The $f$-Chromatic Index of a Graph Whose $f$-Core Has Maximum Degree 2

S. Akbari, M. Chavooshi, M. Ghanbari, and S. Zare

Abstract. Let $G$ be a graph. The minimum number of colors needed to color the edges of $G$ is called the chromatic index of $G$ and is denoted by $\chi'(G)$. It is well known that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, for any graph $G$, where $\Delta(G)$ denotes the maximum degree of $G$. A graph $G$ is said to be class 1 if $\chi'(G) = \Delta(G)$ and class 2 if $\chi'(G) = \Delta(G) + 1$. Also, $G_{\Delta}$ is the induced subgraph on all vertices of degree $\Delta(G)$. Let $f: V(G) \to \mathbb{N}$ be a function. An $f$-coloring of a graph $G$ is a coloring of the edges of $E(G)$ such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. The minimum number of colors needed to $f$-color $G$ is called the $f$-chromatic index of $G$ and is denoted by $\chi^f(G)$. It was shown that for every graph $G$, $\Delta_f(G) \leq \chi^f_1(G) \leq \Delta_f(G) + 1$, where $\Delta_f(G) = \max_{v \in V(G)} \left\lceil \frac{d_G(v)}{f(v)} \right\rceil$. A graph $G$ is said to be $f$-class 1 if $\chi^f_1(G) = \Delta_f(G)$, and $f$-class 2, otherwise. Also, $G_{\Delta_f}$ is the induced subgraph of $G$ on $\{v \in V(G) : \frac{d_G(v)}{f(v)} = \Delta_f(G)\}$. Hilton and Zhao showed that if $G_{\Delta}$ has maximum degree two and $G$ is class 2, then $G$ is critical, $G_{\Delta}$ is a disjoint union of cycles and $\delta(G) = \Delta(G) - 1$, where $\delta(G)$ denotes the minimum degree of $G$, respectively. In this paper, we generalize this theorem to $f$-coloring of graphs. Also, we determine the $f$-chromatic index of a connected graph $G$ with $|G_{\Delta_f}| \leq 4$.

1 Introduction

All graphs considered in this paper are simple and finite. Let $G$ be a graph. The number of vertices of $G$ is called the order of $G$ and is denoted by $|G|$. Also, $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. The degree of vertex $v$ in $G$ is denoted by $d_G(v)$, and $N_G(v)$ denotes the set of all vertices adjacent to $v$. Also, let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of $G$, respectively. A graph $G$ is said to be connected if any two vertices are connected by a path in $G$. If $G$ is not connected, then $G$ is decomposed into connected components that are the maximal connected subgraphs of $G$. A star graph is a graph containing a vertex adjacent to all other vertices and with no other edges. A matching in a graph $G$ is a set of pairwise non-adjacent edges. An edge cut is a set of edges whose removal produces a subgraph with more connected components than the original graph. Moreover, a graph is $k$-edge connected if the minimum number of edges whose removal would disconnect the graph is at least $k$. For a subset $X \subseteq V(G)$, we denote the induced subgraph of $G$ on $X$ by $G[X]$. By $G \setminus H$ we mean the induced subgraph on $V(G) \setminus V(H)$. Also, $G_{\Delta}$ is the induced subgraph on all vertices of degree $\Delta(G)$. For two subgraphs $S$ and $T$ of $G$, where $V(S) \cap V(T) = \emptyset$, $e_G(S, T)$ denotes the number of edges with one end in $S$ and other end in $T$. An edge coloring of a graph in which no two adjacent edges have the same color is called a proper edge coloring. The minimum number of colors needed to color the edges of

Received by the editors June 15, 2011; revised November 14, 2012.
Published electronically March 20, 2013.
AMS subject classification: 05C15, 05C38.
Keywords: $f$-coloring, $f$-core, $f$-class 1.

https://doi.org/10.4153/CMB-2012-046-3 Published online by Cambridge University Press
G properly is called the chromatic index of G and is denoted by \( \chi'(G) \). Vizing [10] proved that \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \), for any graph G. A graph G is said to be class 1 if \( \chi'(G) = \Delta(G) \) and class 2 if \( \chi'(G) = \Delta(G) + 1 \). A graph G is called critical if G is connected, class 2 and \( \chi'(G \setminus e) < \chi'(G) \), for every edge \( e \in E(G) \) and is called overfull when \( |E(G)| > \Delta(G) \lceil \frac{\Delta(G)}{\Delta(G) + 1} \rceil \).

For a function \( f \) that assigns a positive integer \( f(v) \) to each vertex \( v \in V(G) \), an \( f \)-coloring of G is an edge coloring of G such that each vertex v has at most \( f(v) \) edges colored with the same color. The minimum number of colors needed to \( f \)-color G is called the \( f \)-chromatic index of G, and denoted by \( \chi'_f(G) \). If \( f(v) = 1 \) for all \( v \in V(G) \), then \( f \)-coloring is equivalent to proper edge coloring. Let \( \Delta_f(G) = \max_{v \in V(G)} \lceil \frac{d_G(v)}{f(v)} \rceil \). A graph G is said to be \( f \)-class 1 if \( \chi'_f(G) = \Delta_f(G) \) and \( f \)-class 2, otherwise. Also, we say that G has a \( f \)-coloring if G is \( f \)-class 1. A vertex \( v \) is called an \( f \)-maximum vertex if \( d_G(v) = \Delta_f(G) f(v) \). The \( f \)-core of a graph G is the induced subgraph of G on the \( f \)-maximum vertices and denoted by \( G_{\Delta_f} \). A graph G is called \( f \)-overfull when \( |E(G)| > \Delta_f(G) \lceil \frac{\Delta_f(G)}{\Delta_f(G) + 1} \rceil \), where \( f(V) = \sum_{v \in V(G)} f(v) \), and is called \( f \)-critical if G is connected, \( f \)-class 2 and \( \chi'_f(G \setminus e) < \chi'_f(G) \), for every \( e \in E(G) \). The following example introduces an \( f \)-class 1 graph.

**Example 1** Let G be a graph shown in the following figure such that \( f(v_1) = f(v_2) = 2 \) and \( f(v_i) = 1 \) for \( i = 3, \ldots, 7 \). It is easy to see that \( \Delta_f(G) = 2 \), \( G_{\Delta_f} = K_3 \), and G is \( f \)-class 1.

![Figure 1](image)

There are interesting real-life applications of \( f \)-colorings in optimization and network design, such as file transfers in a computer network [4, 5, 9]. Since the classical edge-coloring problem is NP-complete [7], the \( f \)-coloring problem which asks to \( f \)-color a given multigraph G with \( \chi'_f(G) \) colors is also NP-complete.

In [5], Hakimi and Kariv obtained the following results.

**Theorem 1** Let G be a graph. Then

\[
\Delta_f(G) \leq \chi'_f(G) \leq \max_{v \in V(G)} \left\lceil \frac{d_G(v)}{f(v)} + 1 \right\rceil \leq \Delta_f(G) + 1.
\]

**Theorem 2** Let G be a bipartite graph. Then G is \( f \)-class 1.
Theorem 3 Let $G$ be a graph, and let $f(v)$ be even for all $v \in V(G)$. Then $G$ is $f$-class 1.

The following results due to Zhang, Wang, and Liu gave a series of sufficient conditions for a graph $G$ to be $f$-class 1 based on the $f$-core of $G$.

Theorem 4 ([13]) Let $G$ be a graph. If $G_{\Delta f}$ is a forest, then $G$ is $f$-class 1.

A graph $G$ is said to be edge-orderable if the edges of $G$ can be ordered $e_1, \ldots, e_{|E(G)|}$ such that, for $j = 1, \ldots, |E(G)|$, $e_j$ has an end vertex $v_j$ such that in every vertex $u \in N_G(v_j)$, there is an edge $e_i$ with $i \geq j$.

Example 2 The graph in Figure 2 is edge-orderable.

![Figure 2](https://doi.org/10.4153/CMB-2012-046-3)

Theorem 5 ([12]) Let $G$ be a graph. If $G_{\Delta f}$ is edge-orderable, then $G$ is $f$-class 1.

It was shown that every forest is edge-orderable, see [12]. Thus, Theorem 5 is an improvement of Theorem 4.

The following theorem states a condition under which $G$ is $f$-class 2.

Theorem 6 ([11]) Let $G$ be a graph. If $G$ is $f$-overfull, then $G$ is $f$-class 2.

We recall the following properties of $f$-critical graphs, which are proved in [8].

Theorem 7 Let $G$ be an $f$-critical graph and $uv \in E(G)$. If $d_G(v) < \Delta_f(G)f(v)$, then $u$ is adjacent to at least $f(u)(f(v)\Delta_f(G) - d_G(v) + 1)$ $f$-maximum vertices.

Theorem 8 For every vertex $v$ of an $f$-critical graph $G$, $v$ is adjacent to at least $2f(v)$ $f$-maximum vertices and $G$ contains at least three $f$-maximum vertices.

Theorem 9 If $G$ is $f$-class 2, then $G$ contains an $f$-critical subgraph $H$ with $\Delta_f(H) = k$, for each $k$ satisfying $2 \leq k \leq \Delta_f(G)$.

In this article, we will generalize the following five theorems.

Theorem 10 ([6]) Let $G$ be a connected class 2 graph with $\Delta(G_{\Delta}) \leq 2$.

(i) $G$ is critical.
(ii) $\delta(G_{\Delta}) = 2$.
(iii) $\delta(G) = \Delta(G) - 1$, unless $G$ is an odd cycle.

**Theorem 11** ([1]) Let $G$ be a connected graph and $\Delta(G_{\Delta}) \leq 2$. Suppose that $G$ has an edge cut of size at most $\Delta(G) - 2$ which is a matching or a star. Then $G$ is class 1.

**Theorem 12** ([2]) Let $G$ be a connected graph with $|G_{\Delta}| = 3$. Then $G$ is class 2 if and only if for some integer $n$, $G$ is obtained from $K_{2n+1}$ by removing $n - 1$ independent edges.

**Theorem 13** [3] Let $G$ be a 2-edge connected graph of even order with $|G_{\Delta}| = 4$. Then $G$ is class 1.

**Theorem 14** ([3]) Let $G$ be a 2-edge connected graph of order $2n + 1$ with $|G_{\Delta}| = 4$. Then $G$ is class 2 if and only if $|E(G)| \geq n\Delta(G) + 1$.

## 2 Results

Hilton and Zhao in [6] proved the result stated in Theorem 10. In the following lemmas we extend their result to $f$-colorings.

**Lemma 1** Let $G$ be an $f$-critical graph with $\Delta(G_{\Delta}) \leq 2$. Then $G_{\Delta_{f}}$ is a disjoint union of cycles and $d_{G}(v) = f(v)\Delta_{f}(G) - 1$ for every $v \in V(G) \setminus V(G_{\Delta_{f}})$.

**Proof** Since $G$ is $f$-critical, by Theorem 8, for every $u \in V(G_{\Delta_{f}})$, $u$ has at least two neighbors in $G_{\Delta_{f}}$. This implies that $d_{G_{\Delta_{f}}}(u) \geq 2$ and since $\Delta(G_{\Delta_{f}}) \leq 2$, $G_{\Delta_{f}}$ is a disjoint union of cycles. Now, by Theorem 8, for every $u \in V(G_{\Delta_{f}})$, $f(u) = 1$. Let $v \in V(G) \setminus V(G_{\Delta_{f}})$. Clearly, $d_{G}(v) < f(v)\Delta_{f}(G)$ and so $d_{G}(v) \leq f(v)\Delta_{f}(G) - 1$. Now, by Theorem 8, there exists a vertex $u \in V(G_{\Delta_{f}})$ such that $uv \in E(G)$. Then by Theorem 7,

$$2 = d_{G_{\Delta_{f}}}(u) \geq f(v)\Delta_{f}(G) - d_{G}(v) + 1.$$ 

Thus $d_{G}(v) \geq f(v)\Delta_{f}(G) - 1$, and so for every $v \in V(G) \setminus V(G_{\Delta_{f}})$, $d_{G}(v) = f(v)\Delta_{f}(G) - 1$. This completes the proof. \[\blacksquare\]

**Lemma 2** Let $G$ be a connected $f$-class 2 graph with $\Delta(G_{\Delta_{f}}) \leq 2$. Then $G$ is $f$-critical.

**Proof** First note that by Theorem 9, $G$ contains an $f$-critical subgraph $H$ with $\Delta_{f}(H) = \Delta_{f}(G)$. Since $H$ is $f$-critical and $\Delta(H_{\Delta_{f}}) \leq \Delta(G_{\Delta_{f}}) \leq 2$, by Lemma 1 $H_{\Delta_{f}}$ is a disjoint union of cycles and

$$(1) \quad d_{H}(v) = f(v)\Delta_{f}(H) - 1, \quad \text{for every } v \in V(H) \setminus V(H_{\Delta_{f}}).$$

Also, by Theorem 8 each vertex of $H$ is adjacent to at least two $f$-maximum vertices of $H$. Now, if $G$ contains a vertex which is not in $H$, then since $G$ is connected, there would be a vertex $w \in V(G) \setminus V(H)$ and a vertex $x$ in $H$ such that $xw \in E(G)$ and so $d_{G}(x) > d_{H}(x)$. Now, by (1) and noting that $d_{H}(x) = f(x)\Delta_{f}(H)$, for every $x \in V(H_{\Delta_{f}})$ and $\Delta_{f}(H) = \Delta_{f}(G)$ we conclude that $d_{H}(x) \geq f(x)\Delta_{f}(G) - 1$, which
implies that \( d_G(x) = f(x)\Delta_f(G) \). Thus \( x \in V(G_{\Delta_1}) \) and \( x \notin V(H_{\Delta_1}) \). On the other hand, since \( H \) is \( f \)-critical, \( x \) is adjacent to at least two \( f \)-maximum vertices of \( H_{\Delta_1} \). Now, since \( x \in V(G_{\Delta_1}) \) and \( H_{\Delta_1} \) is a disjoint union of cycles and moreover is a subgraph of \( G_{\Delta_1} \), \( G_{\Delta_1} \) is not a disjoint union of paths and cycles, a contradiction. Thus \( V(G) = V(H) \). Clearly, if \( G = H \), then \( G \) is \( f \)-critical and we are done. Since \( H \) is a subgraph of \( G \), \( d_G(v) \geq d_H(v) \), for every \( v \in V(G) \). Thus assume that \( e = st \in E(G) \setminus E(H) \). If \( s \in V(H_{\Delta_1}) \), then \( d_H(s) = f(s)\Delta_f(H) \) and so \( d_G(s) > d_H(s) = f(s)\Delta_f(H) = f(s)\Delta_f(G) \), a contradiction. Hence \( s \notin V(H_{\Delta_1}) \). So by (1) we find that \( d_H(s) = f(s)\Delta_f(G) - 1 \). This implies that \( d_G(s) = f(s)\Delta_f(G) \) and so \( s \in V(G_{\Delta_1}) \).

By Theorem 8, \( s \) is adjacent to at least two \( f \)-maximum vertices of \( H_{\Delta_1} \). Since \( s \in V(G_{\Delta_1}) \setminus V(H_{\Delta_1}) \) and \( V(H_{\Delta_1}) \subseteq V(G_{\Delta_1}) \), there exists a vertex in \( G_{\Delta_1} \), with degree at least 3, contradicting \( \Delta(G_{\Delta_1}) \leq 2 \). Therefore \( G = H \) and so \( G \) is \( f \)-critical.

Now we provide a criterion under which a graph is \( f \)-class 1.

**Theorem 15** Let \( G \) be a connected graph and \( \Delta(G_{\Delta_1}) \leq 2 \). Suppose that \( G \) has an edge cut of size at most \( \Delta_f(G) - 2 \) that is a matching. Then \( G \) is \( f \)-class 1 and \( G \) has a \( \Delta_f \)-coloring in which the edges of the edge cut have different colors.

**Proof** By assumption, there is an edge cut \( F \) of minimum size that is a matching and \( |F| = s \leq \Delta_f(G) - 2 \). By minimality of \( F \), \( G \setminus F \) has exactly two connected components, say \( G_1 \) and \( G_2 \). Again by minimality of \( F \), every edge in \( F \) has one end point in \( G_1 \) and another one in \( G_2 \). Let \( V(G_1) \cap V(F) = \{u_1, \ldots, u_s\} \) and \( V(G_2) \cap V(F) = \{v_1, \ldots, v_s\} \). Now, add two new vertices \( x_1 \) and \( x_2 \) to \( G \setminus F \) and join \( x_1 \) to \( u_1 \) and \( x_2 \) to \( v_1 \), for \( i = 1, \ldots, s \), respectively. Define \( f(x_1) = f(x_2) = 1 \). Let \( H = G[V(G_1) \cup \{x_1\}] \) and \( K = G[V(G_2) \cup \{x_2\}] \). Note that \( H \) and \( K \) are connected. Moreover, \( \max(\Delta_f(H), \Delta_f(K)) = \Delta_f(G) \). We claim that there are \( \Delta_f \)-colorings for both \( H \) and \( K \). If \( \Delta_f(H) < \Delta_f(G) \), then by Theorem 1, \( \chi'(H) \leq \Delta_f(H) + 1 \leq \Delta_f(G) \), and so there exists a \( \Delta_f \)-coloring for \( H \). If \( \Delta_f(H) = \Delta_f(G) \), then \( \Delta_f(H_{\Delta_1}) \leq 2 \). Now, if \( H \) is \( f \)-class 2, then by Lemma 2, \( H \) is \( f \)-critical, and so by Lemma 1, \( d_H(x_1) = f(x_1)\Delta_f(G) - 1 \), but \( d_H(x_1) \leq \Delta_f(G) - 2 \), a contradiction. So there exists a \( \Delta_f \)-coloring \( \phi \) of \( H \) and similarly a \( \Delta_f \)-coloring \( \theta \) of \( K \). Note that since \( f(x_1) = f(x_2) = 1 \), by a suitable permutation of colors, one may assume that \( \phi(u_1u_i) = \theta(x_iv_1) \), for \( i = 1, \ldots, s \). Now, define a \( \Delta_f \)-coloring \( c : E(G) \rightarrow \{1, \ldots, \Delta_f(G)\} \) such that \( c(e) = \phi(e) \) and \( c(e') = \theta(e') \), for every \( e \in E(G_1), e' \in E(G_2) \) and \( c(u_1v_1) = \phi(u_1x_1) \), for \( i = 1, \ldots, s \). Thus \( G \) is \( f \)-class 1. Moreover, since \( f(x_1) = f(x_2) = 1 \), the color of edges of \( F \) are distinct and the proof is complete.

**Lemma 3** Let \( G \) be a connected graph with \( \Delta(G_{\Delta_1}) \leq 2 \). Suppose that \( L = \{uv_1, \ldots, uv_s\} \), \( r \leq \Delta_f(G) - 2 \), is an edge cut of \( G \) and \( f(u) = 1 \). Then \( G \) is \( f \)-class 1.

**Proof** By assumption, there is an edge cut \( F \) of minimum size which is a star and \( |F| = s \leq \Delta_f(G) - 2 \). By minimality of \( F \), \( G \setminus F \) has exactly two connected components, say \( G_1 \) and \( G_2 \). Again by minimality of \( F \), every edge in \( F \) has one end point in \( G_1 \) and another one in \( G_2 \). Let \( V(G_1) \cap V(F) = \{u\} \) and \( V(G_2) \cap V(F) = \{v_1, \ldots, v_s\} \). Clearly, by Theorem 15, we can suppose that \( s \geq 2 \). Now add a new vertex \( z \) to \( G \setminus F \) and join \( z \) to \( v_i \), for \( i = 1, \ldots, s \). Define \( f(z) = 1 \). Let \( K = G[V(G_2) \cup \{z\}] \). Note
that $G_1$ and $K$ are connected. Moreover, $\max(\Delta_f(G_1), \Delta_f(K)) = \Delta_f(G)$. We claim that there are $\Delta_f(G)$-colorings for both $G_1$ and $K$. If $\Delta_f(G_1) < \Delta_f(G)$, then by Theorem 1 $\chi'_f(G_1) \leq \Delta_f(G_1) + 1 \leq \Delta_f(G)$, and so there exists a $\Delta_f$-coloring of $G_1$. If $\Delta_f(G_1) = \Delta_f(G)$ and $G_1$ has no $f$-coloring, then by Lemma 2, $G_1$ is $f$-critical, and so by Lemma 1 and noting that $f(u) = 1$, $d_{G_1}(u) \geq \Delta_f(G_1) - 1$. But since $s \geq 2$, we obtain that $d_{G_1}(u) \leq \Delta_f(G_1) - 2$, a contradiction. So there exists a $\Delta_f$-coloring $\phi$ of $G_1$ and similarly since $s \leq \Delta_f(G) - 2$, there is a $\Delta_f$-coloring $\theta$ of $K$. Now by a suitable permutation of colors in $f$-coloring of $G_1$ one may assume that $\{\theta(zv_1), \ldots, \theta(zv_s)\}$ are those colors that do not appear in the edges incident with $u$. Now define a $\Delta_f$-coloring $c: E(G) \rightarrow \{1, \ldots, \Delta_f(G)\}$ such that $c(e) = \phi(e)$, $c(e') = \theta(e')$, for every $e \in E(G_1)$ and $e' \in E(G_2)$ and $c(uv_i) = \theta(zv_i)$, for $i = 1, \ldots, s$. Thus, $G$ is $f$-class 1 and the proof is complete.

**Theorem 16** Let $G$ be a connected graph with $\Delta_f(G) \geq 4$ and $\Delta(G_{\Delta_f}) \leq 2$. Suppose that $G$ has an edge cut of size at most 2. Then $G$ is $f$-class 1.

**Proof** Let $F$ be an edge cut of $G$ of minimum size such that $|F| \leq 2$. Clearly, by Theorem 15 we can assume that $F = \{uv_1, uv_2\}$, and also by Lemma 3 we can suppose that $f(u) \geq 2$. To the contrary, suppose $G$ is $f$-class 2. Then by Lemma 2 $G$ is $f$-critical, and so by Theorem 8 and noting that $\Delta(G_{\Delta_f}) \leq 2$, $u \notin V(G_{\Delta_f})$. Let $V(G) = X \cup Y$, $X \cap Y = \emptyset$ such that $u \in X$ and $\{v_1, v_2\} \subseteq Y$. Consider two copies of $G[X]$ and $G[X_v]$ and call the corresponding vertices $v \in X$ and $X$ and $v_1$ and $v_2$, respectively. Let $G^*$ be the graph obtained from the union of $G[X_1]$, $G[X_2]$ and $Y$ and adding the edges $uv_1$, $uv_2$ and $u1v_2$. Moreover, let $g: V(G^*) \rightarrow \mathbb{N}$ be a function such that $g(v) = f(v)$ for every $v \in Y$ and $g(v_1) = g(v_2) = f(v)$ for every $v \in X$. Note that $\Delta_f(G^*) = \Delta_f(G)$, and since $u \notin V(G_{\Delta_f})$, we obtain that $\Delta(G^*) \leq 2$. Obviously, $\{uv_1, uv_2\}$ is an edge cut for $G^*$. Since $\Delta_f(G^*) = \Delta_f(G) \geq 4$, using Theorem 15 there exists a $\Delta_x$-coloring $\theta$ of $G^*$ in which $\theta(u1v_1) \neq \theta(u2v_2)$. Now, we claim that there exists a $\Delta_x$-coloring $\phi$ of $G[X_1]$ such that each of the colors $\theta(u1v_1)$ and $\theta(u2v_2)$ appears at most $g(u) - 1$ times in $G[X_1]$. If $\theta(u1v_2) \in \{\theta(u1v_1), \theta(u2v_2)\}$, then with no loss of generality we can assume that $\theta(u1v_1) = \theta(u2v_2)$, and so the claim is proved. If $\theta(u1v_2) \notin \{\theta(u1v_1), \theta(u2v_2)\}$, then define a $\Delta_x$-coloring $\phi$ of $G[X_1]$ such that for every $e \in E(G[X_1]),$

$$\phi(e) = \begin{cases} \theta(u2v_2) & \text{if } \theta(e) = \theta(u1v_1), \\ \theta(u1v_2) & \text{if } \theta(e) = \theta(u2v_2), \\ \theta(e) & \text{otherwise.} \end{cases}$$

Then define a $\Delta_f$-coloring $c: E(G[X_1] \cup Y) \cup \{uv_1v_2\} \rightarrow \{1, \ldots, \Delta_f(G)\}$ such that $c(e) = \phi(e)$ and $c(e') = \phi(e')$ for every $e \in E(G[Y])$ and $e' \in E(G[X_1])$, $c(u1v_1) = \theta(u1v_1)$, and $c(u1v_2) = \theta(u2v_2)$. This completes the proof of the theorem.

**Remark 1** The assumption of $\Delta_f(G) \geq 4$ in Theorem 16 is not superfluous. To see this, let $P^*$ be the Petersen graph with one vertex removed and define $f(v) = 1$, for every $v \in V(P^*)$. It is an easy exercise to show that $\Delta_f(P^*) = 3$ and $\Delta(G_{\Delta_f}^*) \leq 2$. Also it is not hard to see that $P^*$ has an edge cut of size 2 and $P^*$ is $f$-class 2.
Now, we want to generalize Theorem 12 for the $f$-coloring of graphs.

**Theorem 17** Let $G$ be a connected graph with $|G_{δ}| = 3$. Then $G$ is $f$-class 2 if and only if for some integer $n$ $G$ is obtained from $K_{2n+1}$ by removing $n - 1$ independent edges and $f(v) = 1$ for every $v \in V(G)$.

**Proof** Clearly, if $G$ is a graph obtained from $K_{2n+1}$ by removing $n - 1$ independent edges and $f(v) = 1$ for every $v \in V(G)$, then by Theorem 12 we are done. Conversely, suppose that $G$ is $f$-class 2 and $|G_{δ}| = 3$. Then by Lemma 2 $G$ is $f$-critical, and so by Theorem 8 $f(v) = 1$ for every $v \in V(G)$. Now by Theorem 12 we are done.

To extend Theorems 13 and 14 to $f$-coloring of graphs, first we need two lemmas.

**Lemma 4** Let $G$ be a graph with $|G_{δ}| \leq 4$. If $G$ is $f$-critical, then for every $v \in V(G)$, $f(v) = 1$.

**Proof** First suppose that if $|G_{δ}| \leq 3$ and there exists a vertex $v \in V(G)$ such that $f(v) \neq 1$. Then by Theorem 8 $v$ has at least 4 neighbors in $G_{δ}$, a contradiction. So, suppose that $|G_{δ}| = 4$. By Theorem 8 $f(u) = 1$ for every $u \in V(G_{δ})$ and $f(v) \leq 2$ for every $v \in V(G) \setminus V(G_{δ})$. Now, to the contrary assume that there exists $v \in V(G) \setminus V(G_{δ})$ such that $f(v) = 2$. Two cases may be considered.

First assume that $Δ(G_{δ}) \leq 2$. Since $G$ is $f$-critical, by Lemma 1, $δ(G_{δ}) = 2$ and $d_{G}(v) = 2Δ_{f}(G) - 1$, and so $|G| \geq 2Δ_{f}(G)$. On the other hand, since for every $u \in V(G_{δ})$, $d_{G}(u) = Δ_{f}(G)$ and $|G_{δ}| = 4$, $G_{δ}$ is a cycle. Also, by Theorem 8 and noting that $f(v) = 2$, we conclude that $e_{G}(G_{δ}, G[v]) \geq 4$ and

$$e_{G}(G_{δ}, G \setminus (G_{δ} \cup G[v])) \geq 2(|G| - 5).$$

Thus the following holds:

$$4 + 2(|G| - 5) \leq e_{G}(G_{δ}, G[v]) + e_{G}(G_{δ}, G \setminus (G_{δ} \cup G[v]))
= e_{G}(G_{δ}, G \setminus G_{δ}) = 4(Δ_{f}(G) - 2).$$

This implies that $|G| \leq 2Δ_{f}(G) - 1$, a contradiction.

Now, assume that $Δ(G_{δ}) \geq 3$. If $δ(G_{δ}) \leq 1$, then by Theorem 5 and Example 2 $G$ is $f$-class 1, a contradiction. So, suppose that $δ(G_{δ}) \geq 2$. Now, since $f(v) = 2$, for every $u \in V(G_{δ})$, $uv \in E(G)$. Now by Theorem 7 we have

$$3 \geq d_{G_{δ}}(u) \geq 2Δ_{f}(G) - d_{G}(v) + 1.$$

Thus $d_{G}(v) \geq 2Δ_{f}(G) - 2$ and so $|G| \geq 2Δ_{f}(G) - 1$. On the other hand, since $δ(G_{δ}) \geq 2$, it is easy to see that

$$4 + 2(|G| - 5) \leq e_{G}(G_{δ}, G \setminus G_{δ}) \leq 2(Δ_{f} - 2) + 2(Δ_{f} - 3).$$

This implies that $|G| \leq 2Δ_{f} - 2$, a contradiction. This completes the proof.
Lemma 5  Let $G$ be a 2-edge connected graph with $|G_{\Delta_H}| = 4$. If $G$ is $f$-class 2, then $G$ is $f$-critical.

Proof First note that if $\Delta (G_{\Delta_H}) \leq 2$, then by Lemma 2, $G$ is $f$-critical and we are done. So, we can assume that $\Delta (G_{\Delta_H}) \geq 3$. Moreover, we can suppose that $\delta (G_{\Delta_H}) \geq 2$, because otherwise by Theorem 5, $G$ is $f$-class 1, a contradiction. Now by Theorem 9 $G$ contains an $f$-critical subgraph $H$ with $\Delta (H) = \Delta_f (G)$. First note that since $H$ is $f$-critical, by the definition of $f$-critical, $H$ is $f$-class 2 and so by Theorem 4, $H_{\Delta_H}$ is not a forest. Hence $|H_{\Delta_H}| \geq 3$. Then by Lemma 4

\[(2) \quad f(v) = 1, \quad \text{for every } v \in V(H). \]

Thus $\Delta_f (H) = \Delta (H) = \Delta (G)$. Now, two cases can be considered.

First assume that $|H_{\Delta_H}| = 3$. Then since $H$ is $f$-critical, by Lemma 1 we have $d_{\Delta_f} (u) = \Delta_f (H), \quad \text{for every } u \in V(H_{\Delta_H})$ and $d_{\Delta_f} (v) = \Delta_f (H) - 1, \quad \text{for every } v \in V(H) \setminus V(H_{\Delta_H}).$ Now, if $V(H) \neq V(G)$, then since $G$ is 2-edge connected, $\epsilon_f (H, G \setminus H) \geq 2$ which implies that $|G_{\Delta_H}| \geq 5$, a contradiction. So, we can assume that $V(G) = V(H)$. Now, if there exists an edge $e \in E(G) \setminus E(H)$, then two end points of $e$ are in $V(H) \setminus V(H_{\Delta_H})$ and so $|G_{\Delta_H}| \geq 5$, a contradiction. Thus $G = H$ and so $G$ is $f$-critical.

Next, suppose that $|H_{\Delta_H}| = 4$. By (2), $f(v) = 1$, for every $v \in V(H)$. Now, since $H$ is $f$-class 2, by Theorems 13 and 14 we conclude that $|H| = 2n + 1$ and $|E(H)| \geq n\Delta (H) + 1 = n\Delta (G) + 1$. If $\delta (H) \geq \Delta (H) - 1$, then similar to the argument in the previous paragraph, $H = G$ and we are done. So assume that there exists a vertex $v \in V(H)$ such that $d_{\Delta_f} (v) \leq \Delta (H) - 2$. Now by Theorem 7 for every edge $e = uv$, where $u \in V(H_{\Delta_H})$, we have

$$3 \geq d_{\Delta_f} (u) \geq \Delta (H) - d_{\Delta_f} (v) + 1.$$ 

Hence, $d_{\Delta_f} (v) \geq \Delta (H) - 2$, which implies that $d_{\Delta_f} (v) = \Delta (H) - 2$.

Also, for every $u \in V(H_{\Delta_H}), \epsilon_f (H[u], H \setminus H_{\Delta_H}) \geq \Delta (H) - 3$, which implies that $|H[H_{\Delta_H}| \geq \Delta (H) - 3$. Now, since $|H_{\Delta_H}| = 4$, we have $|H| \geq \Delta (H) + 1$. On the other hand, since there are 4 vertices of degree $\Delta (H)$ in $H$, one vertex of degree $\Delta (H) - 2$ and $2n + 1 - 5$ vertices of degree at most $\Delta (H) - 1$, we conclude that

$$n\Delta (H) + 1 \leq |E(H)| \leq \frac{4\Delta (H) + \Delta (H) - 2 + (2n + 1 - 5)(\Delta (H) - 1)}{2}.$$ 

So $|H| = \Delta (H) + 1$. Since the equality holds in the above inequality, we conclude that there are $2n + 1 - 5$ vertices of degree $\Delta (H) - 1$ in $H$, and so for every $x \in V(H) \setminus \{V(H_{\Delta_H}) \cup \{v\}\}$, $d_f (x) = \Delta (H) - 1$. If $G$ contains a vertex not in $H$, then since $G$ is 2-edge connected, $\epsilon_f (H, G \setminus H) \geq 2$ which implies that $|G_{\Delta_H}| \geq 5$, a contradiction. So we can assume that $V(G) = V(H)$. If there exists an edge $e \in E(G) \setminus E(H)$, then two end points of $e$ is in $V(H) \setminus V(H_{\Delta_H})$, and so $|G_{\Delta_H}| \geq 5$, a contradiction. Thus $G = H$ and hence $G$ is $f$-critical and the proof is complete. 

\[\square\]
Remark 2 The assumption of 2-edge connectivity in Lemma 5 is not superfluous. Let \( G \) be the graph in Figure 3 and define \( f(x) = 1 \), for every \( x \in V(G) \). Clearly, \( G_{\Delta_f} = K_4 \) and \( G \) and \( H = G \setminus \{uv\} \) are \( f \)-class 2 (note that since \( f(x) = 1 \), for every \( x \in V(G) \), the \( f \)-coloring and proper edge coloring coincide). Thus \( G \) is not \( f \)-critical.

We close the paper with the following theorem.

**Theorem 18** Let \( G \) be a 2-edge connected graph with \( |G_{\Delta_f}| = 4 \).

(i) If \( G \) has an even order, then \( G \) is \( f \)-class 1.

(ii) If \( G \) has an odd order, then \( G \) is \( f \)-class 2 if and only if \( G \) is \( f \)-overfull.

**Proof**

(i) Assume to the contrary that \( G \) is \( f \)-class 2. Then by Lemma 5 \( G \) is \( f \)-critical, and so by Lemma 4 \( f(v) = 1 \) for every \( v \in V(G) \). Now by Theorem 13 \( G \) is \( f \)-class 1, a contradiction.

(ii) First note that if \( G \) is \( f \)-overfull, then by Theorem 6 we are done. Now suppose that \( G \) is \( f \)-class 2. Then by Lemma 5, \( G \) is \( f \)-critical, and so by Lemma 4 \( f(v) = 1 \), for every \( v \in V(G) \). Thus \( f(V) = |G| \). Now by Theorem 14 \( |E(G)| \geq \frac{|G| - 1}{2} \Delta(G) + 1 > \frac{|G|}{2} \Delta(G) \). Therefore, \( G \) is overfull, and since \( f(v) = 1 \), for every \( v \in V(G) \), \( G \) is \( f \)-overfull, and the proof is complete.

**Acknowledgments** The authors would like to express their deep gratitude to the referee for her/his constructive and fruitful comments. The first, third and fourth authors are indebted to the School of Mathematics, Institute for Research in Fundamental Sciences (IPM) for support. The research of the first author was in part supported by a grant from IPM (No. 91050212).

**References**


https://doi.org/10.4153/CMB-2012-046-3 Published online by Cambridge University Press