset $\Theta \subseteq \Omega$ with $x_{i0} \in \Theta (i = 1, 2, \ldots, N)$, such that $x_i(t, t_0, X_0) \in \Omega$ for all $t > t_0, i = 1, 2, \ldots, N$, and

$$\lim_{t\to\infty} \|x_i(t, t_0, X_0) - x_j(t, t_0, X_0)\| = 0, \quad i, j = 1, 2, \ldots, N,$$

then the controlled network (2.1) is said to achieve asymptotical network synchronization and $\Theta \times \cdots \times \Theta$ is called the region of synchronization for the dynamical network (2.1). If $\Theta = R^n$, then the network (2.1) is said to achieve globally asymptotic network synchronization .

The following lemma characterizes the right and left eigenvectors of the matrix B corresponding to eigenvalue 0, which play key roles in the investigation of the synchronization of the coupled system (2.1).

Lemma 2.3 [14]. *If B is irreducible, then the following statements are valid.*

(i) *The matrix B has an eigenvalue* 0 *with multiplicity* 1, *and has right eigenvector* $[1, 1, \ldots, 1]^T$.
(ii) *If λ is an eigenvalue of B and $\lambda \neq 0$, then* $\mathrm{Re}(\lambda) < 0$.

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(iii) *Suppose* $\xi = [\xi_1, \xi_2, \ldots, \xi_N]^T \in R^N$*(without loss of generality, assume that* $\sum_{i=1}^N \xi_i = 1$*) is the left eigenvector of* B *corresponding to eigenvalue* 0*. Then* $\xi_i > 0$ *holds for all* $i = 1, 2, \ldots, N$.

In the following discussion, we always assume that B is irreducible.

Let $\hat{x}(t) = \sum_{i=1}^N \xi_i x_i(t)$ and $e_i(t) = x_i(t) - \hat{x}(t)$, $i = 1, 2, \ldots, N$. The reference state $\hat{x}(t)$ is introduced by Lu et al. [13], and it is a key point in discussing synchronization of complex networks.

The objective of this paper is to find a state feedback pinning controller, such that the network (2.1) achieves asymptotical synchronization, and the form of the controller is

$$\begin{cases} u_1(t) = -c\varepsilon\Gamma e_1(t), \\ u_i = 0, \quad 2 \le i \le N, \end{cases} \tag{2.4}$$

where $\varepsilon > 0$ is the feedback gain. In order to obtain our main results, the following theorem is presented.

THEOREM 2.4 [9]. *Consider the autonomous system*

$$D_t^\alpha x(t) = A(x)x, \tag{2.5}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in R^n$ *is the state variable with* $0 < \alpha < 1$*. If there exists a real matrix* $P > 0$*, such that*

$$H = x^T(t) P D_t^\alpha x(t) \le 0$$

holds for all $x \in R^n$*, then system* (2.5) *is asymptotically stable.*

3. Pinning synchronization of complex networks

In this section the problem of synchronization of fractional-order complex dynamical networks (2.1) based on state feedback pinning control is investigated. We present some criteria and discuss the range of feedback gain.

Under the control law (2.4), the complex network (2.1) can be written as

$$\begin{cases} D_t^\alpha x_1(t) = f(x_1, t) + c\sum_{j=1}^N b_{ij}\Gamma x_j(t) - c\varepsilon\Gamma e_1(t), \\ D_t^\alpha x_i(t) = f(x_i, t) + c\sum_{j=1}^N b_{ij}\Gamma x_j(t), \quad i = 2, \ldots, N. \end{cases} \tag{3.1}$$

In the following, we present the synchronization conditions.

THEOREM 3.1. *If* B *is a symmetrical matrix, Assumption* 2.1 *holds and*

$$\breve{B}_j = T^T(c\gamma_j(\Xi B)^s - c\gamma_j\Xi\varepsilon_1 + \sigma_j\Xi)T \le 0, \quad j = 1, 2, \ldots, n,$$

where

$$T = [\xi_N I_{N-1}, -\bar{\xi}]^T, \quad \Xi = \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_N),$$
$$\bar{\xi} = [\xi_1, \xi_2, \ldots, \xi_{N-1}]^T, \quad \textit{and} \quad \varepsilon_1 = \mathrm{diag}(\varepsilon, 0, \ldots, 0),$$

then the controlled complex network (2.1) *can achieve asymptotical synchronization under control law* (2.4).

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PROOF. Since $e_i(t) = x_i(t) - \hat{x}(t)$, $i = 1, 2, \ldots, N$, complex network (3.1) can be changed to

$$\begin{cases} D_t^\alpha e_1(t) = f(x_1, t) - \sum_{k=1}^{N} \xi_k f(x_k, t) + c \sum_{j=1}^{N} b_{1j}\Gamma e_j(t) - c\varepsilon(1 - \xi_1)\Gamma e_1(t), \\ D_t^\alpha e_i(t) = f(x_i, t) - \sum_{k=1}^{N} \xi_k f(x_k, t) + c \sum_{j=1}^{N} b_{ij}\Gamma e_j(t) + c\varepsilon\xi_1\Gamma e_1(t),\ i = 2, \ldots, N. \end{cases} \tag{3.2}$$

Denote $\bar{e} = [e_1^T, e_2^T, \ldots, e_N^T]^T$, $P = \Xi \otimes I_n$, and $F(t) = f(\hat{x}, t) - \sum_{k=1}^{N} \xi_k f(x_k, t)$. Then

$$\begin{aligned} H &= \bar{e}^T(t) P D_t^\alpha \bar{e}(t) \\ &= \sum_{i=1}^{N} \xi_i e_i^T D_t^\alpha e_i \\ &= \sum_{i=1}^{N} \xi_i e_i^T \Big(f(x_i, t) - f(\hat{x}, t) + c \sum_{j=1}^{N} b_{ij}\Gamma e_j + F(t) + c\varepsilon\xi_1\Gamma e_1\Big) - c\varepsilon\xi_1 e_1^T \Gamma e_1 \\ &\le \sum_{i=1}^{N} \xi_i e_i^T \Sigma e_i + c \sum_{i=1}^{N} \sum_{j=1}^{N} \xi_i e_i^T b_{ij}\Gamma e_j - c\varepsilon\xi_1 e_1^T \Gamma e_1. \end{aligned}$$

Let

$$\begin{aligned} e_i &= [e_{i1}, e_{i2}, \ldots, e_{in}]^T, \quad i = 1, 2, \ldots, N, \\ \tilde{e}_j &= [e_{1j}, e_{2j}, \ldots, e_{Nj}]^T, \quad j = 1, 2, \ldots, n. \end{aligned}$$

Then $\xi^T \tilde{e}_j = 0$, and $\tilde{e}_j \in L = \{z \in R^n | \xi^T z = 0\}$, for $j = 1, 2, \ldots, n$.

Since column vectors of matrix T form a basis of the subspace L, there exists $y_j \in R^{N-1}$, such that $\tilde{e}_j = T y_j$, $j = 1, 2, \ldots, n$. Then

$$\begin{aligned} H &\le \sum_{j=1}^{n} \tilde{e}_j^T \Xi(c\gamma_j(B - \varepsilon_1) + \sigma_j I)\tilde{e}_j \\ &= \sum_{j=1}^{n} y_j^T T^T (c\gamma_j(\Xi B)^s - c\gamma_j \Xi\varepsilon_1 + \sigma_j \Xi) T y_j \\ &\le 0. \end{aligned}$$

From Theorem 2.4, system (3.2) is asymptotically stable, that is, the controlled complex network (2.1) achieves asymptotical synchronization, and this completes the proof. □

Denote

$$\breve{B}_j = \begin{bmatrix} \breve{B}_{j11} & \breve{B}_{j12} \\ \breve{B}_{j12}^T & \breve{B}_{j22} \end{bmatrix},$$

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where $\breve{B}_{j11} \in R$, $\breve{B}_{j12} \in R^{1\times(N-2)}$, $\breve{B}_{j22} \in R^{(N-2)\times(N-2)}$. Then we have

$$\begin{aligned}
\breve{B}_{j11} &= \xi_1\xi_N(c\gamma_j(b_{11}\xi_N + b_{NN}\xi_1 - b_{1N}\xi_1 - b_{N1}\xi_N - \varepsilon\xi_N) + \sigma_j(\xi_1 + \xi_N)),\\
\breve{B}_{j12} &= (\tfrac{1}{2}c\gamma_j(\xi_1\xi_N\hat{B}_{12} + \xi_N\hat{B}_{21}^T\tilde{\Xi}_1 + T_{21}^T\tilde{\Xi}_1\hat{B}_{22} + T_{21}^T\hat{B}_{22}^T\tilde{\Xi}_1) + \sigma_j T_{21}^T\tilde{\Xi}_1)T_{22},\\
\breve{B}_{j22} &= T_{22}^T(\tfrac{1}{2}c\gamma_j(\tilde{\Xi}_1\hat{B}_{22} + \hat{B}_{22}^T\tilde{\Xi}_1) + \sigma_j\tilde{\Xi}_1)T_{22},
\end{aligned}$$

where

$$\begin{aligned}
\hat{B}_{12} &= [b_{12}, b_{13}, \ldots, b_{1N}],\\
\hat{B}_{21} &= [b_{21}, b_{31}, \ldots, b_{N1}]^T,\\
\hat{B}_{22} &= \begin{bmatrix} b_{22} & b_{23} & \cdots & b_{2N}\\ \cdots & & & \\ b_{N2} & b_{N3} & \cdots & b_{NN}\end{bmatrix},\\
\tilde{\Xi}_1 &= \operatorname{diag}(\xi_2, \ldots, \xi_N),\\
T_{21} &= [0, \ldots, 0, -\xi_1]^T,\\
T_{22} &= \begin{bmatrix} \xi_N & & \\ & \ddots & \\ & & \xi_N \\ -\xi_2 & \cdots & -\xi_{N-1}\end{bmatrix}.
\end{aligned}$$

By the Schur complement [15], the following result holds with respect to state feedback gain ε.

Corollary 3.2. *If Assumption 2.1 holds, $\breve{B}_{j22} < 0$, $j = 1, 2, \ldots, n$, and*

$$\varepsilon \geq \varepsilon_2 = \max_{j=1,2,\ldots,n}\left\{\frac{\xi_1\xi_N(c\gamma_j((b_{NN} - b_{1N})\xi_1 + (b_{11} - b_{N1})\xi_N) + \sigma_j(\xi_1 + \xi_N)) - \breve{B}_{j12}\breve{B}_{j22}^{-1}\breve{B}_{j12}^T}{c\gamma_j\xi_1\xi_N^2}\right\},$$

then the controlled complex network (2.1) can achieve asymptotical synchronization under control law (2.4).

4. Numerical simulation

A numerical example is presented here, to illustrate the pinning control methods for fractional-order complex networks. Consider the fractional-order complex network (2.1) with 10 nodes, where $\alpha = 0.995$, $x_i \in R^3$, $N = 10$, $\Gamma = I_3$, $c = 75$, the

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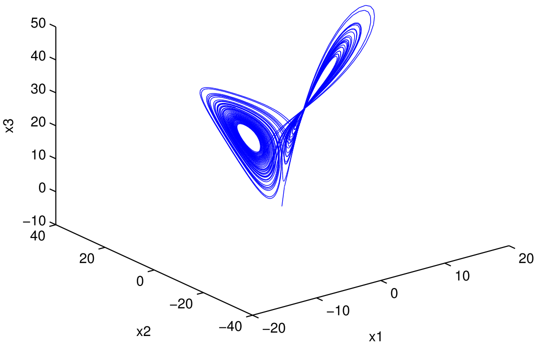

FIGURE 1. The chaotic attractor.

function $f(x,t) = [10(x_2 - x_1), 28x_1 - x_1x_3 - x_2, x_1x_2 - 8x_3/3]^T$ and the matrix

$$B = (b_{ij}) = \begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -3 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

In this network, when there is no coupling relationship, each node is a fractional-order Lorenz system, displaying a chaotic attractor as in Figure 1. Let $\varepsilon = 1$. Then synchronization errors, $e_{i1}, e_{i2}, e_{i3}, \|e(t)\|$, under state feedback control law (2.4) are displayed in Figure 2.

5. Conclusion

In this paper the state feedback pinning synchronization of fractional-order complex networks with irreducible coupling matrix by a single controller has been investigated. A reference state is employed, making use of the left eigenvector associated with the zero eigenvalue of the coupling matrix, and some criteria for asymptotical synchronization are obtained, based on the stability theory of fractional-order differential systems. However, there are many unsolved problems concerning the state feedback pinning synchronization of fractional-order complex networks. For example, in this paper only complex networks with constant coupling matrix were treated. Complex networks with time-varying coupling matrix have been neglected. For the dynamical behaviours at each node, we only considered fractional-order nominal systems; fractional-order systems with time delay need further study.

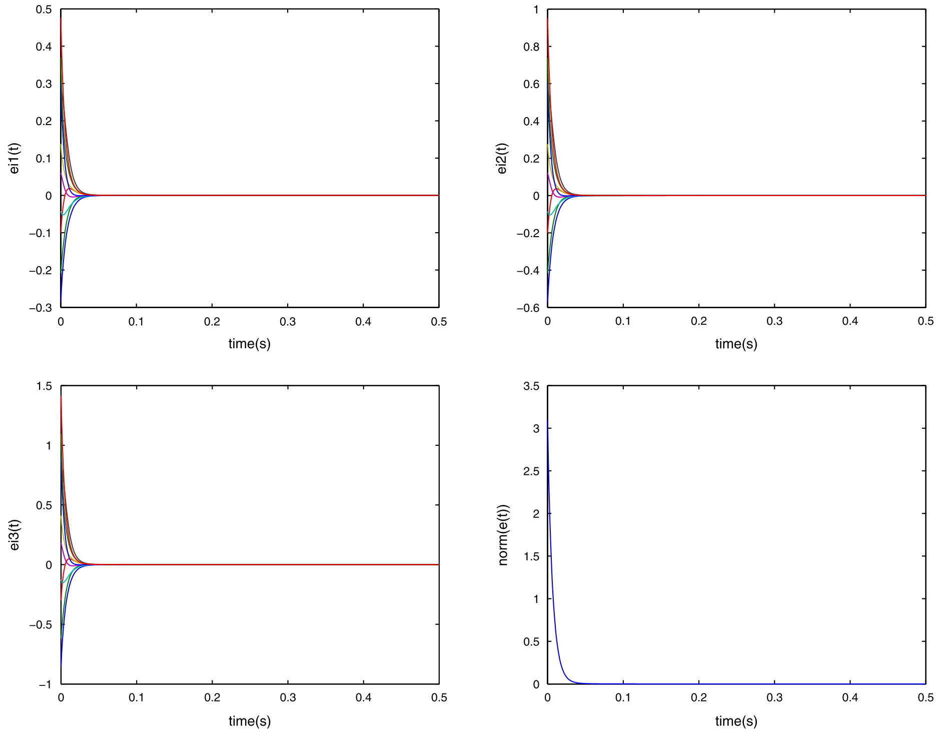

FIGURE 2. Synchronization errors under state feedback pinning control.

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