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THE $\mathcal{K}_{up}$-APPROXIMATION PROPERTY AND ITS DUALITY

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Abstract

We introduce an approximation property ($\mathcal{K}_{up}$-AP, $1 \le p < \infty$), which is weaker than the classical approximation property, and discover the duality relationship between the $\mathcal{K}_{up}$-AP and the $\mathcal{K}_p$-AP. More precisely, we prove that for every $1 < p < \infty$, if the dual space X^* of a Banach space X has the $\mathcal{K}_{up}$-AP, then X has the $\mathcal{K}_p$-AP, and if X^* has the $\mathcal{K}_p$-AP, then X has the $\mathcal{K}_{up}$-AP. As a consequence, it follows that every Banach space has the $\mathcal{K}_{u2}$-AP and that for every $1 < p < \infty$, $p \neq 2$, there exists a separable reflexive Banach space failing to have the $\mathcal{K}_{up}$-AP.

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1. Introduction and main results

A Banach space X is said to have the *approximation property* (AP) if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank and continuous linear map (operator) S on X such that $\sup_{x \in K} \|Sx - x\| \le \varepsilon$. Grothendieck [G] proved that a Banach space X has the AP if and only if for every Banach space Y, $\mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}^{\|\cdot\|}$, where $\mathcal{K}(Y, X)$ and $\mathcal{F}(Y, X)$ respectively are the spaces of all compact operators and finite rank operators from Y to X. Oja [O] extended the criterion to an arbitrary Banach operator ideal $[\mathcal{B}, \|\cdot\|_{\mathcal{B}}]$. A Banach space X is said to have the $\mathcal{B}$-*approximation property* ($\mathcal{B}$-AP) if for every Banach space Y, $\mathcal{B}(Y, X) = \overline{\mathcal{F}(Y, X)}^{\|\cdot\|_{\mathcal{B}}}$. Delgado *et al.* [DPS1] investigated the $\mathcal{K}_p$-AP, where $\mathcal{K}_p$ is the ideal of p-compact operators for $1 \le p \le \infty$.

The notion of the p-compact operator was introduced by Sinha and Karn [SK], and stems from Grothendieck's description [G] of compactness in Banach spaces. It was shown in [G] that a subset K of a Banach space X is relatively compact if and only if

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there exists a null sequence (x_n) in X such that

$$K \subset \Big\{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_1}\Big\},$$

where B_Z is the unit ball of a Banach space Z. This criterion was extended in [SK] as follows. For $1 \le p \le \infty$, a subset K of X is said to be relatively *p-compact* if there exists an $(x_n) \in \ell_p(X)$ ($c_0(X)$ if $p = \infty$) such that

$$K \subset p\text{-}co(\{x_n\}) := \Big\{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}}\Big\},$$

where $1/p + 1/p^* = 1$ and $\ell_p(X)$ (respectively, $c_0(X)$) is the Banach space with the norm $\|\cdot\|_p$ (respectively, $\|\cdot\|_\infty$) of all X-valued absolutely p-summable (respectively, null) sequences.

For $1 \le p \le \infty$, a linear map $T : X \to Y$ is said to be *p-compact* if $T(B_X)$ is a relatively p-compact subset of Y. Delgado *et al.* [DPS1] defined a norm on the space $\mathcal{K}_p(X, Y)$ of all p-compact operators from X to Y. For $T \in \mathcal{K}_p(X, Y)$, let

$$\kappa_p(T) := \inf\{\|(y_n)\|_p : (y_n) \in \ell_p(Y) \text{ and } T(B_X) \subset p\text{-}co(\{y_n\})\}.$$

Then $[\mathcal{K}_p, \kappa_p]$ is a Banach operator ideal (see the note preceding [DPS2, Proposition 3.11]).

We need another vector-valued sequence to introduce the main notion of the paper. For $1 \le p \le \infty$, the space $\ell^u_p(X)$, which is a closed subspace of the Banach space $\ell^w_p(X)$ with the norm $\|\cdot\|^w_p$ of all X-valued weakly p-summable sequences, consists of all sequences (x_n) satisfying

$$\|(0, \ldots, 0, x_m, x_{m+1}, \ldots)\|^w_p \longrightarrow 0$$

as $m \to \infty$. In [K1], this sequence was called the *unconditionally p-summable sequence*, and the relatively *unconditionally p-compact* (u-p-compact) set and the u-p-compact operator were defined by replacing the space $\ell_p(X)$, in the definition of p-compactness, by the space $\ell^u_p(X)$. The space of all u-p-compact operators from X to Y is denoted by $\mathcal{K}_{up}(X, Y)$ and the norm u_p on $\mathcal{K}_{up}(X, Y)$ is defined by

$$u_p(T) := \inf\{\|(y_n)\|^w_p : (y_n) \in \ell^u_p(Y) \text{ and } T(B_X) \subset p\text{-}co(\{y_n\})\}.$$

Then $[\mathcal{K}_{up}, u_p]$ is a Banach operator ideal [K1, Theorem 2.1]. The main goal of this paper is to study the $\mathcal{K}_{up}$-AP, and the principal result is the following theorem.

THEOREM 1.1. *Let $1 < p < \infty$. If the dual space X^* of a Banach space X has the $\mathcal{K}_{up}$-AP, then X has the $\mathcal{K}_p$-AP, and if X^* has the $\mathcal{K}_p$-AP, then X has the $\mathcal{K}_{up}$-AP.*

We do not know whether Theorem 1.1 would be also true for the case $p = 1$. We prove Theorem 1.1 in Section 4 after studying the $\mathcal{K}_{up}$-AP and the $\mathcal{K}_p$-AP. First, we present some applications of Theorem 1.1. Since every Banach space has the $\mathcal{K}_2$-AP (see [DPS1, Corollary 3.6]), from Theorem 1.1, we have the following corollaries.

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Corollary 1.2. *Every Banach space has the $\mathcal{K}_{u2}$-AP.*

Corollary 1.3. *For every $1 < p < \infty$, $p \neq 2$, there exists a separable reflexive Banach space failing to have the $\mathcal{K}_{up}$-AP.*

Proof. Let $1 < p < \infty$, $p \neq 2$. Then the dual space S^* of the Szankowski space S [S], failing to have the AP, which is a subspace of ℓ_p, does not have the $\mathcal{K}_{up}$-AP. Indeed, if S^* had the $\mathcal{K}_{up}$-AP, then, by Theorem 1.1, S would have the $\mathcal{K}_p$-AP. Since the $\mathcal{K}_p$-AP is equivalent to the AP for subspaces of ℓ_p (see [O, Theorem 1]), we have a contradiction. □

We do not know whether Corollary 1.3 would be also true for the case $p = 1$. Moreover, we ask:

Problem. Is there any Banach space failing to have the $\mathcal{K}_{u1}$-AP?

The final application shows that the converses of the duality results between the $\mathcal{K}_{up}$-AP and the $\mathcal{K}_p$-AP do not hold in general.

Corollary 1.4. *For every $1 < p < \infty$, $p \neq 2$, there exists a Banach space Y_p (respectively, Z_p) such that Y_p^{**} (respectively, Z_p^{**}) has a boundedly complete basis but Y_p^{***} (respectively, Z_p^{***}) is separable and does not have the $\mathcal{K}_{up}$-AP (respectively, $\mathcal{K}_p$-AP).*

Proof. Let $1 < p < \infty$, $p \neq 2$. Then by Corollary 1.3 there exists a separable reflexive Banach space X_p failing to have the $\mathcal{K}_{up}$-AP. Since X_p^* is separable, by a result of Lindenstrauss [L] (cf. [C, Proposition 1.3]) there exists a Banach space Y_p such that Y_p^{**} has a boundedly complete basis and Y_p^{***} is isomorphic to $Y_p^* \oplus X_p^{**}$. Suppose that Y_p^{***} had the $\mathcal{K}_{up}$-AP. Then we see that X_p would have the $\mathcal{K}_{up}$-AP, which is a contradiction. One may take the dual space X_p^* in the above procedure to show the other part using Theorem 1.1. □

2. Reformulations of the $\mathcal{K}_{up}$-AP and $\mathcal{K}_p$-AP

We define two vector topologies τ_{up} and $\tau_{\kappa p}$ on the space $\mathcal{L}(X, Y)$ of all operators from X to Y by the convergence of nets. Let $1 \leq p \leq \infty$. For a T and a net (T_α) in $\mathcal{L}(X, Y)$, we say that $T_\alpha \xrightarrow{\tau_{up}} T$ if $\lim_\alpha \|((T_\alpha - T)x_n)\|_p^w = 0$ for every $(x_n) \in \ell_p^u(X)$. For $\hat{x} := (x_n) \in \ell_p(X)$ ($c_0(X)$ if $p = \infty$) and $R \in \mathcal{L}(X, Y)$, the p-compact operator $E_{\widehat{Rx}} : \ell_{p^*} \to Y$ is defined by

$$E_{\widehat{Rx}}(\alpha_n) = \sum_n \alpha_n R x_n.$$

We say that $T_\alpha \xrightarrow{\tau_{\kappa p}} T$ if $\lim_\alpha \kappa_p(E_{\widehat{(T_\alpha - T)x}}) = 0$ for every $(x_n) \in \ell_p(X)$. We see that the topologies τ_{up} and $\tau_{\kappa p}$ are Hausdorff locally convex. We denote by τ_c the topology of uniformly compact convergence on $\mathcal{L}(X, Y)$ and recall the topology τ_p of uniformly p-compact convergence for $1 \leq p < \infty$ (see [SK, CK]).

Proposition 2.1. *For every $1 \leq p < \infty$, $\tau_c \geq \tau_{up} \geq \tau_p$ and $\tau_c \geq \tau_{\kappa p} \geq \tau_p$.*

PROOF. τ_{up} is actually the topology of uniformly u-p-compact convergence because, if K is a u-p-compact subset of X, then we may assume that $K = p\text{-}co(\{x_n\})$ for some $(x_n) \in \ell_p^u(X)$, and for every $R \in \mathcal{L}(X, Y)$ it is easily seen that $\sup_{x\in K} \|Rx\| = \|(Rx_n)\|_p^w$. Thus $\tau_{up} \geq \tau_p$ follows.

For every $1 \leq p < \infty$ and $(x_n) \in \ell_p^u(X)$, one may check that the above map $E_{\hat{x}} : \ell_{p^*} \to X$ is a compact operator. Thus every relatively u-p-compact set is relatively compact and so $\tau_c \geq \tau_{up}$ follows.

Now let (T_α) be a net in $\mathcal{L}(X, Y)$. Suppose that $T_\alpha \xrightarrow{\tau_c} 0$. To show that $T_\alpha \xrightarrow{\tau_{\kappa p}} 0$, let $(x_n) \in \ell_p(X)$. Choose a sequence (β_n) of positive numbers with $\beta_n \longrightarrow \infty$ such that $\sum_n \beta_n^p \|x_n\|^p < \infty$. Consider the relatively compact subset $\{x_n/(\beta_n\|x_n\|)\}$ of X. Since for every α, $E_{\widehat{T_\alpha x}}(B_{\ell_{p^*}}) = p\text{-}co(\{T_\alpha x_n\})$,

$$\kappa_p(E_{\widehat{T_\alpha x}}) \leq \|(T_\alpha x_n)\|_p \leq \|(\beta_n\|x_n\|)\|_p \sup_n \|T_\alpha(x_n/(\beta_n\|x_n\|))\| \longrightarrow 0.$$

Hence $T_\alpha \xrightarrow{\tau_{\kappa p}} 0$.

Suppose that $T_\alpha \xrightarrow{\tau_{\kappa p}} 0$. To show that $T_\alpha \xrightarrow{\tau_p} 0$, let $(x_n) \in \ell_p(X)$ and let $\varepsilon > 0$ be given. Let β be such that $\alpha \geq \beta$ implies that $\kappa_p(E_{\widehat{T_\alpha x}}) \leq \varepsilon/2$. Now for every α, there exists a $(y_n^\alpha) \in \ell_p(Y)$ such that $E_{\widehat{T_\alpha x}}(B_{\ell_{p^*}}) \subset p\text{-}co(\{y_n^\alpha\})$ and $\|(y_n^\alpha)\|_p \leq \kappa_p(E_{\widehat{T_\alpha x}}) + \varepsilon/2$. Hence $\alpha \geq \beta$ implies that

$$\|(T_\alpha x_n)\|_p^w \leq \|(y_n^\alpha)\|_p \leq \kappa_p(E_{\widehat{T_\alpha x}}) + \frac{\varepsilon}{2} \leq \varepsilon. \qquad \square$$

The purpose of this section is to characterize the $\mathcal{K}_{up}$-AP (respectively, $\mathcal{K}_p$-AP) in terms of the topology τ_{up} (respectively, $\tau_{\kappa p}$). The following lemma is well known and easily verified by a standard argument.

LEMMA 2.2. *Let K be a collection of sequences of positive numbers. If*

$$\lim_l \sup_{(k_j)\in K} \sum_{j\geq l} k_j = 0,$$

then there exists a sequence (b_j) of real numbers with $b_j \nearrow \infty$ and $b_j > 1$ for all j such that

$$\lim_l \sup_{(k_j)\in K} \sum_{j\geq l} k_j b_j = 0.$$

THEOREM 2.3. *Let $T \in \mathcal{L}(X) := \mathcal{L}(X, X)$ and let $1 \leq p < \infty$. The following statements are equivalent.*

(a) $T \in \overline{\mathcal{F}(X)}^{\tau_{up}}$.

(b) *For every Banach space Y and every $R \in \mathcal{K}_{up}(Y, X)$,*

$$TR \in \overline{\{SR : S \in \mathcal{F}(X)\}}^{u_p}.$$

(c) *For every quotient space Y of ℓ_{p^*} and every injective u-p-compact operator $R : Y \to X$, we have $TR \in \overline{\mathcal{F}(Y, X)}^{\tau_{up}}$.*

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PROOF. (b) implies (c) is trivial.

(a) implies (b). Let Y be a Banach space and let $R \in \mathcal{K}_{up}(Y, X)$. Let $\varepsilon > 0$ be given. Then there exists an $(x_n) \in \ell_p^u(X)$ such that $R(B_Y) \subset p\text{-}co(\{x_n\})$. By (a) there exists an $S \in \mathcal{F}(X)$ such that

$$\|((S - T)x_n)\|_p^w \le \varepsilon.$$

Since $(SR - TR)(B_Y) \subset p\text{-}co(\{(S - T)x_n\})$ and $((S - T)x_n) \in \ell_p^u(X)$,

$$u_p(SR - TR) \le \|((S - T)x_n)\|_p^w \le \varepsilon.$$

Hence $TR \in \overline{\{SR : S \in \mathcal{F}(X)\}}^{u_p}$.

(c) implies (a). This proof is essentially due to that of [DOPS, Theorem 2.1]. Let $(x_n) \in \ell_p^u(X)$ and let $\varepsilon > 0$ be given. Then by Lemma 2.2 there exists a sequence (β_n) of positive numbers with $\beta_n \longrightarrow 0$ such that $(z_n) := (x_n/\beta_n) \in \ell_p^u(X)$. Define the operators $D_\beta : \ell_{p^*} \to \ell_{p^*}$ and $E_z : \ell_{p^*} \to X$ by $D_\beta\alpha = (\alpha_n\beta_n)$ and $E_z\alpha = \sum_n \alpha_n z_n$, respectively. The injective operator $\hat{E}_z : \ell_{p^*}/\ker(E_z) \to X$ is defined by $\hat{E}_z[\alpha] = E_z\alpha$. A simple verification shows that the operators D_β and $\hat{E}_z$ are u-p-compact. Let $\pi : \ell_{p^*} \to \ell_{p^*}/\ker(E_z)$ be the quotient operator. Then $\pi D_\beta(B_{\ell_{p^*}})$ is a relatively u-p-compact subset of $\ell_{p^*}/\ker(E_z)$:

$$\ell_{p^*} \xrightarrow{D_\beta} \ell_{p^*} \xrightarrow{\pi} \ell_{p^*}/\ker(E_z) \xrightarrow{\hat{E}_z} X.$$

Then by (c) there exists an $S \in \mathcal{F}(\ell_{p^*}/\ker(E_z), X)$ such that

$$\sup_{y\in\pi D_\beta(B_{\ell_{p^*}})} \|Sy - T\hat{E}_z y\| \le \frac{\varepsilon}{2}.$$

We may write $S = \sum_{k=1}^m y_k^* \otimes x_k$, where $y_k^* \in (\ell_{p^*}/\ker(E_z))^*$, $x_k \in X$ for each $k = 1, \ldots, m$ and $\sum_{k=1}^m \|x_k\| = 1$. Since $\hat{E}_z$ is injective, $(\ell_{p^*}/\ker(E_z))^* = \overline{\hat{E}_z^{\,*}(X^*)}^{weak^*} = \overline{\hat{E}_z^{\,*}(X^*)}^{\tau_c}$. The second equality follows from $(Z^*, weak^*)^* = (Z^*, \tau_c)^*$ for every Banach space Z (cf. [M, Theorem 2.7.8]). Thus for each $k = 1, \ldots, m$, there exists an $x_k^* \in X^*$ such that

$$\sup_{y\in\pi D_\beta(B_{\ell_{p^*}})} |y_k^*(y) - \hat{E}_z^{\,*}(x_k^*)(y)| \le \frac{\varepsilon}{2}.$$

Consider the operator $\sum_{k=1}^m x_k^* \otimes x_k \in \mathcal{F}(X)$. Then as in the proof of [K2, Theorem 5.5(d) implies (a)], for every $(\alpha_n) \in B_{\ell_{p^*}}$,

$$\left\|\sum_{k=1}^m x_k^*\Big(\sum_n \alpha_n x_n\Big)x_k - T\Big(\sum_n \alpha_n x_n\Big)\right\| \le \varepsilon.$$

Hence $T \in \overline{\mathcal{F}(X)}^{\tau_{up}}$. □

THEOREM 2.4. *Let $T \in \mathcal{L}(X)$ and let $1 \le p < \infty$. The following statements are equivalent.*

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(a) $T \in \overline{\mathcal{F}(X)}^{\tau_{\kappa p}}$.

(b) *For every Banach space Y and every* $R \in \mathcal{K}_p(Y, X)$,

$$TR \in \overline{\{SR : S \in \mathcal{F}(X)\}}^{\kappa_p}.$$

(c) *For every quotient space Y of* ℓ_{p^*} *and every injective p-compact operator* $R : Y \to X$, *we have* $TR \in \overline{\mathcal{F}(Y, X)}^{\kappa_p}$.

Proof. (b) implies (c) is trivial.

(a) implies (b). Let Y be a Banach space and let $R \in \mathcal{K}_p(Y, X)$. Let $\varepsilon > 0$ be given. There exists an $(x_n) \in \ell_p(X)$ such that $R(B_Y) \subset p\text{-}co(\{x_n\})$. By (a) there exists an $S \in \mathcal{F}(X)$ such that $\kappa_p(E_{\widehat{(S-T)x}}) \le \varepsilon/2$. Now let $(z_n) \in \ell_p(X)$ such that $E_{\widehat{(S-T)x}}(B_{\ell_{p^*}}) \subset p\text{-}co(\{z_n\})$ and $\|(z_n)\|_p \le \kappa_p(E_{\widehat{(S-T)x}}) + \varepsilon/2$. Since $(SR - TR)(B_Y) \subset p\text{-}co(\{(S-T)x_n\}) \subset p\text{-}co(\{z_n\})$,

$$\kappa_p(SR - TR) \le \|(z_n)\|_p \le \kappa_p(E_{\widehat{(S-T)x}}) + \frac{\varepsilon}{2} \le \varepsilon.$$

Hence $TR \in \overline{\{SR : S \in \mathcal{F}(X)\}}^{\kappa_p}$.

(c) implies (a). This proof comes from a combination of those of [DOPS, Theorem 2.1] and [DPS1, Proposition 2.1]. Let $(x_n) \in \ell_p(X)$ and let $\varepsilon > 0$ be given. We should find an $S \in \mathcal{F}(X)$ such that $\kappa_p(E_{\widehat{(S-T)x}}) \le \varepsilon$. Choose a sequence (β_n) of positive numbers with $\beta_n \le 1$ and $\beta_n \longrightarrow 0$ such that $(z_n) := (x_n/\beta_n) \in \ell_p(X)$.

Now let $D_\beta : \ell_{p^*} \to \ell_{p^*}$, $E_z : \ell_{p^*} \to X$, $\hat{E}_z : \ell_{p^*}/\ker(E_z) \to X$, and $\pi : \ell_{p^*} \to \ell_{p^*}/\ker(E_z)$ be the operators in the proof of Theorem 2.3(c) implies (a). Since the map $\hat{E}_z$ is an injective p-compact operator, by (c) there exists a $U \in \mathcal{F}(\ell_{p^*}/\ker(E_z), X)$ such that

$$\kappa_p(U - T\hat{E}_z) \le \frac{\varepsilon}{2}.$$

Put $U = \sum_{k=1}^m y_k^* \otimes x_k$, where $y_k^* \in (\ell_{p^*}/\ker(E_z))^*$ and $x_k \in X$ for each $k = 1, \ldots, m$. We may assume that $(\sum_{k=1}^m \|x_k\|^p)^{1/p} = \varepsilon/2$. Since $\hat{E}_z$ is injective, $(\ell_{p^*}/\ker(E_z))^* = \overline{\hat{E}_z^{\,*}(X^*)}^{\tau_c}$. Thus for each $k = 1, \ldots, m$, there exists an $x_k^* \in X^*$ such that

$$\sup_{y \in \pi D_\beta(B_{\ell_{p^*}})} |y_k^*(y) - \hat{E}_z^{\,*}(x_k^*)(y)| \le \frac{1}{m^{1/p^*}}.$$

We show that $S := \sum_{k=1}^m x_k^* \otimes x_k$ is the desired operator.

Now, for every $(\alpha_n) \in B_{\ell_{p^*}}$,

$$(S\hat{E}_z\pi D_\beta - U\pi D_\beta)(\alpha_n) = \sum_{k=1}^m (((\hat{E}_z^{\,*} x_k^*)\pi D_\beta - y_k^*\pi D_\beta)(\alpha_n))x_k$$

and

$$\sum_{k=1}^m |((\hat{E}_z^{\,*} x_k^*)\pi D_\beta - y_k^*\pi D_\beta)(\alpha_n)|^{p^*} \le 1.$$

Thus $(S\hat{E}_z\pi D_\beta - U\pi D_\beta)(B_{\ell_{p^*}}) \subset p\text{-}co(\{x_1, \ldots, x_m, 0, \ldots\})$ and so

$$\kappa_p(S\hat{E}_z\pi D_\beta - U\pi D_\beta) \le \Big(\sum_{k=1}^{m} \|x_k\|^p\Big)^{1/p} = \frac{\varepsilon}{2}.$$

Hence we have

$$\begin{aligned}\kappa_p(E_{\widehat{(S-T)x}}) &= \kappa_p(SE_zD_\beta - TE_zD_\beta)\\ &= \kappa_p(S\hat{E}_z\pi D_\beta - T\hat{E}_z\pi D_\beta)\\ &\le \kappa_p(S\hat{E}_z\pi D_\beta - U\pi D_\beta) + \kappa_p(U\pi D_\beta - T\hat{E}_z\pi D_\beta)\\ &\le \frac{\varepsilon}{2} + \kappa_p(U - T\hat{E}_z) \le \varepsilon.\end{aligned}$$ □

Remark 2.5. Let $T = id_X$, the identity map, in Theorems 2.3 and 2.4. It follows from Proposition 2.1 that

$$\text{AP} \Longrightarrow \mathcal{K}_{up}\text{-AP}, \quad \mathcal{K}_p\text{-AP} \Longrightarrow p\text{-AP}.$$

In view of Theorem 2.3(c), we also see that every Banach space has the $\mathcal{K}_{u2}$-AP because every Hilbert space has the AP.

3. Dual spaces of $\mathcal{L}(X, Y)$

The purpose of this section is to establish some representations of dual spaces of $\mathcal{L}(X, Y)$ endowed with the topologies τ_{up} and $\tau_{\kappa p}$, which are crucial tools in the proof of Theorem 1.1. We need the following lemma, which is a consequence of [DPS1, Proposition 3.3] and [P, Corollary 1], to obtain a representation of $(\mathcal{L}(X, Y), \tau_{\kappa p})^*$.

Lemma 3.1. *Let $1 < p < \infty$. The space $\mathcal{N}_{p^*}(\ell_p, X^*)$ of p^*-nuclear operators from ℓ_p to X^* is isometrically isomorphic to $(\mathcal{K}_p(\ell_{p^*}, X), \kappa_p)^*$ via $S \mapsto \operatorname{trace}(R^*S)$ for all $R \in \mathcal{K}_p(\ell_{p^*}, X)$.*

Note that $f \in (\mathcal{L}(X, Y), \tau_{\kappa p})^*$ if and only if there exist $C > 0$ and $(x_n) \in \ell_p(X)$ such that $|f(T)| \le C\kappa_p(E_{\widehat{Tx}})$ for every $T \in \mathcal{L}(X, Y)$.

Theorem 3.2. *Let $1 < p < \infty$. Then the dual space $(\mathcal{L}(X, Y), \tau_{\kappa p})^*$ consists of all functionals of the form*

$$f(T) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j y_j^*(Tx_n),$$

where $(x_n) \in \ell_p(X)$, $((\lambda_n^j)_n)_{j=1}^{\infty} \in \ell_{p^*}(\ell_{p^*})$ and $(y_j^*) \in \ell_p^w(Y^*)$.

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PROOF. Suppose that f is of the above form. Let $T \in \mathcal{L}(X, Y)$. Consider the operators $\sum_j (\lambda_n^j)_n \otimes y_j^* \in \mathcal{N}_{p^*}(\ell_p, Y^*)$ and $\sum_n e_n \otimes Tx_n = E_{\widehat{Tx}} \in \mathcal{K}_p(\ell_{p^*}, Y)$. Then by Lemma 3.1,

$$|f(T)| = \left|\operatorname{trace}\left(\left(\sum_n e_n \otimes Tx_n\right)^* \left(\sum_j (\lambda_n^j)_n \otimes y_j^*\right)\right)\right|$$
$$\leq \nu_{p^*}\left(\sum_j (\lambda_n^j)_n \otimes y_j^*\right)\kappa_p(E_{\widehat{Tx}}).$$

Hence $f \in (\mathcal{L}(X, Y), \tau_{\kappa p})^*$.

Conversely, suppose that $f \in (\mathcal{L}(X, Y), \tau_{\kappa p})^*$. Then there exist $C > 0$ and $(x_n) \in \ell_p(X)$ such that $|f(T)| \leq C\kappa_p(E_{\widehat{Tx}})$ for every $T \in \mathcal{L}(X, Y)$. Consider the linear subspace $\mathcal{Y} := \{E_{\widehat{Tx}} : T \in \mathcal{L}(X, Y)\}$ of $\mathcal{K}_p(\ell_{p^*}, Y)$ and the functional φ on $\mathcal{Y}$ given by $\varphi(E_{\widehat{Tx}}) = f(T)$. We see that φ is well defined and linear, and $\|\varphi\|_{(\mathcal{Y},\kappa_p)^*} \leq C$. Thus there exists a Hahn–Banach extension $\hat{\varphi} \in (\mathcal{K}_p(\ell_{p^*}, Y), \kappa_p)^*$ of φ such that $f(T) = \varphi(E_{\widehat{Tx}}) = \hat{\varphi}(E_{\widehat{Tx}})$ for every $T \in \mathcal{L}(X, Y)$. By Lemma 3.1 there exist $((\lambda_n^j)_n)_{j=1}^\infty \in \ell_{p^*}(\ell_{p^*})$ and $(y_j^*) \in \ell_p^w(Y^*)$ such that for every $T \in \mathcal{L}(X, Y)$,

$$f(T) = \hat{\varphi}(E_{\widehat{Tx}}) = \operatorname{trace}\left((E_{\widehat{Tx}})^*\left(\sum_j (\lambda_n^j)_n \otimes y_j^*\right)\right) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j y_j^*(Tx_n). \qquad \square$$

We can use the proof of [CK, Theorem 2.5] by replacing $\ell_p(X)$ by $\ell_p^u(X)$ to obtain the following representation of $(\mathcal{L}(X, Y), \tau_{up})^*$.

THEOREM 3.3. *Let $1 < p < \infty$. Then the dual space $(\mathcal{L}(X, Y), \tau_{up})^*$ consists of all functionals of the form*

$$f(T) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j y_j^*(Tx_n),$$

where $(x_n) \in \ell_p^u(X)$, $z_j := (\lambda_n^j)_{n=1}^\infty \in \ell_{p^*}$ for each $j \in \mathbf{N}$ and (y_j^*) in Y^* with $\sum_{j=1}^\infty \|z_j\|_{p^*}\|y_j^*\| < \infty$.

COROLLARY 3.4. *Let $1 < p < \infty$. Then the dual space $(\mathcal{L}(X, Y), \tau_{up})^*$ consists of all functionals of the form*

$$f(T) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j y_j^*(Tx_n),$$

where $(x_n) \in \ell_p^w(X)$, $((\lambda_n^j)_n)_{j=1}^\infty \in \ell_{p^*}(\ell_{p^*})$ and $(y_j^*) \in \ell_p(Y^*)$.

PROOF. Let f be of the above form. Since $\sum_{n=1}^\infty \sum_{j=1}^\infty |\lambda_n^j|^{p^*} < \infty$, there exists a sequence $(\beta_n)_n$ of positive numbers with $\beta_n \longrightarrow 0$ such that

$$\sum_{n=1}^\infty \left(\sum_{j=1}^\infty |\lambda_n^j|^{p^*}\right) \Big/ \beta_n^{p^*} < \infty.$$

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Then we see that $(\beta_n x_n) \in \ell_p^u(X)$ and $((\lambda_n^j/\beta_n)_n)_{j=1}^\infty \in \ell_{p^*}(\ell_{p^*})$. For each $j \in \mathbf{N}$, put $z_j := (\lambda_n^j/\beta_n)_n$. Since $\sum_{j=1}^\infty \|z_j\|_{p^*}\|y_j^*\| < \infty$, we obtain a representation of f in Theorem 3.3.

Now, let f be of the form in Theorem 3.3. We may assume that $\sum_{j=1}^\infty \|z_j\|_{p^*} < \infty$ and $\|y_j^*\| = 1$ for every $j \in \mathbf{N}$. Consider

$$\|z_j\|_{p^*}^{-1/p} z_j \quad \text{and} \quad \|z_j\|_{p^*}^{1/p} y_j^*$$

for each $j \in \mathbf{N}$. Then it follows that $(\|z_j\|_{p^*}^{-1/p} z_j) \in \ell_{p^*}(\ell_{p^*})$ and $(\|z_j\|_{p^*}^{1/p} y_j^*) \in \ell_p(Y^*)$. Hence we obtain the desired representation of f. □

We can also use the proof of Corollary 3.4 using [CK, Theorem 2.5] to obtain an analogue of Corollary 3.4 for $(\mathcal{L}(X,Y), \tau_p)^*$ by only replacing $\ell_p^w(X)$ by $\ell_p(X)$.

4. Proof of Theorem 1.1

We need the following lemmas which were motivated by [K2, Theorem 3.1].

LEMMA 4.1. *Let $1 < p < \infty$. Let $(S_\beta)_{\beta\in I}$ be a net in $\mathcal{L}(X,Y)$ and let $T \in \mathcal{L}(X,Y)$. If $S_\beta^* \xrightarrow{\tau_{up}} T^*$ in $\mathcal{L}(Y^*,X^*)$, then there exists a net (T_α) in the convex hull $co(\{S_\beta\}_{\beta\in I})$ of the set $\{S_\beta\}_{\beta\in I}$ such that*

$$T_\alpha \xrightarrow{\tau_{\kappa p}} T \quad \text{and} \quad T_\alpha^* \xrightarrow{\tau_{up}} T^*.$$

PROOF. If $S_\beta^* \xrightarrow{\tau_{up}} T^*$ in $\mathcal{L}(Y^*,X^*)$, then $g(S_\beta^*) \longrightarrow g(T^*)$ for every $g \in (\mathcal{L}(Y^*,X^*), \tau_{up})^*$. Let $\overline{\tau_{\kappa p}}$ be the topology on $\mathcal{L}(X,Y)$ induced by $(\mathcal{L}(X,Y), \tau_{\kappa p})^*$ in Theorem 3.2. We show that $S_\beta \xrightarrow{\overline{\tau_{\kappa p}}} T$. Then by passing to convex combinations, we complete the proof.

Let $f \in (\mathcal{L}(X,Y), \tau_{\kappa p})^*$ in Theorem 3.2. Then there exist $(x_n) \in \ell_p(X)$, $((\lambda_n^j)_n)_{j=1}^\infty \in \ell_{p^*}(\ell_{p^*})$ and $(y_j^*) \in \ell_p^w(Y^*)$ such that

$$f(R) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j y_j^*(Rx_n)$$

for every $R \in \mathcal{L}(X,Y)$. By Corollary 3.4,

$$g := \sum_{n=1}^\infty \sum_{j=1}^\infty \lambda_n^j i_X(x_n)(\,\cdot\, y_j^*) \in (\mathcal{L}(Y^*,X^*), \tau_{up})^*,$$

where $i_X : X \to X^{**}$ is the canonical isometry. Hence we have

$$f(S_\beta) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j y_j^*(S_\beta x_n) = g(S_\beta^*) \longrightarrow g(T^*) = f(T). \qquad \square$$

LEMMA 4.2. *Let $1 < p < \infty$. Let $(S_\beta)_{\beta\in I}$ be a net in $\mathcal{L}(X,Y)$ and let $T \in \mathcal{L}(X,Y)$. If $S_\beta^* \xrightarrow{\tau_{\kappa p}} T^*$ in $\mathcal{L}(Y^*,X^*)$, then there exists a net (T_α) in $co(\{S_\beta\})_{\beta\in I}$ such that*

$$T_\alpha \xrightarrow{\tau_{up}} T \quad \text{and} \quad T_\alpha^* \xrightarrow{\tau_{\kappa p}} T^*.$$

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PROOF. If $S_\beta^* \xrightarrow{\tau_{\kappa p}} T^*$ in $\mathcal{L}(Y^*, X^*)$, then $g(S_\beta^*) \longrightarrow g(T^*)$ for every $g \in (\mathcal{L}(Y^*, X^*), \tau_{\kappa p})^*$. Let $\overline{\tau_{up}}$ be the topology on $\mathcal{L}(X, Y)$ induced by $(\mathcal{L}(X, Y), \tau_{up})^*$ in Corollary 3.4. We should show that $S_\beta \xrightarrow{\overline{\tau_{up}}} T$. Let $f \in (\mathcal{L}(X, Y), \tau_{up})^*$. Then there exist $(x_n) \in \ell_p^w(X)$, $((\lambda_n^j)_n)_{j=1}^\infty \in \ell_{p^*}(\ell_{p^*})$ and $(y_j^*) \in \ell_p(Y^*)$ such that

$$f(R) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j y_j^*(Rx_n)$$

for every $R \in \mathcal{L}(X, Y)$. Since $((\lambda_n^j)_j)_{n=1}^\infty \in \ell_{p^*}(\ell_{p^*})$, by Theorem 3.2,

$$g := \sum_{n=1}^\infty \sum_{j=1}^\infty \lambda_n^j i_X(x_n)(\,\cdot\, y_j^*) \in (\mathcal{L}(Y^*, X^*), \tau_{\kappa p})^*.$$

Hence

$$f(S_\beta) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j y_j^*(S_\beta x_n) = g(S_\beta^*) \longrightarrow g(T^*) = f(T). \qquad \square$$

We are now ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. We only prove the first part because the second part has essentially the same proof. If X^* has the $\mathcal{K}_{up}$-AP, then, by Theorem 2.3, $id_{X^*} \in \overline{\mathcal{F}(X^*)}^{\tau_{up}}$. Since $\mathcal{F}(X^*) \subset \overline{\{S^* : S \in \mathcal{F}(X)\}}^{\tau_c}$ (cf. [LT, Lemma 1.e.17]) and $\tau_c \geq \tau_{up}$ (see Proposition 2.1), we have $id_{X^*} \in \overline{\{S^* : S \in \mathcal{F}(X)\}}^{\tau_{up}}$. By Lemma 4.1, $id_X \in \overline{\mathcal{F}(X)}^{\tau_{\kappa p}}$. It follows from Theorem 2.4 that X has the $\mathcal{K}_p$-AP. $\square$

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References

[C] P. G. Casazza, 'Approximation properties', in: *Handbook of the Geometry of Banach Spaces*, Vol. 1 (eds. W. B. Johnson and J. Lindenstrauss) (Elsevier, Amsterdam, 2001), 271–316.

[CK] Y. S. Choi and J. M. Kim, 'The dual space of $(\mathcal{L}(X, Y), \tau_p)$ and the p-approximation property', *J. Funct. Anal.* **259** (2010), 2437–2454.

[DOPS] J. M. Delgado, E. Oja, C. Piñeiro and E. Serrano, 'The p-approximation property in terms of density of finite rank operators', *J. Math. Anal. Appl.* **354** (2009), 159–164.

[DPS1] J. M. Delgado, C. Piñeiro and E. Serrano, 'Density of finite rank operators in the Banach space of p-compact operators', *J. Math. Anal. Appl.* **370** (2010), 498–505.

[DPS2] J. M. Delgado, C. Piñeiro and E. Serrano, 'Operators whose adjoints are quasi p-nuclear', *Studia Math.* **197** (2010), 291–304.

[G] A. Grothendieck, 'Produits tensoriels topologiques et espaces nucléaires', *Mem. Amer. Math. Soc.* **16** (1955).

[K1] J. M. Kim, 'Unconditionally p-null sequences and unconditionally p-compact operators', *Studia Math.*, to appear.

[K2] J. M. Kim, 'The approximation properties via the Grothendieck p-compact sets', *Math. Nachr.* **286** (2013), 360–373.

[L] J. Lindenstrauss, ‘On James’ paper “Separable conjugate spaces”’, *Israel J. Math.* **9** (1971), 279–284.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces* (Springer, Berlin, 1977).

[M] R. E. Megginson, *An Introduction to Banach Space Theory* (Springer, New York, 1998).

[O] E. Oja, ‘A remark on the approximation of p-compact operators by finite-rank operators’, *J. Math. Anal. Appl.* **387** (2012), 949–952.

[P] A. Persson, ‘On some properties of p-nuclear and p-integral operators’, *Studia Math.* **33** (1969), 213–222.

[SK] D. P. Sinha and A. K. Karn, ‘Compact operators whose adjoints factor through subspaces of l_p’, *Studia Math.* **150** (2002), 17–33.

[S] A. Szankowski, ‘Subspaces without the approximation property’, *Israel J. Math.* **30** (1978), 123–129.

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