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Abstract

Recently Houdayer and Isono have proved, among other things, that every biexact group Γ has the property that for any non-singular strongly ergodic essentially free action $\Gamma \curvearrowright (X, \mu)$ on a standard measure space, the group measure space von Neumann algebra $\Gamma \ltimes L^\infty(X)$ is full. In this paper, we prove the same property for a wider class of groups, notably including $\mathrm{SL}(3, \mathbb{Z})$. We also prove that for any connected simple Lie group G with finite center, any lattice $\Gamma \leqslant G$, and any closed non-amenable subgroup $H \leqslant G$, the non-singular action $\Gamma \curvearrowright G/H$ is strongly ergodic and the von Neumann factor $\Gamma \ltimes L^\infty(G/H)$ is full.

1. Introduction

Recall that a von Neumann factor N is said to be *full* [Con74] if the *central sequence algebra* $N' \cap N^\omega$ is trivial for a non-principal ultrafilter ω. For more information on fullness, we refer the reader to [AH14, HI16] and the references therein. Recently, Houdayer and Isono [HI16] have studied which group has the property that the group measure space factor $\Gamma \ltimes L^\infty(X)$ is full for every non-singular strongly ergodic essentially free action $\Gamma \curvearrowright (X, \mu)$ on a standard measure space, and they have proved, among other things, that *biexact* groups (e.g., hyperbolic groups) have this property. Recall that a non-singular action $\Gamma \curvearrowright (X, \mu)$ on a probability space (if μ is not a probability measure, replace it with a probability measure in the same measure class) is said to be *strongly ergodic* if any sequence $(E_n)_n$ of measurable subsets such that $\mu(E_n \triangle sE_n) \to 0$ for every $s \in \Gamma$ has to be trivial: $\mu(E_n)(1 - \mu(E_n)) \to 0$. Unless the action is strongly ergodic, the von Neumann algebra $\Gamma \ltimes L^\infty(X,)$ cannot be full. We note that in the case where the strongly ergodic action $\Gamma \curvearrowright (X, \mu)$ is *probability measure preserving*, Choda [Cho82] obtained in 1982 the rather satisfactory result that the factor $\Gamma \ltimes L^\infty(X)$ is full whenever Γ is not *inner amenable*. In this note, we combine Choda's proof with Zimmer's notion of amenable action [Zim77] and prove Houdayer and Isono's property for a wider class of groups, notably including $\mathrm{SL}(3, \mathbb{Z})$, which is not biexact [Sak09]. We also prove that for any connected simple Lie group G with finite center, any lattice $\Gamma \leqslant G$, and any closed non-amenable subgroup $H \leqslant G$, the non-singular action $\Gamma \curvearrowright G/H$ is strongly ergodic and the von Neumann algebra $\Gamma \ltimes L^\infty(G/H)$ is full as long as it is a factor.

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2. Groups with Houdayer and Isono's property

We recall the notion of amenability in the sense of Zimmer and generalize it to a relative situation. The definition below is different from the original, but is equivalent to it [Zim77, Proposition 4.1]. We also note that amenability of a non-singular action $\Gamma \curvearrowright (X,\mu)$ is equivalent to injectivity of the von Neumann algebra $\Gamma \ltimes L^\infty(X)$ [Zim77, Ana87].

Definition 1. Let Γ be a discrete group and $\mathcal{C}$ be a non-empty family of its subgroups, and consider the set $K=\bigsqcup_{\Lambda\in\mathcal{C}}\Gamma/\Lambda$ on which Γ acts by translation. A non-singular action $\Gamma \curvearrowright (X,\mu)$ of Γ on a standard measure space (X,μ) is said to be *amenable* (in the sense of Zimmer) if there is a Γ-equivariant conditional expectation Φ from $L^\infty(\Gamma\times X)$ onto $L^\infty(X)$, where Γ acts on $\Gamma\times X$ diagonally. (Recall that Φ is called a *conditional expectation* if it is positive and satisfies $\Phi(1\otimes f)=f$ for every $f\in L^\infty(X)$.) We say $\Gamma \curvearrowright (X,\mu)$ is *amenable relative to* $\mathcal{C}$ if there is a Γ-equivariant conditional expectation from $L^\infty(K\times X)$ onto $L^\infty(X)$, where Γ acts on $K\times X$ diagonally.

Remark 2. We note that amenable actions on non-atomic measure spaces are never strongly ergodic by the Connes–Feldman–Weiss theorem ([CFW81]; see also [Sch81, Theorem 2.4]). We collect here some simple observations. Amenability does not change when one replaces the measure μ with another in the same measure class. By definition, amenability is same as amenability relative to $\{\mathbf{1}\}$, where $\mathbf{1}$ is the trivial subgroup consisting of the neutral element e. Let $\Gamma \curvearrowright (X,\mu)$ and $\mathcal{C}$ be given. If $\Gamma \curvearrowright (X,\mu)$ is amenable, then it is amenable relative to $\mathcal{C}$. The converse also holds true if the family $\mathcal{C}$ consists entirely of amenable subgroups. If $\ell_\infty(\bigsqcup_{\Lambda\in\mathcal{C}}\Gamma/\Lambda)$ admits a Γ-invariant state, then the action $\Gamma \curvearrowright (X,\mu)$ is amenable relative to $\mathcal{C}$. The converse also holds true if $L^\infty(X)$ admits a Γ-invariant state (e.g., if the action is probability measure preserving). Let K be a set on which Γ acts. Then there is a Γ-equivariant conditional expectation from $L^\infty(K\times X)$ onto $L^\infty(X)$ if and only if $\Gamma \curvearrowright (X,\mu)$ is amenable relative to the family $\{\Gamma_p : p\in K\}$ of the stabilizer subgroups $\Gamma_p=\{s\in\Gamma : sp=p\}$. We will be interested in the conjugation action of Γ on Γ. The stabilizer subgroup of the conjugation action $\Gamma\curvearrowright\Gamma$ at $t\in\Gamma$ is the centralizer subgroup $C_\Gamma(t):=\{s\in\Gamma : st=ts\}$.

Theorem 3. *Let Γ be a countable group, $\Omega\subset\Gamma$ be a conjugacy invariant subset, and $\mathcal{C}_\Omega=\{C_\Gamma(t) : t\in\Omega\}$. Let $\Gamma \curvearrowright (X,\mu)$ be a non-singular action of Γ on a standard measure space, which does not have a non-null $\mathcal{C}_\Omega$-relatively amenable component. Let $(w_n)_n$ be a bounded sequence in $\Gamma\ltimes L^\infty(X)$ such that $w_n-\lambda(s)w_n\lambda(s)^*\to 0$ ultrastrongly for every $s\in\Gamma$, and expand them as $w_n=\sum_t\lambda(t)w_n^t$. Then one has $\sum_{t\in\Omega}|w_n^t|^2\to 0$ ultrastrongly in $L^\infty(X)$.*

Proof. We put $s\cdot f:=\lambda(s)f\lambda(s)^*$ for $f\in L^\infty(X)$ and $s\in\Gamma$, or equivalently $(s\cdot f)(x)=f(s^{-1}x)$. We may assume that μ is a probability measure. Put $h_n^t:=|w_n^t|^2$ and $C:=\sup_n\|w_n\|^2$. Then the h_n^t are non-negative functions such that $0\leqslant\sum_t h_n^t\leqslant C$ and

$$\begin{aligned}
&\sum_{t\in\Gamma}\|h_n^t-s\cdot h_n^{s^{-1}ts}\|_{L^1(X)}\\
&\leqslant\Big(\sum_{t\in\Gamma}\|w_n^t-s\cdot w_n^{s^{-1}ts}\|_{L^2(X)}^2\Big)^{1/2}\Big(\sum_{t\in\Gamma}\||w_n^t|+|s\cdot w_n^{s^{-1}ts}|\|_{L^2(X)}^2\Big)^{1/2}\\
&\leqslant\|(w_n-\lambda(s)w_n\lambda(s)^*)(\delta_e\otimes 1_X)\|_{\ell_2(\Gamma\times X)}\cdot 2\|w_n\|\\
&\to 0
\end{aligned}$$

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for every $s \in \Gamma$. It suffices to prove that if there are functions h_n^t as above such that $h_n := \sum_{t\in\Omega} h_n^t$ does not converge to zero, then a non-null $\mathcal{C}_\Omega$-relatively amenable component exists. By compactness, we may assume that the ultraweak limit $h := \lim_n h_n$ exists and, after scaling, that $X_0 := \{x \in X : 1 \leqslant h(x) \leqslant 2\}$ has positive measure. By passing to convex combinations, we may further assume that $h_n \to h$ ultrastrongly. Since Ω is conjugacy invariant, the function h is Γ-invariant and so is X_0. We consider the conditional expectations Φ_n from $L^\infty(\Omega \times X_0)$ onto $L^\infty(X_0)$ given by

$$\Phi_n(f)(x) = \frac{1}{h_n(x)} \sum_{t\in\Omega} h_n^t(x) f(t,x),$$

and claim that it is approximately Γ-equivariant. (Let us neglect the points x where $h_n(x) = 0$.) Let $s \in \Gamma$ and $f \in L^\infty(\Omega \times X_0)$ with $\|f\|_\infty \leqslant 1$ be given. Then one has

$$\varepsilon_n := \mu(\{x \in X_0 : |h_n(x) - h(x)| + |h_n(s^{-1}x) - h(x)| \geqslant 1/n\}) \to 0$$

(recall that h is Γ-invariant) and

$$\begin{aligned}
&\|\Phi_n(s\cdot f) - s\cdot\Phi_n(f)\|_{L^1(X_0)}\\
&\quad\leqslant \int_{X_0} \frac{1}{h(x)} \left| \sum_{t\in\Omega} h_n^t(x) f(s^{-1}ts, s^{-1}x) - \sum_{t\in\Omega} h_n^t(s^{-1}x) f(t, s^{-1}x)\right| d\mu(x) + 2\varepsilon_n + \frac{2}{n}\\
&\quad\leqslant \int_{X_0} \sum_{t\in\Omega} |h_n^t(x) - (s\cdot h_n^{s^{-1}ts})(x)||f(s^{-1}ts, s^{-1}x)|\, d\mu(x) + 2\varepsilon_n + \frac{2}{n}\\
&\quad\to 0.
\end{aligned}$$

This proves the claim. Hence any pointwise ultraweak limit of $(\Phi_n)_n$ is a Γ-equivariant conditional expectation and the action $\Gamma \curvearrowright (X_0, \mu|_{X_0})$ is amenable relative to $\mathcal{C}_\Omega$. □

The following corollary gives a criterion of Γ for which every non-singular strongly ergodic essentially free action $\Gamma \curvearrowright (X,\mu)$ gives rise to a full factor $\Gamma \ltimes L^\infty(X)$. The essential freeness assumption is only used to assure $\Gamma \ltimes L^\infty(X)$ (and perhaps $\Lambda \ltimes L^\infty(X_0)$ in the proof) is a factor and probably can be greatly relaxed.

Corollary 4. *Assume that Γ is a countable group which has a finite-index subgroup Λ such that $\Lambda_{\mathrm{nac}} := \{t \in \Lambda : C_\Lambda(t)$ is not amenable$\}$ is finite. Then, for any non-singular strongly ergodic essentially free action $\Gamma \curvearrowright (X,\mu)$ on a standard measure space, the von Neumann factor $\Gamma \ltimes L^\infty(X)$ is full.*

Proof. Let $N = \Gamma \ltimes L^\infty(X)$ and $M = \Lambda \ltimes L^\infty(X)$. Then $M \subset N$ is a finite-index inclusion with a canonical normal conditional expectation E from N onto M. We first observe that it suffices to show that $M' \cap M^\omega$ is finite-dimensional for a non-principal ultrafilter ω. (We refer the reader to [HI16, § 2] for an account of the central sequence algebra $M' \cap M^\omega$.) Indeed, if it is so, then $N' \cap N^\omega$ is finite-dimensional since the map E^ω from $N' \cap N^\omega$ into $M' \cap M^\omega$ satisfies the Pimsner–Popa inequality [PP86]. Since N is a factor, this implies that N is full by [HR15, Corollary 2.6].

Thus we are left to show that M is a direct sum of finitely many full factors (and hence $M' \cap M^\omega$ is finite-dimensional). For this, we note that if $\Lambda \curvearrowright (X_0, \mu_0)$ is a strongly ergodic and essentially free action, then the crossed product $M_0 := \Lambda \ltimes L^\infty(X_0, \mu_0)$ is a full factor.

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Indeed, by [HI16, Lemma 5.1], if M_0 were not full, then there would be a unitary central sequence $(w_n)_n$ in M_0 such that $\sum_{t\in F}|w_n^t|^2 \to 0$ ultrastrongly for every finite subset $F \subset \Lambda$. But then Theorem 3, applied to $\Omega = \Lambda\backslash\Lambda_{\mathrm{nac}}$ (note that we may assume that (X_0, μ_0) is non-atomic and thus the Λ-action on it is non-amenable), implies that $1_{M_0} = \sum_{t\in\Lambda}|w_n^t|^2 \to 0$, which is absurd. This proves M_0 is a full factor. Therefore the proof of the corollary is complete once we prove the following claim, which is probably known to experts.

Claim. *The restriction $\Lambda \curvearrowright (X, \mu)$ of the strongly ergodic action of Γ to a finite-index subgroup Λ decomposes into finitely many ergodic components, and each of the ergodic components is strongly ergodic.*

If $\Lambda \curvearrowright (X, \mu)$ is not a union of finitely many ergodic components, then for any $\varepsilon > 0$, there is a Λ-invariant measurable subset $E \subset X$ such that $0 < \mu(E) < \varepsilon$. Also, if there is a non-null ergodic component $X_0 \subset X$ which is not strongly ergodic, then for any $\varepsilon > 0$, there is an asymptotically Λ-invariant sequence $(E_n)_n$ of measurable subsets of X_0 such that $0 < \lim_n \mu(E_n) < \varepsilon$ (see, for example, the proof of [JS87, Lemma 2.3]). Therefore, to prove the claim, it suffices to show that there is $\varepsilon > 0$ such that any asymptotically Λ-invariant sequence $(E_n)_n$ with $\inf_n \mu(E_n) > 0$ satisfies $\liminf_n \mu(E_n) \geqslant \varepsilon$. Now, let $\{t_0, \ldots, t_d\}$ be a system of representatives of the left cosets Γ/Λ, and take $\varepsilon > 0$ such that $\mu(E) \leqslant \varepsilon$ implies $\sum_i \mu(t_i E) \leqslant 1/2$. Then for any asymptotically Λ-invariant sequence $(E_n)_n$ with $\inf_n \mu(E_n) > 0$, the functions $f_n := \sum_i 1_{t_i E_n}$ satisfy $\|s \cdot f_n - f_n\|_{L^1(\mu)} \to 0$ for every $s \in \Gamma$, and so by strong ergodicity, $\|f_n - \int f_n\, d\mu\|_{L^1(\mu)} \to 0$. This implies $\liminf \mu(E_n) \geqslant \varepsilon$ and the claim is proved. □

Example 5. If Γ is a torsion-free hyperbolic group or a torsion-free discrete subgroup of a simple Lie group of rank 1, then $\Gamma_{\mathrm{nac}} \subset \{e\}$. Indeed, any non-trivial centralizer subgroup is elementary and hence is abelian. More generally, if Γ is a group acting with finite quotient on a fine hyperbolic graph K in such a way that every vertex stabilizer is amenable and the neutral element is the only element which fixes infinitely many points in the boundary ∂K of K, then $\Gamma_{\mathrm{nac}} \subset \{e\}$. For the proof recall that the Bowditch compactification $\Delta K := K \cup \partial K$ is topologically amenable [Oza06b, Theorem 1]. Hence any subgroup $\Lambda \leqslant \Gamma$ which admits a non-empty Λ-invariant finite subset of ΔK is amenable. Let $t \in \Gamma$ be such that $C_\Gamma(t)$ is non-amenable and denote by $\mathrm{Fix}(t)$ the t-fixed points of ΔK. Since $C_\Gamma(t)$ has unbounded orbits in K, the closed $C_\Gamma(t)$-invariant subset $\mathrm{Fix}(t)$ contains infinitely many boundary points. By assumption, $t = e$. This applies to a torsion-free relatively hyperbolic group and an amalgamated free product $\Gamma := \Gamma_1 *_\Lambda \Gamma_2$ of amenable groups Γ_i over a common malnormal subgroup Λ.

Example 6. If Γ is a subgroup of $\mathrm{SL}(2, F)$ for a field F, then $\Gamma_{\mathrm{nac}} \subset \{\pm I\}$. Indeed, by considering the Jordan normal form, it is not hard to see that the centralizer subgroup $C_{\mathrm{SL}(2,F)}(g)$ of any element $g \in \mathrm{SL}(2, F)\backslash\{\pm I\}$ is abelian.

Example 7. If Λ is a subgroup of $\mathrm{SL}(3, \mathbb{Z})$ which contains no elements of order 2, then $\Lambda_{\mathrm{nac}} \subset \{e\}$. In particular, the finite-index subgroup $\Gamma(3) = \ker(\mathrm{SL}(3, \mathbb{Z}) \to \mathrm{SL}(3, \mathbb{Z}/3\mathbb{Z}))$ satisfies $\Gamma(3)_{\mathrm{nac}} = \{e\}$. For a proof, suppose that $g \in \mathrm{SL}(3, \mathbb{Z})$ has a non-abelian centralizer subgroup in $\mathrm{SL}(3, \mathbb{R})$. Then g must be diagonalizable in $\mathrm{SL}(3, \mathbb{R})$ and has eigenvalues $(\lambda, \lambda, \lambda^{-2})$ for some $\lambda \in \mathbb{C}$. Its characteristic polynomial $p(t) = t^3 + at^2 + bt - 1$ has integer coefficients and satisfies $0 = 9p(\lambda) - (3\lambda + a)p'(\lambda) = 2(3b - a^2)\lambda - (9 + ab)$. So, if $3b - a^2 \neq 0$, then λ is rational. If $3b - a^2 = 0$, then $0 = 3p'(\lambda) = (3\lambda + a)^2$ and λ is again rational. Hence in either case $p(\lambda) = 0$ implies $\lambda \in \{\pm 1\}$. This proves $g^2 = I$. Now, suppose for a contradiction that there is an element

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$g \in \Gamma(3)$ of order 2. Then the largest integer m such that $g - I \in 3^m \mathbb{M}_3(\mathbb{Z})$ is positive, but this is in contradiction to $(g-I)^2 = -2(g-I)$. In fact it is well known that the finite-index subgroups $\Gamma(m)$ are torsion-free for all $m \geqslant 3$. By Corollary 4, the factor $\mathrm{SL}(3,\mathbb{Z}) \ltimes L^\infty(X)$ is full for every strongly ergodic essentially free action $\mathrm{SL}(3,\mathbb{Z}) \curvearrowright (X,\mu)$. It is not clear if the same conclusion holds for $\mathrm{SL}(n \geqslant 4, \mathbb{Z})$.

Example 8. In Corollary 4 one can replace the condition that Λ_{nac} is finite with existence of a map $\zeta : \Lambda \to \mathrm{Prob}(\Lambda)$ such that $\lim_{t\to\infty} \|\zeta_{sts^{-1}} - s\zeta_t\| = 0$ for every $s \in \Lambda$. (In fact the latter condition is more general.) Indeed, if there is a unitary central sequence $(w_n)_n$ in $\Lambda \ltimes L^\infty(X)$ such that $\sum_{t\in F} |w_n^t|^2 \to 0$ for every finite subset $F \subset \Lambda$, then the action is amenable since the maps $\Phi_n : L^\infty(\Lambda \times X) \to L^\infty(X)$ given by $\Phi_n(f)(x) = \sum_{t,p} |w_n^t(x)|^2 \zeta_t(p) f(p,x)$ are approximately Λ-equivariant. We note that biexact groups satisfy the above property by [Oza06a, Proposition 4.1].

3. Actions of lattices on homogeneous spaces

In this section we consider the non-singular action $\Gamma \curvearrowright G/H$ of a lattice Γ in a second countable locally compact group G on a homogeneous space G/H. Here $H \leqslant G$ is a closed subgroup and G/H is equipped with a G-quasi-invariant measure, which is unique up to equivalence. Generally speaking, one can relate the action $\Gamma \curvearrowright G/H$ to the action $\Gamma\backslash G \curvearrowleft H$ (see [PV11, § 3]). For example, if one is amenable, then so is the other [PV11, Remark 4.2]. The following is the relative version of this fact.

Lemma 9. *Let $\Gamma \curvearrowright G/H$ be as above and assume that it is amenable relative to a non-empty family $\mathcal{C}$ of subgroups of Γ. Then there is an H-invariant state ψ on $L^\infty(\bigsqcup_{\Lambda\in\mathcal{C}} \Lambda\backslash G)$, where H acts on $\bigsqcup_{\Lambda\in\mathcal{C}} \Lambda\backslash G$ diagonally from the right. Moreover, if there is a normal Γ-equivariant conditional expectation from $L^\infty(K \times G/H)$ onto $L^\infty(G/H)$, where $K = \bigsqcup_{\Lambda\in\mathcal{C}} \Gamma/\Lambda$, then the H-invariant state ψ can be taken to be normal.*

Proof. Let Φ denote a Γ-equivariant conditional expectation from $L^\infty(K\times G/H)$ onto $L^\infty(G/H)$. We fix a lifting $\sigma : K \to \Gamma$ such that $\sigma(p)\Lambda = p$ for $p \in \Gamma/\Lambda \subset K$ and a measurable lifting $\tau : G/H \to G$. The corresponding measure space isomorphisms and cocycles are denoted as follows:

$$\theta : K \times G/H \times H \to K \times G, \quad \theta(p,x,y) = (p, \sigma(p)^{-1}\tau(x)y);$$
$$\theta' : G \to G/H \times H, \quad \theta'(g) = (gH, \tau(gH)^{-1}g);$$
$$\alpha : \Gamma \times K \ni (s,p) \mapsto \sigma(sp)^{-1} s\sigma(p) \in \Lambda \quad \text{for } s \in \Gamma \quad \text{and} \quad p \in \Gamma/\Lambda \subset K;$$
$$\beta : \Gamma \times G/H \ni (s,x) \mapsto \tau(sx)^{-1} s\tau(x) \in H.$$

These maps satisfy the following relations. Let $s \cdot (p,x,y) \cdot h := (sp, sx, \beta(s,x)yh)$ for $(p,x,y) \in K \times G/H \times H$ and $s \cdot (p,g) \cdot h := (sp, \alpha(s,p)gh)$ for $(p,g) \in K \times G$. Then one has $\theta(s \cdot (p,x,y) \cdot h) = s \cdot \theta(p,x,y) \cdot h$. Let $s \cdot (x,y) \cdot h := (sx, \beta(s,x)yh)$ for $(x,y) \in G/H \times H$. Then one has $\theta'(sgh) = s \cdot \theta'(g) \cdot h$.

We claim that the conditional expectation $\Phi \otimes \mathrm{id}_{L^\infty(H)}$ from $L^\infty(K \times G/H \times H)$ onto $L^\infty(G/H \times H)$ satisfies $(\Phi \otimes \mathrm{id}_{L^\infty(H)})(s \cdot f \cdot h) = s \cdot (\Phi \otimes \mathrm{id}_{L^\infty(H)})(f) \cdot h$ for every $s \in \Gamma$ and $h \in H$, where $(s \cdot f \cdot h)(p,x,y) := f(s^{-1} \cdot (p,x,y) \cdot h^{-1})$, etc. Note that even though Φ may not be normal, the map $\Phi \otimes \mathrm{id}_{L^\infty(H)}$ is well defined via the defining relation

$$(\xi \otimes \eta)((\Phi \otimes \mathrm{id}_{L^\infty(H)})(f)) = \xi(\Phi((\mathrm{id} \otimes \eta)(f))) \quad \text{for } (\xi,\eta) \in L^\infty(G/H)_* \times L^\infty(H)_*.$$

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(Of course $L^\infty(X,\mu)_* = L^1(X,\mu)$ under the duality coupling $\langle f,\xi\rangle = \int f\xi\,d\mu$, but we often regard $\xi\in L^1(X,\mu)$ as an ultraweakly continuous linear functional on $L^\infty(X)$.) We observe that for any countable partition $(q_i)_i$ of unity in $L^\infty(G/H)$ and any bounded sequence $(f_i)_i$ in $L^\infty(K\times G/H)$, one has $\Phi(\sum_i f_i(1_K\otimes q_i)) = \sum_i \Phi(f_i)q_i$. Indeed, this follows from the fact that for any state $\xi\in L^\infty(G/H)_*$ one has

$$\begin{aligned}\left|\xi\left(\Phi\left(\sum_{i\geqslant n} f_i(1_K\otimes q_i)\right)\right)\right| &\leqslant (\xi\circ\Phi)\left(\left|\sum_{i\geqslant n} f_i(1_K\otimes q_i)\right|\right)\\ &\leqslant (\xi\circ\Phi)\left((\sup_i \|f_i\|)\sum_{i\geqslant n} 1_K\otimes q_i\right)\\ &= (\sup_i\|f_i\|)\xi\left(\sum_{i\geqslant n} q_i\right)\to 0.\end{aligned}$$

Now, let $s\in\Gamma$ be given and take $\eta\in L^1(H,\nu)$ and $\varepsilon>0$ arbitrary. Here ν denotes the left Haar measure of H. Let $(Y_i)_i$ be a countable measurable partition of H together with $h_i\in Y_i$ such that $\|h\eta - h_i\eta\|_1<\varepsilon$ for $h\in Y_i$, and put $X_i=\{x\in G/H : \beta(s^{-1},x)\in H_i\}$. Then, for every $f\in L^\infty(K\times G/H\times H)$ with $\|f\|\leqslant 1$, one has

$$\left\|(\mathrm{id}_{L^\infty(K\times G/H)}\otimes\eta)\left(s\cdot f - \sum_i ((s\otimes h_i^{-1})f)1_{K\times X_i\times H}\right)\right\|_{L^\infty(K\times G/H)} < \varepsilon,$$

where s acts on $L^\infty(K\times G/H)$ diagonally. Indeed, for every i and almost every $(p,x)\in K\times X_i$, one has

$$\begin{aligned}((\mathrm{id}\otimes\eta)(s\cdot f))(p,x) &= \int_H f(s^{-1}p, s^{-1}x, \beta(s^{-1},x)y)\eta(y)\,d\nu(y)\\ &\approx_\varepsilon \int_H f(s^{-1}p, s^{-1}x, h_i y)\eta(y)\,d\nu(y)\\ &= (\mathrm{id}\otimes\eta)((s\otimes h_i^{-1})f)(p,x).\end{aligned}$$

Similarly, for every $g\in L^\infty(G/H\times H)$ with $\|g\|\leqslant 1$, one has

$$\left\|(\mathrm{id}_{L^\infty(G/H)}\otimes\eta)\left(s\cdot g - \sum_i ((s\otimes h_i^{-1})g)1_{X_i\times H}\right)\right\|_{L^\infty(G/H)} < \varepsilon.$$

Hence

$$\begin{aligned}(\mathrm{id}\otimes\eta)((\Phi\otimes\mathrm{id})(s\cdot f)) &= \Phi((\mathrm{id}\otimes\eta)(s\cdot f))\\ &\approx_\varepsilon \Phi\left((\mathrm{id}\otimes\eta)\left(\sum_i ((s\otimes h_i^{-1})f)1_{K\times X_i\times H}\right)\right)\\ &= \sum_i \Phi((\mathrm{id}\otimes\eta)((s\otimes h_i^{-1})f))1_{X_i}\\ &= \sum_i (\mathrm{id}\otimes\eta)((s\otimes h_i^{-1})((\Phi\otimes\mathrm{id})(f)))1_{X_i}\\ &\approx_\varepsilon (\mathrm{id}\otimes\eta)(s\cdot(\Phi\otimes\mathrm{id})(f)).\end{aligned}$$

Since η and $\varepsilon>0$ were arbitrary, one obtains $(\Phi\otimes\mathrm{id})(s\cdot f) = s\cdot(\Phi\otimes\mathrm{id})(f)$. That $(\Phi\otimes\mathrm{id})(f\cdot h) = (\Phi\otimes\mathrm{id})(f)\cdot h$ for $h\in H$ is obvious.

We define $\Psi : L^\infty(K\times G) \to L^\infty(G)$ by $\Psi = \theta'_* \circ (\Phi \otimes \mathrm{id}_{L^\infty(H)}) \circ \theta_*$. It may not be a conditional expectation, but it is a unital positive map, which is $(\Gamma \times H)$-equivariant:

$$\Psi(s\cdot f\cdot h) = s\Psi(f)h \quad \text{for } s\in\Gamma, h\in H, \text{ and } f\in L^\infty(K\times G),$$

where $(s\Psi(f)h)(g) = \Psi(f)(s^{-1}gh^{-1})$ for $g \in G$. We denote by ι the embedding of $L^\infty(\bigsqcup_{\Lambda\in\mathcal{C}}\Lambda\backslash G)$ into $L^\infty(K\times G)$ given by $\iota(f)(p,g) = f(\Lambda g)$ for $\Lambda\in\mathcal{C}$, $p\in\Gamma/\Lambda\subset K$, and $g\in G$. The map ι is $(\Gamma\times H)$-equivariant, where Γ acts trivially on $\bigsqcup_{\Lambda\in\mathcal{C}}\Lambda\backslash G$ and H acts on it from the right. Thus $\Psi\circ\iota$ is an H-equivariant unital positive map from $L^\infty(\bigsqcup_{\Lambda\in\mathcal{C}}\Lambda\backslash G)$ into $L^\infty(G)^\Gamma \cong L^\infty(\Gamma\backslash G)$. An H-invariant state ψ can be obtained by composing $\Psi\circ\iota$ with the G-invariant probability measure on $\Gamma\backslash G$. If Φ is normal, then so is ψ. □

Lemma 10 (cf. [Ioa14, Proposition G]). *Let G be a second countable locally compact group, $\Gamma \leqslant G$ be a lattice, and $H\leqslant G$ be a closed subgroup such that the action $\Gamma\backslash G \curvearrowleft H$ is strongly ergodic. Then $\Gamma \curvearrowright G/H$ is strongly ergodic.*

Proof. We first recall that a non-singular action $H\curvearrowright(Y,\nu)$ of a locally compact group H on a probability space (Y,ν) is said to be strongly ergodic if any sequence $(E_n)_n$ of measurable subsets of Y that is approximately H-invariant (i.e., $\nu(E_n \triangle hE_n)\to 0$ uniformly on compact subsets of H) is trivial in the sense that $\nu(E_n)(1-\nu(E_n))\to 0$.

Now, take a Borel lifting $\tau:\Gamma\backslash G\to G$ and let $Y=\tau(\Gamma\backslash G)$ be the corresponding Γ-fundamental domain. Since G is σ-compact, we may assume that $\tau(L)$ is relatively compact for every compact subset $L\subset\Gamma\backslash G$. The Haar measure λ_G of G is normalized so that $\lambda_G(Y)=1$. Then the formula $\lambda(\Gamma A)=\lambda_G(\Gamma A\cap Y)$ for measurable subsets $A\subset G$ defines the G-invariant probability measure λ on $\Gamma\backslash G$. Assume that there is a non-trivial approximately Γ-invariant sequence $(E_n)_n$ of measurable subsets of G/H. We will prove that $(\Gamma(E_nH\cap Y))_n$ is a non-trivial approximately H-invariant sequence of measurable subsets of $\Gamma\backslash G$.

Recall that $L^\infty(G,\lambda_G)\cong L^\infty(G/H\times H,\mu\otimes\nu)$, where μ and ν are quasi-invariant probability measures. Since $(\mu\otimes\nu)(sE_nH\triangle E_nH)=\mu(sE_n\triangle E_n)\to 0$ for every $s\in\Gamma$, one has $\lambda_G((sE_nH\triangle E_nH)\cap Z)\to 0$ for any Z with $\lambda_G(Z)<\infty$. Thus, if $\lambda_G(E_nH\cap Y)\to 0$, then $\lambda_G(E_nH\cap sY)\approx\lambda_G(sE_nH\cap sY)=\lambda_G(E_nH\cap Y)\to 0$ for every $s\in\Gamma$, which means that $\mu(E_n)\to 0$. Therefore $(\Gamma(E_nH\cap Y))_n$ is non-trivial if $(E_n)_n$ is non-trivial.

Let $\beta:\Gamma\backslash G\times H\ni(y,h)\mapsto\tau(y)h\tau(yh)^{-1}\in\Gamma$ be the cocycle associated with τ that satisfies $\tau(y)h=\beta(y,h)\tau(yh)$ for $y\in\Gamma\backslash G$ and $h\in H$. Let a compact subset $K\subset H$ and $\varepsilon>0$ be given. Take a compact subset $L\subset\Gamma\backslash G$ such that $\lambda(L)>1-\varepsilon$. Then $F:=\{\beta(y,h): y\in L, h\in K\}$ is a finite subset of Γ, since it is relatively compact in G. Thus, if n is large enough, then one has $\lambda_G((FE_nH\cap Y)\backslash(E_nH\cap Y))<\varepsilon$. So, for every $h\in K$, one has

$$\begin{aligned}\lambda(\Gamma(E_nH\cap Y)h\backslash\Gamma(E_nH\cap Y)) &\approx_\varepsilon \lambda(\Gamma(E_nH\cap\tau(L))h\backslash\Gamma(E_nH\cap Y))\\ &\leqslant \lambda_G((FE_nH\cap Y)\backslash(E_nH\cap Y))\\ &\approx_\varepsilon 0.\end{aligned}$$

This means that $(\Gamma(E_nH\cap Y))_n$ is approximately H-invariant. □

Theorem 11. *Let G be a connected simple Lie group with finite center $\mathcal{Z}(G)$, $\Gamma\leqslant G$ be a lattice, and $H\leqslant G$ be a closed non-amenable subgroup. Then the action $\Gamma\curvearrowright G/H$ is strongly ergodic, and if $\Gamma\cap\mathcal{Z}(G)\cap H=\{e\}$, then $\Gamma\ltimes L^\infty(G/H)$ is a full factor.*

Proof. Strong ergodicity of $\Gamma \curvearrowright G/H$ follows from that of $\Gamma\backslash G \curvearrowleft H$ by Lemma 10. Since the latter action is finite measure preserving, strong ergodicity follows if the unitary representation of H on $L^2_0(\Gamma\backslash G)$ does not weakly contain the trivial representation $\mathbf{1}_H$ [CW80, Sch81]. By [BV95, Proposition 4.1] the latter holds true as long as H is non-amenable.

For the second assertion, we note that the assumption $\Gamma \cap \mathcal{Z}(G) \cap H = \{e\}$ is equivalent to the action $\mathcal{Z}(\Gamma) \curvearrowright G/H$ being free. Here we note that

$$\Gamma \cap \mathcal{Z}(G) = \mathcal{Z}(\Gamma) = \{t \in \Gamma : [\Gamma : C_\Gamma(t)] < \infty\},$$

by the Borel density theorem (applied to the lattice $C_\Gamma(t)$). Let $\mathcal{C}_\infty$ denote the family of infinite-index subgroups of Γ and $K := \bigsqcup_{\Lambda \in \mathcal{C}_\infty} \Gamma/\Lambda$. To prove that $\Gamma \ltimes L^\infty(G/H)$ is a factor, take a unitary central element w, and expand it as $w = \sum_{t\in\Gamma} \lambda(t) w^t$. Since $\mathcal{Z}(\Gamma) \curvearrowright G/H$ is free, one has $w^t = 0$ for all $t \in \mathcal{Z}(\Gamma)$ except that $w^e \in \mathbb{C}1$ (by ergodicity). If $w \neq w^e$, then by the proof of Theorem 3, there is a normal Γ-equivariant conditional expectation from $L^\infty(K \times G/H)$ onto $L^\infty(G/H)$. By Lemma 9, this gives rise to a normal H-invariant state ψ on $L^\infty(\bigsqcup_{\Lambda\in\mathcal{C}_\infty} \Lambda\backslash G)$, which in turn provides a non-zero H-invariant vector in $L^2(\bigsqcup_{\Lambda\in\mathcal{C}_\infty} \Lambda\backslash G)$. But such a vector is also G-invariant by Moore's ergodicity theorem, in contradiction to the fact that all $\Lambda \in \mathcal{C}_\infty$ have infinite covolume in G.

Next, we prove that the factor $\Gamma \ltimes L^\infty(G/H)$ is full. The case of a rank-1 Lie group is already covered by Corollary 4 and Example 5 (note that the essential freeness assumption in Corollary 4 is only used to assure factoriality of $\Gamma \ltimes L^\infty(X)$ and $\Lambda \ltimes L^\infty(X_0)$, and thus can be dispensed with by the above result). Thus we may assume that G has Kazhdan's property (T). Then, by [Cow79, Moo84], the unitary representation of H on $\bigoplus_{\Lambda\in\mathcal{C}_\infty} L^2(\Lambda\backslash G)$ does not weakly contain $\mathbf{1}_H$. (We note that we can avoid the use of this heavy machinery if H has a non-compact subgroup with relative property (T).) This means that $L^\infty(\bigsqcup_{\Lambda\in\mathcal{C}_\infty} \Lambda\backslash G)$ does not admit an H-invariant state (see [Sch81]), and hence by Lemma 9 the action $\Gamma \curvearrowright G/H$ is not amenable relative to $\mathcal{C}_\infty$. Now, suppose for a contradiction that the factor $\Gamma \ltimes L^\infty(G/H)$ is not full and there is a unitary central sequence $(w_n)_n$ in $\Gamma \ltimes L^\infty(G/H)$ such that $\sum_{t\in\mathcal{Z}(\Gamma)} |w_n^t|^2 \to 0$ [HI16, Lemma 5.1]. Then, by Theorem 3, this would imply $1 = \sum_{t\in\Gamma} |w_n^t|^2 \to 0$, which is absurd. □

Example 12. Consider the linear action of $\Gamma := \mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{R}^n$. Since the action extends to a measure-preserving and essentially transitive action of $G := \mathrm{SL}(n, \mathbb{R})$, it is isomorphic to the action $\Gamma \curvearrowright G/H$, where $H \cong \mathrm{SL}(n-1, \mathbb{R}) \ltimes \mathbb{R}^{n-1}$ is the stabilizer subgroup of $G \curvearrowright \mathbb{R}^n$ at $(1, 0, \ldots, 0)^T \in \mathbb{R}^n$. If $n \geqslant 3$, then H is non-amenable and $\mathrm{SL}(n, \mathbb{Z}) \ltimes L^\infty(\mathbb{R}^n)$ is a full factor by Theorem 11. (If $n \leqslant 2$, then H is amenable and so is the action $\Gamma \curvearrowright \mathbb{R}^n$, see [PV11, Remark 4.2].) Moreover, since $\mathrm{SL}(n, \mathbb{Z})$-action is measure preserving, $\mathrm{SL}(n, \mathbb{Z}) \ltimes L^\infty(\mathbb{R}^n)$ is a type II_∞ factor with a continuous trace-scaling $\mathbb{R}^\times_+$-action, coming from the diagonal $\mathbb{R}^\times_+$-action on $\mathbb{R}^n$ by multiplication. The corresponding crossed product is a type III_1 full factor that is isomorphic to $\mathrm{SL}(n, \mathbb{Z}) \ltimes L^\infty(S^{n-1})$, where $\mathrm{SL}(n, \mathbb{Z})$ acts naturally on the sphere $S^{n-1} \cong (\mathbb{R}^n \backslash \{0\})/\mathbb{R}^\times_+$ (which is also isomorphic to a homogeneous space of $\mathrm{SL}(n, \mathbb{R})$). Indeed, since the $\mathbb{R}^\times_+$-action is smooth and commutes with the $\mathrm{SL}(n, \mathbb{Z})$-action, one has natural isomorphisms

$$\begin{aligned}(\mathrm{SL}(n, \mathbb{Z}) \ltimes L^\infty(\mathbb{R}^n)) \rtimes \mathbb{R}^\times_+ &\cong \mathrm{SL}(n, \mathbb{Z}) \ltimes (L^\infty(S^{n-1} \times \mathbb{R}^\times_+) \rtimes \mathbb{R}^\times_+) \\ &\cong \mathrm{SL}(n, \mathbb{Z}) \ltimes (L^\infty(S^{n-1}) \otimes \mathbb{B}(L^2(\mathbb{R}^\times_+))) \\ &\cong (\mathrm{SL}(n, \mathbb{Z}) \ltimes L^\infty(S^{n-1})) \otimes \mathbb{B}(L^2(\mathbb{R}^\times_+)).\end{aligned}$$

Remark 13. Adrian Ioana and Jesse Peterson kindly pointed out to the author that the action $\Gamma \curvearrowright G/H$ as in Theorem 11 is essentially free, and moreover that if G is a connected Lie group and $H \leqslant G$ is a closed subgroup with $N := \bigcap_{g\in G} gHg^{-1}$, then the action $G/N \curvearrowright G/H$

is essentially free. Indeed, this follows from the fact that $\bigcap_{i=1}^{d+1} g_i H g_i^{-1} = N$ for almost every $(g_1, \ldots, g_{d+1}) \in G^{d+1}$, where $d = \dim G$. This fact ought to be well known, but since we did not find a reference, we sketch a proof here (see also [CP13, 7.1]). Since G is a connected Lie group, for any closed subgroup K whose connected component is not normal in G, one has $\dim(K \cap gKg^{-1}) < \dim K$ for almost every $g \in G$. It follows that the connected component of $\bigcap_{i=1}^{d} g_i H g_i^{-1}$ is normal in G almost surely (note that zero-dimensional subgroups are discrete and countable), and hence $\bigcap_{i=1}^{d+1} g_i H g_i^{-1}$ is normal in G almost surely.

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