ON A THEOREM OF CUTLER

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In [1] Cutler proved the following theorem.

THEOREM. *If* G *and* K *are abelian groups such that* $nG \cong nK$ *for some positive integer* n, *then there are abelian groups* U *and* V *such that* $U \oplus G \cong V \oplus K$ *and* $nU = 0 = nV$.

Cutler's proof is long and fairly involved. Walker [3] obtains the theorem rather elegantly as a corollary of his results on n-extensions. We give here a proof that is extremely simple both in conception and execution. Our proof relies on the notion of p-basic subgroups introduced by Fuchs in [2]. Therefore we shall first recall certain pertinent facts from [2].

Let p be a fixed prime. A subgroup B of an abelian group G is said to be a p-basic subgroup of G if:

(1) B is a direct sum of cyclic groups of infinite and p-power orders;

(2) B is p-pure in G (that is, $p^n G \cap B = p^n B$ for all positive integers n);

(3) G/B is p-divisible (that is, $p(G/B) = G/B$).

Fuchs [2] calls a family $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements of G p-pure independent if (i) the family is independent, (ii) the subgroup generated by the family is p-pure, and (iii) each x_λ has either infinite or p-power order. He then shows (a) that every p-pure-independent family can be expanded to a maximal p-pure-independent family, (b) that the generators of a p-basic subgroup form a maximal p-pure-independent family and, conversely, (c) that the subgroup generated by a maximal p-pure-independent family is a p-basic subgroup. Although we do not require the fact, we mention that any two p-basic subgroups of G are isomorphic. We need two very simple lemmas about p-basic subgroups.

LEMMA 1. *If* B *is a p-basic subgroup of* G, *then* pB *is a p-basic subgroup of* pG.

Proof. pB surely satisfies condition (1). pG/pB is p-divisible since

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$pG/pB = pG/pG \cap B \cong pG + B/B = p(G/B) = G/B$. Finally, pB is p-pure in pG since

$$p^n(pG) \cap pB \subseteq p^{n+1} G \cap B = p^{n+1} B = p^n(pB)$$

for all positive integers n.

LEMMA 2. <u>Suppose C is a subgroup of the abelian group G and that C is a direct sum of cyclic groups of infinite and p-power orders. If pC is a p-basic subgroup of pG, then there is a subgroup A of G such that $pA = 0$ and $A \oplus C$ is a p-basic subgroup of G.</u>

Proof. To insure the existence of an A such that $A \oplus C$ is a p-basic subgroup of G, it suffices by (a) and (c) to show that C is a p-pure subgroup of G. Let $C = \bigoplus_{\mu \in M} \langle c_\mu \rangle$ and suppose $t_1 c_{\mu_1} + \ldots + t_n c_{\mu_n} \in pG$. Since pC is a p-basic subgroup of pG, $p(t_1 c_{\mu_1} + \ldots + t_n c_{\mu_n}) \in p^2 G \cap pC = p^2 C$. Therefore each t_i is divisible by p and $t_1 c_{\mu_1} + \ldots + t_n c_{\mu_n} \in pC$. For $n > 1$ we have

$$p^n G \cap C \subseteq p^{n-1}(pG) \cap pC = p^{n-1}(pC) = p^n C,$$

since pC is p-pure in pG. Now $A \oplus C$ is a p-basic subgroup of G, and $pA \oplus pC$ is a p-basic subgroup of pG by Lemma 1. But then (b) implies that $pA = 0$.

We now turn to the proof of Cutler's theorem. First, we observe, by iteration, that it suffices to prove the theorem in the case n is an arbitrary prime p. Let B be a p-basic subgroup of G and let ϕ be an isomorphism of pG onto pK. Choose C_1 to be a direct sum of cyclic groups of infinite and p-power orders without a p-bounded summand and such that $pC_1 = \phi(pB)$. Then clearly pC_1 is a p-basic subgroup of pK and, by Lemma 2, there is a subgroup A such that $pA = 0$ and $C = A \oplus C_1$ is a p-basic subgroup of K. We can write $B = D \oplus B_1$, where $pD = 0$ and B_1 contains no p-bounded direct summand. Then $pB_1 = pB$ and there is obviously an isomorphism ψ of B_1 onto C_1 that extends $\phi \mid pB$. Clearly then there exist p-bounded abelian groups U and V (one of which can be chosen to be 0) such that $U \oplus B \cong V \oplus C$ under an extension $\bar{\psi}$ of ψ. Then $U \oplus B$ and $V \oplus C$ are p-basic subgroups of $U \oplus G$ and $V \oplus K$ respectively. Since $U \oplus G = (U \oplus B) + pG$ and $V \oplus K = (V \oplus C) + pK$ and since $\bar{\psi}$ and ϕ agree on $(U \oplus B) \cap pG = pB$, there is an obvious isomorphism $\bar{\phi}$ of $U \oplus G$ onto $V \oplus K$ that extends both $\bar{\psi}$ and ϕ.

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REFERENCES

1. D.O. Cutler, Quasi-isomorphism for infinite abelian p-groups. Pacific J. Math. 16 (1966) 25-45.

2. L. Fuchs, Notes on abelian groups II. Acta. Math. Acad. Sci. Hungar. 11 (1960) 117-125.

3. E.A. Walker, On n-extensions of abelian groups. Annales Univ. Sci. Budapest 8 (1965) 71-74.

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