NORMAL FUNCTIONS AND NON-TANGENTIAL BOUNDARY ARCS

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1. Introduction. Let D and C denote respectively the open unit disk and the unit circle in the complex plane. Further, $\gamma = z(t)$, $0 \leqslant t \leqslant 1$, will denote a simple continuous arc lying in D except for $\tau = z(1) \in C$, and we shall say that γ is a *boundary arc* at τ.

We use extensively the notions of non-Euclidean hyperbolic geometry in D and employ the usual metric

$$\rho(a, b) = \tfrac{1}{2} \log \frac{|1 - a\bar{b}| + |a - b|}{|1 - a\bar{b}| - |a - b|},$$

where a and b are elements of D. For $a \in D$ and $r > 0$ let

$$B(a, r) = \{z \in D | \rho(a, z) < r\}.$$

For details we refer the reader to **(4)**.

If γ is any boundary arc of D and $0 < r < \infty$

$$H(\gamma, r) = [z \in D | \rho(z, \gamma) < r],$$

where $\rho(\gamma, z)$ is the non-Euclidean distance of z to γ defined in the usual way. Consequently, $H(\gamma, r)$ is an open, connected (but not necessarily simply connected) subset of D. The boundary of $H(\gamma, r)$ is seen to contain two distinct boundary arcs at τ. Let ρ_τ, $\tau \in C$, denote the diameter of C ending at τ. In this case, $H(\rho_\tau, r)$ is a connected domain bounded by two hypercycles from τ to $-\tau$ which form the angle $2 \arctan r$ and $-2 \arctan r$, respectively, with ρ_τ at τ. If $\gamma = z(t)$ is a boundary arc at τ such that for some $0 \leqslant t_0 < 1$, and some $0 < r < \infty$, $z(t) \in H(\rho_\tau, r)$, $t \geqslant t_0$, we say γ *approaches* τ *in a non-tangential manner*, and the set of all such non-tangential boundary arcs at τ we denote by $\Lambda(\tau)$.

We consider functions $f(z)$ defined in D and taking values in the extended complex plane W. For a set $\Omega \subset D$, with the closure of Ω intersecting C at a single point τ, $C_\Omega(f, \tau)$ indicates the set of all values $w \in W$ with the property that there is a sequence $\{z_n\}$ in Ω with $z_n \to \tau$, $n \to \infty$ and $f(z_n) \to w$, $n \to \infty$.

Finally

$$\Pi_{H(\gamma,\, r)}(f, \tau) = \bigcap\nolimits_{\gamma^*} C_{\gamma^*}(f, \tau),$$

where γ^* ranges over all boundary arcs at τ that lie within $H(\gamma, r)$, and

$$\Pi_{\mathfrak{A}}(f, \tau) = \bigcap_{\gamma \in \Lambda(\tau)} C_\gamma(f, \tau).$$

Received October 22, 1964.

For an elaboration of the theory of cluster sets see, for example, **(8)**.

2. Boundary behaviour of normal functions. Of course,

$$\Pi_{\mathfrak{A}}(f, \tau) \subseteq \Pi_{H(\gamma, r)}(f, \tau),$$

for any $\gamma \in \Lambda(\tau)$ and any $r > 0$. Our main result is that

$$\Pi_{\mathfrak{A}}(f, \tau) = \Pi_{H(\gamma, r)}(f, \tau),$$

where τ is any point in C, γ is any curve in $\Lambda(\tau)$, and r is any positive number—*provided that f is a normal function in D.*

DEFINITION. *A function $f(z)$ meromorphic in D is said to be a normal function in D if the family $\{f(S(z))\}$ is normal in the sense of Montel, where $S(z)$ is an arbitrary one-to-one conformal map of D onto itself; see* **(6**, p. 53**)**.

Before proving this result we set forth two lemmas, which will expedite the proof. For any $w \in W$ and $d \geqslant 0$, let $Z_f(w, d) = \bigcup_z B(z, d)$, where the union is taken over all $z \in D$ such that $f(z) = w$.

LEMMA 1. *Let f be a normal function in D and $\gamma \in \Lambda(t)$, $\tau \in C$. If $w \in C_\gamma(f, \tau)$ and if there exists $d > 0$ such that $\gamma \cap Z_f(w, d) = \emptyset$, then*

$$w \in C_{\gamma'}(f, \tau) \textit{ for any } \gamma' \in \Lambda(\tau).$$

Proof. Since $w \in C_\gamma(f, \tau)$, let $\{z_n\}$ be a sequence on γ such that $z_n \to \tau$, $f(z_n) \to w$, as $n \to \infty$. Let

$$f(S_n(\zeta)) = g_n(\zeta)$$

with

$$S_n(\zeta) = \frac{\zeta + z_n}{1 + \zeta \bar{z}_n}, \qquad |\zeta| < 1, n = 1, 2, \ldots ;$$

and let $\{g_{n_k}(\zeta)\}$ be the convergent subsequence guaranteed by the normalcy of f. Let $g(\zeta)$ be the limit function. Now

$$g(0) = \lim_{k\to\infty} g_{n_k}(0) = \lim_{k\to\infty} f(z_{n_k}) = w;$$

but for $|\zeta| < \tanh d$ the equation

$$g_{n_k}(\zeta) = w$$

is satisfied for no value of k. By Hurwitz's theorem we conclude that $g(\zeta) \equiv w$. Hence for any fixed $0 < d' < \infty$, f tends to the value w as z tends to τ on the set

$$\bigcup_{k=1}^{\infty} B(z_{nk}, d').$$

If $\gamma' \in \Lambda(\tau)$, then

$$\gamma' \cap B(z_{nk}, d') \neq \emptyset$$

for suitable $d' > 0$ and all $k = 1, 2, \ldots$; and we conclude that $w \in C_{\gamma'}(f, \tau)$, which proves the lemma.

LEMMA 2. *Let f be a normal function in D. Suppose for some $\tau \in C$ and for each $n = 1, 2, \ldots$, that there is a set of distinct points*

$$\{\xi_i^{(n)}\}, \qquad i = 1, 2, \ldots, m_n,$$

with the following properties:

(i) *for some $r > 0$ and all $n = 1, 2, \ldots, \xi_1^{(n)} \in H(\rho_\tau, r)$;*

(ii) *also $\xi_1^{(n)} \to \tau, \qquad n \to \infty$;*

(iii) *further $\rho(\xi_i^{(n)}, \xi_{i+1}^{(n)}) < k_n, \qquad i = 1, 2, \ldots, m_n - 1$,*
with $k_n \to 0, n \to \infty$;

(iv) *there exists a positive number A independent of n such that*

$$\rho(\xi_1^{(n)}, \xi_{m_n}^{(n)}) \geqslant A > 0;$$

(v) *lastly*

$$f(\xi_i^{(n)}) = w, \qquad i = 1, 2, \ldots, m_n;\ n = 1, 2, \ldots.$$

In this case $w \in C_\gamma(f, \tau)$ for all $\gamma \in \Lambda(\tau)$.

Proof. Again we let

$$f(S_n(\zeta)) = g_n(\zeta), \qquad n = 1, 2, \ldots,$$

where

$$z = S_n(\zeta) = \frac{\zeta + \xi_1^{(n)}}{1 + \bar{\xi}_1^{(n)}\zeta}.$$

Let $\{g_{n_k}(\zeta)\}$ be that convergent subsequence with

$$g_{n_k}(\zeta) \to g(\zeta), \qquad k \to \infty.$$

Without loss of generality and for ease of notation we assume that $g_n(\zeta)$ is the desired subsequence.

Since

$$g_n(0) = f(\xi_1^{(n)}) \to w$$

as $n \to \infty$, we have $g(0) = w$. We now show that the set of points $Z_g(w, 0)$ which also lie in $|\zeta| < \tanh A \equiv B$ is infinite. Consequently $g(\zeta) \equiv w$ in D, and we can argue as in the last paragraph of the proof of Lemma 1 to obtain this lemma.

Suppose there is a ring R, $0 < r' \leqslant |\zeta| \leqslant r'' < B$, $r'' > r'$, which contains no points of $Z_g(w, 0)$. Since $g(0) = 0$, we take $r' > 0$. For any fixed n the set

$$\{\xi_i^{(n)}: i = 1, \ldots, m_n\}$$

is transformed by

$$S_n^{-1}(z)$$

onto a set of points we label

$$\{\zeta_i^{(n)}: i = 1, \ldots, m_n\}$$

https://doi.org/10.4153/CJM-1966-028-9 Published online by Cambridge University Press

with

$$(1)\qquad \begin{aligned} &\text{(i)}\quad \zeta_1^{(n)} = 0, \qquad |\zeta_{m_n}^{(n)}| \geqslant B;\\ &\text{(ii)}\quad \rho(\zeta_i^{(n)}, \zeta_{i+1}^{(n)}) < k_n, \qquad i = 1, 2, \ldots, m_n - 1;\\ &\text{(iii)}\quad g_n(\zeta_i^{(n)}) = w, \qquad i = 1, 2, \ldots, m_n. \end{aligned}$$

There must be at most a finite number of such

$$\zeta_i^{(n)}, \qquad i = 1, 2, \ldots, m_n,\ n = 1, 2, \ldots,$$

within R; otherwise this set would have a limit point, say ζ_0, and by the continuous convergence of $g_n(\zeta)$ to $g(\zeta)$ we would have $g(\zeta_0) = w$, contrary to assumption.

Thus there is an index N_0 such that for $n > N_0$ no point of the form

$$\zeta_i^{(n)}, \qquad i = 1, 2, \ldots, m_n,$$

lies in R. If $n_1 > N_0$ is chosen so that

$$k_{n_1} < \rho(0, r'') - \rho(0, r'),$$

this is incompatible with (1) and the definition of R, whence $g(\zeta) \equiv w$.

We proceed to the main theorem. We demonstrate that if $w \in \Pi_{H(\gamma, r)}(f, \tau)$, $\gamma \in \Lambda(\tau)$, either Lemma 1 or Lemma 2 applies and conclude that

$$w \in \Pi_{H(\gamma', r')}(f, \tau) \text{ for all } \gamma' \in \Lambda(\tau) \text{ and any } r' > 0.$$

Consequently, $\Pi_{\mathfrak{A}}(f, \tau) = \Pi_{H(\gamma, r)}(f, \tau)$.

THEOREM 1. *Assume that $f(z)$ is normal in D. Then for any γ, $\gamma' \in \Lambda(\tau)$ and $r, r' > 0$,*

$$\Pi_{H(\gamma, r)}(f, \tau) = \Pi_{H(\gamma', r')}(f, \tau) = \Pi_{\mathfrak{A}}(f, \tau).$$

Proof. For any given fixed curve $\gamma \in \Lambda(\tau)$ and a fixed $r > 0$ let

$$Z'_f(w, 1/n) = Z_f(w, 1/n) \cap H(\gamma, r), \qquad n = 1, 2, \ldots.$$

There are two cases according as $\gamma \cap Z'_f(w, 1/n) = \emptyset$ for some n or

$$\gamma \cap Z'_f(w, 1/n) \neq \emptyset$$

for all n. In the first case we refer to Lemma 1 and the theorem is immediate.

We now consider the second case. For each value of n decompose $Z'_f(w, 1/n)$ into its components

$$\{Y_i^{(n)}: i = 1, \ldots, j_n;\ 1 \leqslant j_n \leqslant \infty\}.$$

We classify a component $Y_i^{(n)}$ as a crosscut if the boundary of this component meets both γ and the boundary of $H(\gamma, r)$. If for each value of n there is at least one component, say $Y_{i_n}^{(n)}$, which is a crosscut, we apply Lemma 2 in the following manner.

Since the boundary of $Y_{i_n}^{(n)}$ meets both γ and the boundary of $H(\gamma, r)$ for each value n, there is a finite set of points

$$\{\xi_j^{(n)}: j = 1, \ldots, h_n\}$$

with

(i) $\xi_1^{(n)} \in H(\rho_\tau, r)$;
(ii) $\xi_1^{(n)} \to \tau, \qquad n \to \infty$;
(iii) $\rho(\xi_j^{(n)}, \xi_{j+1}^{(n)}) < 2/n, \qquad j = 1, 2, \ldots, h_n - 1$;
(iv) $\rho(\xi_1^{(n)}, \gamma) < 1/n, \qquad \rho(\xi_{h_n}^{(n)}, \mathrm{Bd}\, H(\gamma, r)) < 1/n$

and so

$$\rho(\xi_1^{(n)}, \xi_{h_n}^{(n)}) \geqslant r - 2/n;$$

(v) $f(\xi_j^{(n)}) = w, \qquad j = 1, 2, \ldots, h_n, \; n = 1, 2, \ldots.$

These five properties follow easily from the construction of $Y_{i_n}^{(n)}$.

If we choose n_0 so that $2/n_0 < r/2$, then for $n \geqslant n_0$ the requirements of Lemma 2 are satisfied with $A = r/2$ and $k_n = 2/n$ and the theorem is proved.

To conclude the proof, suppose there is an n_0 such that no $Y_i^{(n_0)}$, $i = 1, \ldots, j_{n_0}$, is a crosscut. Let V denote the union of all those components of $Z'_f(w, 1/n_0)$ that meet γ together with the set γ itself. This is a connected set lying entirely within $H(\gamma, r)$ and such that the closure of V meets C only at τ. There is a subset β of the boundary of V which is a boundary arc approaching τ within $H(\gamma, r)$ and, of course, with $\beta \cap Z'_f(w, 1/n_0) = \emptyset$. Since $w \in C_\beta(f, \tau)$, an application of Lemma 1 completes the proof of the theorem.

THEOREM 2. *Let $f(z)$ be a function from D into W. Let $\gamma \in \Lambda(\tau)$, $\tau \in C$, and let $r > 0$ be given. Then there is a countable collection of boundary arcs at τ, $\{\gamma_n\}$, $\gamma_n \subset H(\gamma, r)$, $n = 1, 2, \ldots$, and*

$$\Pi_{H(\gamma, r)}(f, \tau) = \bigcap_{i=1}^{\infty} C_{\gamma_n}(f, \tau).$$

Proof. Let F be the family of all boundary arcs at τ contained in $H(\gamma, r)$. For $\gamma \in F$ let $B_\gamma = W - C_\gamma(f, \tau)$, which is an open set in W. Now

$$\bigcup_{\gamma \in F} B_\gamma = W - \Pi_{H(\gamma, r)}(f, \tau),$$

and by the Lindelöf covering property there is a countable subcovering $\{B_{\gamma_n}\}$ of $W - \Pi_{H(\gamma, r)}(f, \tau)$. Consequently

$$\bigcup_{n=1}^{\infty} B_{\gamma_n} = W - \Pi_{H(\gamma, r)}(f, \tau),$$

whence

$$\bigcap_{n=1}^{\infty} C_{\gamma_n}(f, \tau) = \Pi_{H(\gamma, r)}(f, \tau).$$

If γ is a boundary arc at τ and f is defined in D taking values in W, define

$R_{H(\gamma,r)}(f,\tau)$, $r>0$, as the set of all $w \in W$ such that there is a sequence $\{z_n\}$ in $H(\gamma, r)$, $z_n \to \tau$ as $n \to \infty$, and $f(z_n) = w$, $n = 1, 2, \ldots$. Let int $R_{H(\gamma,r)}(f, \tau)$ denote the interior of $R_{H(\gamma,r)}(f,\tau)$ relative to W.

Finally set

$$\tilde{R}_{H(\gamma,r)}(f,\tau) = \bigcap_{r'>r} \operatorname{int} R_{H(\gamma,r')}(f,\tau).$$

A theorem of Rung (**8**, p. 44) can be formulated as follows:

THEOREM A. *If $f(z)$ is a normal function in D then for any $\tau \in C$, any boundary arc γ at τ, and $r > 0$ we have*

$$C_{H(\gamma,r)}(f,\tau) - \tilde{R}_{H(\gamma,r)}(f,\tau) \subseteq \Pi_{H(\gamma,r)}(f,\tau).$$

(This theorem was originally proved in the case where γ was a rectilinear segment at τ: but it is easily seen that the method of proof yields the more general Theorem A.) In conjunction with Theorem 1, Theorem A yields the following theorem.

THEOREM 3. *If $f(z)$ is normal in D, then, for any $\tau \in C$, any $\gamma \in \Lambda(\tau)$, and any $r > 0$,*

$$C_{H(\gamma,r)}(f,\tau) - \tilde{R}_{H(\gamma,r)}(f,\tau) \subseteq \Pi_{\mathfrak{A}}(f,\tau).$$

3. Examples. If $f(z)$ is merely required to be meromorphic and not normal, Theorem 1 fails. This can be seen by the following example. Let $\{\gamma_n\}$, $\gamma_n \in \Lambda(\tau)$, $n = 1, 2, \ldots$, be a sequence of mutually exclusive boundary arcs at $\tau = 1$, such that for some $r_0 > 0$ each boundary arc $\gamma \subseteq H(\rho_1, r_0)$ intersects each γ_n infinitely often. Such a construction is obviously possible. Let $\{w_n\}$ be a sequence of distinct points that are dense in W. According to Bagemihl and Seidel (**2**, p. 1251), there exists a function holomorphic in D such that

$$C_{\gamma_n}(f, 1) = w_n, \qquad n = 1, 2, \ldots,$$

thus

$$\Pi_{H(\rho_1, r_0)}(f, 1) = W,$$

but

$$\Pi_{H(\gamma_1, r)}(f, 1) \subseteq \{w_1\},$$

for any $r > 0$.

In Theorem 3 (and also in Theorem A), the set $\tilde{R}_{H(\gamma,r)}(f,\tau)$ cannot be replaced by $R_{H(\gamma,r)}(f,\tau)$. To verify this, let $\{z_n\}$ be any sequence of points in D tending to $\tau = 1$ such that, for some $r > 0$, no z_n lies in $H(\rho_1, r)$, but there exists a sequence $\{\zeta_n\}$, contained in $H(\rho_1, r)$, for which $\rho(z_n, \zeta_n) \to 0$ as $n \to \infty$. Cargo has shown (**5**, p. 142) that from the sequence $\{z_n\}$ we can extract a subsequence $\{z_{n_k}\}$ such that the corresponding Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{z_{nk}}{|z_{nk}|}\left(\frac{z - z_{nk}}{1 - z\bar{z}_{nk}}\right)$$

converges in D and

(1) $$|B(z)| \geqslant A > 0, \qquad z \in \rho_1.$$

https://doi.org/10.4153/CJM-1966-028-9 Published online by Cambridge University Press

Since $\rho(z_n, \zeta_n) \to 0, n \to \infty$, then

$$|B(z_{nk}) - B(\zeta_{nk})| \to 0, \qquad k \to \infty,$$

and hence

$$0 \in C_{H(\rho_1, r)}(f, 1) - R_{H(\rho_1, r)}(f, 1),$$

while

$$0 \notin \Pi_{H(\rho_1, r)}(f, 1)$$

on account of (1).

We conclude with a few remarks about the set $\Pi_{\mathfrak{A}}(f, \tau)$, where we assume that f is a normal meromorphic function in D. This set may be empty for all $\tau \in C$. Consider a Schwarzian triangle function $U(z)$ in D whose fundamental triangle has angles of, say, $\pi/2$, $\pi/7$, $\pi/3$. Let its system of triangles be that displayed in (**4**, p. 444, Fig. 122) where we assume $U(0) = \infty$ and that at each vertex of the system, $U(z)$ assumes one of the values 0, 1, ∞. Now $U(z)$ is known to be a meromorphic normal function in D. For any value $w \in W$ the set $Z_U(w, r)$ consists of a countable number of disjoint disks in D for suitably small $r > 0$. Since $U(z)$ assumes the same values in each disk of $Z_U(w, r)$, $U(Z_U(w, r))$ is a neighbourhood N of w in W. Further no point of N is assumed by U in $D - Z_U(w, r)$. Thus if τ is any point of C, then $\Pi_{\mathfrak{A}}(f, \tau) = \emptyset$. Let us now consider the elliptic modular function $\mu(z)$. The above argument shows that $\Pi_{\mathfrak{A}}(\mu, \tau) \subseteq \{0, 1, \infty\}$. However, if τ is a Plessner point of $\mu(z)$ (i.e. the equation

$$C_{H(\gamma, r)}(\mu, \tau) = W$$

is satisfied for any rectilinear segment γ at τ and any $r > 0$), then Theorem 3 shows that $\Pi_{\mathfrak{A}}(\mu, \tau) = \{0, 1, \infty\}$.

If we define

$$\Pi_{\mathrm{T}}(f, \tau) = \bigcap_{\gamma} C_{\gamma}(f, \tau),$$

where γ is any rectilinear segment at τ, Bagemihl (**1**, p. 4) and Rung (**9**, p. 48) showed independently that at a Plessner point τ of a normal function, $\Pi_{\mathrm{T}}(f, \tau) = W$. Thus the set $\Pi_{\mathfrak{A}}(f, \tau)$ may be considerably smaller than $\Pi_{\mathrm{T}}(f, \tau)$.

Finally we construct a normal function f for which $\Pi_{\mathfrak{A}}(f, \tau) = W$ for $\tau = 1$. We begin by defining a simply connected domain H in Im $(z) > 0$. First we construct a sequence of disjoint rectangles in the $z = x + iy$ plane symmetric about the imaginary axis with sides parallel to the real and imaginary axis. In

$$S_1 = \{z | 0 \leqslant \mathrm{Im}\,(z) < 2\pi\},$$

set $R_{1,1}$ equal to the open rectangle symmetric about the line Im $(z) = 2\pi/2$ with the sides parallel to the real axis of length 1 and with the width of $R_{1,1}$ equal to 1/4. In

$$S_2 = \{2\pi \leqslant \mathrm{Im}\,(z) < 2(2\pi)\},$$

let $R_{i,2}$, $i = 1, 2, 3$, denote the open rectangle symmetric about the line $I(z) = 2\pi + 2\pi i/2^2$ of length 2 and width $1/2^3$. For arbitrary $n = 1, 2, \ldots$, let $R_{i,n}$, $i = 1, 2, \ldots, 2^n - 1$, be the open rectangle in

$$S_n = (n - 1)2\pi \leqslant \text{Im}\,(z) < n(2\pi)$$

symmetric about the line $\text{Im}\,(z) = (n - 1)2\pi + 2\pi i/2^n$ of length n and width $1/2^{n+1}$. Now connect each rectangle to the one above as follows. Let $R_{1,1}$ be joined to $R_{1,2}$ by an open set bounded by two parallel straight-line segments of distance 1/4 apart starting from the upper right corner of $R_{1,1}$ to the lower right corner of $R_{1,2}$ and including appropriate boundary segments of the rectangles so that the union of $R_{1,1}$, $R_{1,2}$, and the connecting strip is a simply connected domain. We now join $R_{1,2}$ to $R_{2,2}$ by an open strip with sides parallel to the imaginary axis beginning at the upper left corner of $R_{1,2}$ and terminating at the lower left corner of $R_{2,2}$. Join $R_{2,2}$ to $R_{3,2}$ by a similar strip on the right corners and connect $R_{3,2}$ to $R_{1,3}$ by a rectilinear strip from the left upper corner of $R_{3,2}$ to the left lower corner of $R_{1,3}$. Having joined the adjacent rectangles together by connecting strips in such a manner that the union of all the rectangles $R_{i,n}$ and the connecting strips is a simply connected domain H, we map the unit circle $|\zeta| < 1$ onto H by $z = f(\zeta)$ so that $\zeta = 1$ corresponds to the prime end at ∞ of H. Then $g(\zeta) = \exp f(\zeta)$ clearly omits three values (for example the values $e^{i\pi}$, $e^{i\pi/2}$, $e^{i\pi/4}$) and therefore is normal. If A is any set of points in the $z = x + iy$ plane, let $S_1(A)$ denote the set of those $z \in S_1$ for which $z + 2\pi i n \in A$ for some integer n. Any curve γ^* in H that tends to $z = \infty$ has the property that $S_1(\gamma^*)$ is a dense subset of S. Consequently for any curve γ that approaches $\zeta = 1$ from within $|\zeta| < 1$, we have $C_\gamma(g, 1) = W$.

There exists a function $f(z)$ holomorphic in D (**2**, p. 1251) such that for every $\tau \in C$, any $\gamma_1, \gamma_2 \in \Lambda(\tau)$, and any $r_1, r_2 > 0$,

$$\Pi_{H(\gamma_1, r_1)}(f, \tau) = \Pi_{H(\gamma_2, r_2)}(f, \tau) = W.$$

Thus the conclusion of Theorem 1 may hold without $f(z)$ necessarily being normal, since the above function, having no Fatou points, cannot be normal (**3**, p. 16).

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