On certain new connections between Legendre and Bessel Functions

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Let \( n \) be a positive integer. Then we know that\(^1\), if \( m > -1 \),

\[
\int_0^1 P_n(1 - 2y^2) y^{2m+1} dy = \frac{1}{2} (-1)^n \frac{\{\Gamma(m+1)\}^2}{\Gamma(m-n+1) \Gamma(m+n+2)}.
\]  

(1)

Consider the integral

\[
I = \int_0^1 P_n(1 - 2y^2) J_0(2yz) y dy,
\]

which is equal to\(^2\)

\[
\sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{(\Gamma(m+1))^2} \int_0^1 P_n(1 - 2y^2) y^{2m+1} dy.
\]

On integrating term by term, we get

\[
I = \frac{1}{2} (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\Gamma(m-n+1) \Gamma(m+n+2)}
\]

\[
= \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+2n}}{\Gamma(m+1) \Gamma(m+2n+2)}
\]

\[
= (2z)^{-1} J_{2n+1}(2z).
\]

(2)

In a similar manner, we can prove the following results:

\[
\int_0^1 P_n(1 - 2y^2) J_0(2yz) I_0(2yz) y^3 dy
\]

\[
= (8z)^{-1} \frac{d}{dz} \{J_{2n+1}(2z) I_{2n+1}(2z)\},
\]

(3)

\[
\int_0^1 P_n(1 - 2y^4) \frac{d}{dy} [y^2 \{\text{ber}_1^2(2yz) + \text{bei}_1^2(2yz)\}] dy
\]

\[
= (-1)^n \{\text{ber}_1^2(2z) + \text{bei}_1^2(2z)\},
\]

(4)

and

\[
\int_0^1 P_n(1 - 2y^4) \frac{d}{dy} [y^2 J_1(2yz) I_1(2yz)] dy
\]

\[
= J_{2n+1}(2z) I_{2n+1}(2z).
\]

(5)

\(^1\) Equation (1) follows at once by putting \( x = 1 - 2y^2 \), using Rodrigues' formula for \( P_n(x) \), and integrating \( n \) times by parts. Cf. Cooke, Proc. London Math. Soc., 23 (1924), xix, equ. (3).

\(^2\) The process of arrangement and term by term integration can be easily justified.