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The f-Chromatic Index of a Graph Whose f-Core Has Maximum Degree 2

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Abstract. Let G be a graph. The minimum number of colors needed to color the edges of G is called the *chromatic index* of G and is denoted by $\chi'(G)$. It is well known that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, for any graph G, where $\Delta(G)$ denotes the maximum degree of G. A graph G is said to be *class* 1 if $\chi'(G) = \Delta(G)$ and *class* 2 if $\chi'(G) = \Delta(G)+1$. Also, G_Δ is the induced subgraph on all vertices of degree $\Delta(G)$. Let $f\colon V(G) \to \mathbb{N}$ be a function. An f*-coloring* of a graph G is a coloring of the edges of $E(G)$ such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. The minimum number of colors needed to f-color G is called the f*-chromatic index* of G and is denoted by $\chi'_f(G)$. It was shown that for every graph G, $\Delta_f(G) \leq \chi'_f(G) \leq \Delta_f(G)+1$, where $\Delta_f(G) = \max_{v \in V(G)} \lceil d_G(v)/f(v) \rceil$. A graph G is said to be f*-class* 1 if $\chi'_f(G) = \Delta_f(G)$, and f*-class* 2, otherwise. Also, G_{Δ_f} is the induced subgraph of G on $\{v \in V(G) : d_G(v)/f(v) = \Delta_f(G)\}$. Hilton and Zhao showed that if G_Δ has maximum degree two and G is class 2, then G is critical, G_Δ is a disjoint union of cycles and $\delta(G) = \Delta(G)-1$, where $\delta(G)$ denotes the minimum degree of G, respectively. In this paper, we generalize this theorem to f-coloring of graphs. Also, we determine the f-chromatic index of a connected graph G with $|G_{\Delta_f}| \leq 4$.

1 Introduction

All graphs considered in this paper are simple and finite. Let G be a graph. The number of vertices of G is called the order of G and is denoted by $|G|$. Also, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The degree of vertex v in G is denoted by $d_G(v)$, and $N_G(v)$ denotes the set of all vertices adjacent to v. Also, let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. A graph G is said to be *connected* if any two vertices are connected by a path in G. If G is not connected, then G is decomposed into *connected components* that are the maximal connected subgraphs of G. A *star graph* is a graph containing a vertex adjacent to all other vertices and with no other edges. A *matching* in a graph G is a set of pairwise non-adjacent edges. An *edge cut* is a set of edges whose removal produces a subgraph with more connected components than the original graph. Moreover, a graph is k*-edge connected* if the minimum number of edges whose removal would disconnect the graph is at least k. For a subset $X \subseteq V(G)$, we denote the induced subgraph of G on X by $G[X]$. By $G \setminus H$ we mean the induced subgraph on $V(G) \setminus V(H)$. Also, G_Δ is the induced subgraph on all vertices of degree $\Delta(G)$. For two subgraphs S and T of G, where $V(S) \cap V(T) = \varnothing$, $e_G(S,T)$ denotes the number of edges with one end in S and other end in T. An edge coloring of a graph in which no two adjacent edges have the same color is called a *proper edge coloring*. The minimum number of colors needed to color the edges of

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G properly is called the *chromatic index* of G and is denoted by $\chi'(G)$. Vizing [10] proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, for any graph G. A graph G is said to be *class* 1 if $\chi'(G) = \Delta(G)$ and *class* 2 if $\chi'(G) = \Delta(G) + 1$. A graph G is called *critical* if G is connected, class 2 and $\chi'(G \setminus e) < \chi'(G)$, for every edge $e \in E(G)$ and is called *overfull* when $|E(G)| > \Delta(G)\lfloor \frac{|G|}{2} \rfloor$.

For a function f that assigns a positive integer $f(v)$ to each vertex $v \in V(G)$, an f*-coloring* of G is an edge coloring of G such that each vertex v has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called the f*-chromatic index* of G, and denoted by $\chi'_f(G)$. If $f(v) = 1$ for all $v \in V(G)$, then f-coloring is equivalent to proper edge coloring. Let $\Delta_f(G) = \max_{v \in V(G)} \lceil \frac{d_G(v)}{f(v)} \rceil$. A graph G is said to be f*-class* 1 if $\chi'_f(G) = \Delta_f(G)$ and f*-class* 2, otherwise. Also, we say that G has a Δ_f-coloring if G is f-class 1. A vertex v is called an f*-maximum vertex* if $d_G(v) = \Delta_f(G) f(v)$. The f*-core* of a graph G is the induced subgraph of G on the f-maximum vertices and denoted by G_{Δ_f}. A graph G is called f*-overfull* when $|E(G)| > \Delta_f(G)\lfloor \frac{f(V)}{2} \rfloor$, where $f(V) = \sum_{v \in V(G)} f(v)$, and is called f*-critical* if G is connected, f-class 2 and $\chi'_f(G \setminus e) < \chi'_f(G)$, for every $e \in E(G)$. The following example introduces an f-class 1 graph.

Example 1 Let G be a graph shown in the following figure such that $f(v_1) = f(v_2) = 2$ and $f(v_i) = 1$ for $i = 3, \ldots, 7$. It is easy to see that $\Delta_f(G) = 2$, $G_{\Delta_f} = K_3$, and G is f-class 1.

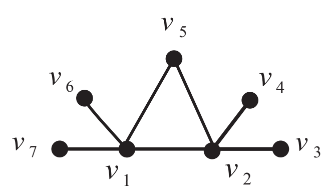

Figure 1

There are interesting real-life applications of f-colorings in optimization and network design, such as file transfers in a computer network [4, 5, 9]. Since the classical edge-coloring problem is NP-complete [7], the f-coloring problem which asks to f-color a given multigraph G with $\chi'_f(G)$ colors is also NP-complete.

In [5], Hakimi and Kariv obtained the following results.

Theorem 1 *Let G be a graph. Then*

$$\Delta_f(G) \leq \chi'_f(G) \leq \max_{v \in V(G)} \left\lceil \frac{d_G(v) + 1}{f(v)} \right\rceil \leq \Delta_f(G) + 1.$$

Theorem 2 *Let G be a bipartite graph. Then G is f-class* 1.

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Theorem 3 *Let G be a graph, and let $f(v)$ be even for all $v \in V(G)$. Then G is f-class* 1.

The following results due to Zhang, Wang, and Liu gave a series of sufficient conditions for a graph G to be f-class 1 based on the f-core of G.

Theorem 4 ([13]) *Let G be a graph. If G_{Δ_f} is a forest, then G is f-class* 1.

A graph G is said to be *edge-orderable* if the edges of G can be ordered $e_1, \dots, e_{|E(G)|}$ such that, for $j = 1, \dots, |E(G)|$, e_j has an end vertex v_j such that in every vertex $u \in N_G(v_j)$, there is an edge e_i with $i \geq j$.

Example 2 The graph in Figure 2 is edge-orderable.

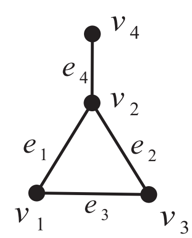

Figure 2

Theorem 5 ([12]) *Let G be a graph. If G_{Δ_f} is edge-orderable, then G is f-class* 1.

It was shown that every forest is edge-orderable, see [12]. Thus, Theorem 5 is an improvement of Theorem 4.

The following theorem states a condition under which G is f-class 2.

Theorem 6 ([11]) *Let G be a graph. If G is f-overfull, then G is f-class* 2.

We recall the following properties of f-critical graphs, which are proved in [8].

Theorem 7 *Let G be an f-critical graph and $uv \in E(G)$. If $d_G(v) < \Delta_f(G)f(v)$, then u is adjacent to at least $f(u)\big(f(v)\Delta_f(G) - d_G(v) + 1\big)$ f-maximum vertices.*

Theorem 8 *For every vertex v of an f-critical graph G, v is adjacent to at least $2f(v)$ f-maximum vertices and G contains at least three f-maximum vertices.*

Theorem 9 *If G is f-class 2, then G contains an f-critical subgraph H with $\Delta_f(H) = k$, for each k satisfying $2 \leq k \leq \Delta_f(G)$.*

In this article, we will generalize the following five theorems.

Theorem 10 ([6]) *Let G be a connected class 2 graph with $\Delta(G_\Delta) \leq 2$.*

(i) *G is critical.*

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(ii) $\delta(G_\Delta) = 2$.
(iii) $\delta(G) = \Delta(G) - 1$, *unless G is an odd cycle.*

Theorem 11 ([1]) *Let G be a connected graph and $\Delta(G_\Delta) \le 2$. Suppose that G has an edge cut of size at most $\Delta(G) - 2$ which is a matching or a star. Then G is class* 1.

Theorem 12 ([2]) *Let G be a connected graph with $|G_\Delta| = 3$. Then G is class* 2 *if and only if for some integer n, G is obtained from K_{2n+1} by removing $n-1$ independent edges.*

Theorem 13 [3] *Let G be a 2-edge connected graph of even order with $|G_\Delta| = 4$. Then G is class* 1.

Theorem 14 ([3]) *Let G be a 2-edge connected graph of order $2n+1$ with $|G_\Delta| = 4$. Then G is class* 2 *if and only if $|E(G)| \ge n\Delta(G) + 1$.*

2 Results

Hilton and Zhao in [6] proved the result stated in Theorem 10. In the following lemmas we extend their result to f-colorings.

Lemma 1 *Let G be an f-critical graph with $\Delta(G_{\Delta_f}) \le 2$. Then G_{Δ_f} is a disjoint union of cycles and $d_G(v) = f(v)\Delta_f(G) - 1$ for every $v \in V(G) \setminus V(G_{\Delta_f})$.*

Proof Since G is f-critical, by Theorem 8, for every $u \in V(G_{\Delta_f})$, u has at least two neighbors in G_{Δ_f}. This implies that $d_{G_{\Delta_f}}(u) \ge 2$ and since $\Delta(G_{\Delta_f}) \le 2$, G_{Δ_f} is a disjoint union of cycles. Now, by Theorem 8, for every $u \in V(G_{\Delta_f})$, $f(u) = 1$. Let $v \in V(G) \setminus V(G_{\Delta_f})$. Clearly, $d_G(v) < f(v)\Delta_f(G)$ and so $d_G(v) \le f(v)\Delta_f(G) - 1$. Now, by Theorem 8, there exists a vertex $u \in V(G_{\Delta_f})$ such that $uv \in E(G)$. Then by Theorem 7,

$$2 = d_{G_{\Delta_f}}(u) \ge f(v)\Delta_f(G) - d_G(v) + 1.$$

Thus $d_G(v) \ge f(v)\Delta_f(G) - 1$, and so for every $v \in V(G) \setminus V(G_{\Delta_f})$, $d_G(v) = f(v)\Delta_f(G) - 1$. This completes the proof. ■

Lemma 2 *Let G be a connected f-class* 2 *graph with $\Delta(G_{\Delta_f}) \le 2$. Then G is f-critical.*

Proof First note that by Theorem 9, G contains an f-critical subgraph H with $\Delta_f(H) = \Delta_f(G)$. Since H is f-critical and $\Delta(H_{\Delta_f}) \le \Delta(G_{\Delta_f}) \le 2$, by Lemma 1 H_{Δ_f} is a disjoint union of cycles and

$$(1) \qquad d_H(v) = f(v)\Delta_f(H) - 1, \quad \text{for every } v \in V(H) \setminus V(H_{\Delta_f}).$$

Also, by Theorem 8 each vertex of H is adjacent to at least two f-maximum vertices of H. Now, if G contains a vertex which is not in H, then since G is connected, there would be a vertex $w \in V(G) \setminus V(H)$ and a vertex x in H such that $xw \in E(G)$ and so $d_G(x) > d_H(x)$. Now, by (1) and noting that $d_H(x) = f(x)\Delta_f(H)$, for every $x \in V(H_{\Delta_f})$ and $\Delta_f(H) = \Delta_f(G)$ we conclude that $d_H(x) \ge f(x)\Delta_f(G) - 1$, which

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implies that $d_G(x) = f(x)\Delta_f(G)$. Thus $x \in V(G_{\Delta_f})$ and $x \notin V(H_{\Delta_f})$. On the other hand, since H is f-critical, x is adjacent to at least two f-maximum vertices of H_{Δ_f}. Now, since $x \in V(G_{\Delta_f})$ and H_{Δ_f} is a disjoint union of cycles and moreover is a subgraph of G_{Δ_f}, G_{Δ_f} is not a disjoint union of paths and cycles, a contradiction. Thus $V(G) = V(H)$. Clearly, if $G = H$, then G is f-critical and we are done. Since H is a subgraph of G, $d_G(v) \geq d_H(v)$, for every $v \in V(G)$. Thus assume that $e = st \in E(G) \setminus E(H)$. If $s \in V(H_{\Delta_f})$, then $d_H(s) = f(s)\Delta_f(H)$ and so $d_G(s) > d_H(s) = f(s)\Delta_f(H) = f(s)\Delta_f(G)$, a contradiction. Hence $s \notin V(H_{\Delta_f})$. So by (1) we find that $d_H(s) = f(s)\Delta_f(G) - 1$. This implies that $d_G(s) = f(s)\Delta_f(G)$ and so $s \in V(G_{\Delta_f})$. By Theorem 8, s is adjacent to at least two f-maximum vertices of H_{Δ_f}. Since $s \in V(G_{\Delta_f}) \setminus V(H_{\Delta_f})$ and $V(H_{\Delta_f}) \subseteq V(G_{\Delta_f})$, there exists a vertex in G_{Δ_f} with degree at least 3, contradicting $\Delta(G_{\Delta_f}) \leq 2$. Therefore $G = H$ and so G is f-critical. ∎

Now we provide a criterion under which a graph is f-class 1.

Theorem 15 *Let G be a connected graph and $\Delta(G_{\Delta_f}) \leq 2$. Suppose that G has an edge cut of size at most $\Delta_f(G) - 2$ that is a matching. Then G is f-class 1 and G has a Δ_f-coloring in which the edges of the edge cut have different colors.*

Proof By assumption, there is an edge cut F of minimum size that is a matching and $|F| = s \leq \Delta_f(G) - 2$. By minimality of F, $G \setminus F$ has exactly two connected components, say G_1 and G_2. Again by minimality of F, every edge in F has one end point in G_1 and another one in G_2. Let $V(G_1) \cap V(F) = \{u_1, \dots, u_s\}$ and $V(G_2) \cap V(F) = \{v_1, \dots, v_s\}$. Now, add two new vertices x_1 and x_2 to $G \setminus F$ and join x_1 to u_i and x_2 to v_i, for $i = 1, \dots, s$, respectively. Define $f(x_1) = f(x_2) = 1$. Let $H = G[V(G_1) \cup \{x_1\}]$ and $K = G[V(G_2) \cup \{x_2\}]$. Note that H and K are connected. Moreover, $\max\big(\Delta_f(H), \Delta_f(K)\big) = \Delta_f(G)$. We claim that there are $\Delta_f(G)$-colorings for both H and K. If $\Delta_f(H) < \Delta_f(G)$, then by Theorem 1, $\chi'_f(H) \leq \Delta_f(H) + 1 \leq \Delta_f(G)$, and so there exists a Δ_f-coloring for H. If $\Delta_f(H) = \Delta_f(G)$, then $\Delta(H_{\Delta_f}) \leq 2$. Now, if H is f-class 2, then by Lemma 2, H is f-critical, and so by Lemma 1, $d_H(x_1) = f(x_1)\Delta_f(G) - 1$, but $d_H(x_1) \leq \Delta_f(G) - 2$, a contradiction. So there exists a Δ_f-coloring ϕ of H and similarly a Δ_f-coloring θ of K. Note that since $f(x_1) = f(x_2) = 1$, by a suitable permutation of colors, one may assume that $\phi(x_1u_i) = \theta(x_2v_i)$, for $i = 1, \dots, s$. Now, define a Δ_f-coloring $c\colon E(G) \longrightarrow \{1, \dots, \Delta_f(G)\}$ such that $c(e) = \phi(e)$ and $c(e') = \theta(e')$, for every $e \in E(G_1)$, $e' \in E(G_2)$ and $c(u_iv_i) = \phi(u_ix_1)$, for $i = 1, \dots, s$. Thus G is f-class 1. Moreover, since $f(x_1) = f(x_2) = 1$, the color of edges of F are distinct and the proof is complete. ∎

Lemma 3 *Let G be a connected graph with $\Delta(G_{\Delta_f}) \leq 2$. Suppose that $L = \{uv_1, \dots, uv_r\}$, $r \leq \Delta_f(G) - 2$, is an edge cut of G and $f(u) = 1$. Then G is f-class 1.*

Proof By assumption, there is an edge cut F of minimum size which is a star and $|F| = s \leq r \leq \Delta_f(G) - 2$. By minimality of F, $G \setminus F$ has exactly two connected components, say G_1 and G_2. Again by minimality of F, every edge in F has one end point in G_1 and another one in G_2. Let $V(G_1) \cap V(F) = \{u\}$ and $V(G_2) \cap V(F) = \{v_1, \dots, v_s\}$. Clearly, by Theorem 15, we can suppose that $s \geq 2$. Now add a new vertex z to $G \setminus F$ and join z to v_i, for $i = 1, \dots, s$. Define $f(z) = 1$. Let $K = G[V(G_2) \cup \{z\}]$. Note

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that G_1 and K are connected. Moreover, $\max\big(\Delta_f(G_1), \Delta_f(K)\big) = \Delta_f(G)$. We claim that there are $\Delta_f(G)$-colorings for both G_1 and K. If $\Delta_f(G-1) < \Delta_f(G)$, then by Theorem 1 $\chi'_f(G_1) \leq \Delta_f(G_1) + 1 \leq \Delta_f(G)$, and so there exists a Δ_f-coloring of G_1. If $\Delta_f(G_1) = \Delta_f(G)$ and G_1 has no Δ_f-coloring, then by Lemma 2, G_1 is f-critical, and so by Lemma 1 and noting that $f(u) = 1$, $d_{G_1}(u) \geq \Delta_f(G_1) - 1$. But since $s \geq 2$, we obtain that $d_{G_1}(u) \leq \Delta_f(G_1) - 2$, a contradiction. So there exists a Δ_f-coloring ϕ of G_1 and similarly since $s \leq \Delta_f(G) - 2$, there is a Δ_f-coloring θ of K. Now by a suitable permutation of colors in f-coloring of G_1 one may assume that $\{\theta(zv_1), \dots, \theta(zv_s)\}$ are those colors that do not appear in the edges incident with u. Now define a Δ_f-coloring $c\colon E(G) \longrightarrow \{1, \dots, \Delta_f(G)\}$ such that $c(e) = \phi(e)$, $c(e') = \theta(e')$, for every $e \in E(G_1)$ and $e' \in E(G_2)$ and $c(uv_i) = \theta(zv_i)$, for $i = 1, \dots, s$. Thus, G is f-class 1 and the proof is complete. ■

Theorem 16 *Let G be a connected graph with $\Delta_f(G) \geq 4$ and $\Delta(G_{\Delta_f}) \leq 2$. Suppose that G has an edge cut of size at most 2. Then G is f-class 1.*

Proof Let F be an edge cut of G of minimum size such that $|F| \leq 2$. Clearly, by Theorem 15 we can assume that $F = \{uv_1, uv_2\}$, and also by Lemma 3 we can suppose that $f(u) \geq 2$. To the contrary, suppose G is f-class 2. Then by Lemma 2 G is f-critical, and so by Theorem 8 and noting that $\Delta(G_{\Delta_f}) \leq 2$, $u \notin V(G_{\Delta_f})$. Let $V(G) = X \cup Y$, $X \cap Y = \varnothing$ such that $u \in X$ and $\{v_1, v_2\} \subseteq Y$. Consider two copies of $G[X]$, say $G[X_1]$ and $G[X_2]$, and call the corresponding vertices $v \in X$ in X_1 and X_2 by v_1 and v_2, respectively. Let G^* be the graph obtained from the union of $G[X_1]$, $G[X_2]$ and $G[Y]$ and adding the edges u_1v_1, u_2v_2 and u_1u_2. Moreover, let $g\colon V(G^*) \longrightarrow \mathbb{N}$ be a function such that $g(v) = f(v)$ for every $v \in Y$ and $g(v_1) = g(v_2) = f(v)$, for every $v \in X$. Note that $\Delta_g(G^*) = \Delta_f(G)$, and since $u \notin V(G_{\Delta_f})$, we obtain that $\Delta(G^*_{\Delta_g}) \leq 2$. Obviously, $\{u_1v_1, u_2v_2\}$ is an edge cut for G^*. Since $\Delta_g(G^*) = \Delta_f(G) \geq 4$, using Theorem 15 there exists a Δ_g-coloring θ of G^* in which $\theta(u_1v_1) \neq \theta(u_2v_2)$. Now, we claim that there exists a Δ_g-coloring ϕ of $G[X_1]$ such that each of the colors $\theta(u_1v_1)$ and $\theta(u_2v_2)$ appears at most $g(u_1) - 1$ times in $G[X_1]$. If $\theta(u_1u_2) \in \{\theta(u_1v_1), \theta(u_2v_2)\}$, then with no loss of generality we can assume that $\theta(u_1u_2) = \theta(u_2v_2)$, and so the claim is proved. If $\theta(u_1u_2) \notin \{\theta(u_1v_1), \theta(u_2v_2)\}$, then define a Δ_g-coloring ϕ of $G[X_1]$ such that for every $e \in E(G[X_1])$,

$$\phi(e) = \begin{cases} \theta(u_2v_2) & \text{if } \theta(e) = \theta(u_1u_2), \\ \theta(u_1u_2) & \text{if } \theta(e) = \theta(u_2v_2), \\ \theta(e) & \text{otherwise.} \end{cases}$$

Then define a Δ_f-coloring $c\colon E(G[X_1 \cup Y]) \cup \{u_1v_2\} \longrightarrow \{1, \dots, \Delta_f(G)\}$ such that $c(e) = \theta(e)$ and $c(e') = \phi(e')$ for every $e \in E(G[Y])$ and $e' \in E(G[X_1])$, $c(u_1v_1) = \theta(u_1v_1)$, and $c(u_1v_2) = \theta(u_2v_2)$. This completes the proof of the theorem. ■

Remark 1 The assumption of $\Delta_f(G) \geq 4$ in Theorem 16 is not superfluous. To see this, let P^* be the Petersen graph with one vertex removed and define $f(v) = 1$, for every $v \in V(P^*)$. It is an easy exercise to show that $\Delta_f(P^*) = 3$ and $\Delta(P^*_{\Delta_f}) \leq 2$. Also it is not hard to see that P^* has an edge cut of size 2 and P^* is f-class 2.

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Now, we want to generalize Theorem 12 for the f-coloring of graphs.

Theorem 17 *Let G be a connected graph with $|G_{\Delta_f}| = 3$. Then G is f-class 2 if and only if for some integer n G is obtained from K_{2n+1} by removing $n-1$ independent edges and $f(v) = 1$ for every $v \in V(G)$.*

Proof Clearly, if G is a graph obtained from K_{2n+1} by removing $n-1$ independent edges and $f(v) = 1$ for every $v \in V(G)$, then by Theorem 12 we are done. Conversely, suppose that G is f-class 2 and $|G_{\Delta_f}| = 3$. Then by Lemma 2 G is f-critical, and so by Theorem 8 $f(v) = 1$ for every $v \in V(G)$. Now by Theorem 12 we are done. ■

To extend Theorems 13 and 14 to f-coloring of graphs, first we need two lemmas.

Lemma 4 *Let G be a graph with $|G_{\Delta_f}| \leq 4$. If G is f-critical, then for every $v \in V(G)$, $f(v) = 1$.*

Proof First suppose that if $|G_{\Delta_f}| \leq 3$ and there exists a vertex $v \in V(G)$ such that $f(v) \neq 1$. Then by Theorem 8 v has at least 4 neighbors in G_{Δ_f}, a contradiction. So, suppose that $|G_{\Delta_f}| = 4$. By Theorem 8 $f(u) = 1$ for every $u \in V(G_{\Delta_f})$ and $f(v) \leq 2$ for every $v \in V(G) \setminus V(G_{\Delta_f})$. Now, to the contrary assume that there exists $v \in V(G) \setminus V(G_{\Delta_f})$ such that $f(v) = 2$. Two cases may be considered.

First assume that $\Delta(G_{\Delta_f}) \leq 2$. Since G is f-critical, by Lemma 1, $\delta(G_{\Delta_f}) = 2$ and $d_G(v) = 2\Delta_f(G) - 1$, and so $|G| \geq 2\Delta_f(G)$. On the other hand, since for every $u \in V(G_{\Delta_f})$, $d_G(u) = \Delta_f(G)$ and $|G_{\Delta_f}| = 4$, G_{Δ_f} is a cycle. Also, by Theorem 8 and noting that $f(v) = 2$, we conclude that $e_G(G_{\Delta_f}, G[v]) \geq 4$ and

$$e_G\big(G_{\Delta_f}, G \setminus (G_{\Delta_f} \cup G[v])\big) \geq 2(|G| - 5).$$

Thus the following holds:

$$\begin{aligned} 4 + 2(|G| - 5) &\leq e_G(G_{\Delta_f}, G[v]) + e_G\big(G_{\Delta_f}, G \setminus (G_{\Delta_f} \cup G[v])\big) \\ &= e_G(G_{\Delta_f}, G \setminus G_{\Delta_f}) = 4\big(\Delta_f(G) - 2\big). \end{aligned}$$

This implies that $|G| \leq 2\Delta_f(G) - 1$, a contradiction.

Now, assume that $\Delta(G_{\Delta_f}) \geq 3$. If $\delta(G_{\Delta_f}) \leq 1$, then by Theorem 5 and Example 2 G is f-class 1, a contradiction. So, suppose that $\delta(G_{\Delta_f}) \geq 2$. Now, since $f(v) = 2$, for every $u \in V(G_{\Delta_f})$, $uv \in E(G)$. Now by Theorem 7 we have

$$3 \geq d_{G_{\Delta_f}}(u) \geq 2\Delta_f(G) - d_G(v) + 1.$$

Thus $d_G(v) \geq 2\Delta_f(G) - 2$ and so $|G| \geq 2\Delta_f(G) - 1$. On the other hand, since $\delta(G_{\Delta_f}) \geq 2$, it is easy to see that

$$4 + 2(|G| - 5) \leq e_G(G_{\Delta_f}, G \setminus G_{\Delta_f}) \leq 2(\Delta_f - 2) + 2(\Delta_f - 3).$$

This implies that $|G| \leq 2\Delta_f - 2$, a contradiction. This completes the proof. ■

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Lemma 5 *Let G be a 2-edge connected graph with $|G_{\Delta_f}| = 4$. If G is f-class 2, then G is f-critical.*

Proof First note that if $\Delta(G_{\Delta_f}) \leq 2$, then by Lemma 2, G is f-critical and we are done. So, we can assume that $\Delta(G_{\Delta_f}) \geq 3$. Moreover, we can suppose that $\delta(G_{\Delta_f}) \geq 2$, because otherwise by Theorem 5, G is f-class 1, a contradiction. Now by Theorem 9 G contains an f-critical subgraph H with $\Delta_f(H) = \Delta_f(G)$. First note that since H is f-critical, by the definition of f-critical, H is f-class 2 and so by Theorem 4, H_{Δ_f} is not a forest. Hence $|H_{\Delta_f}| \geq 3$. Then by Lemma 4

$$f(v) = 1, \quad \text{for every } v \in V(H). \tag{2}$$

Thus $\Delta_f(H) = \Delta(H) = \Delta(G)$. Now, two cases can be considered.

First assume that $|H_{\Delta_f}| = 3$. Then since H is f-critical, by Lemma 1 we have $d_H(u) = \Delta_f(H)$, for every $u \in V(H_{\Delta_f})$ and $d_H(v) = \Delta_f(H) - 1$, for every $v \in V(H) \setminus V(H_{\Delta_f})$. Now, if $V(H) \neq V(G)$, then since G is 2-edge connected, $e_G(H, G \setminus H) \geq 2$ which implies that $|G_{\Delta_f}| \geq 5$, a contradiction. So, we can assume that $V(G) = V(H)$. Now, if there exists an edge $e \in E(G) \setminus E(H)$, then two end points of e are in $V(H) \setminus V(H_{\Delta_f})$ and so $|G_{\Delta_f}| \geq 5$, a contradiction. Thus $G = H$ and so G is f-critical.

Next, suppose that $|H_{\Delta_f}| = 4$. By (2), $f(v) = 1$, for every $v \in V(H)$. Now, since H is f-class 2, by Theorems 13 and 14 we conclude that $|H| = 2n + 1$ and $|E(H)| \geq n\Delta(H) + 1 = n\Delta(G) + 1$. If $\delta(H) \geq \Delta(H) - 1$, then similar to the argument in the previous paragraph, $H = G$ and we are done. So assume that there exists a vertex $v \in V(H)$ such that $d_H(v) \leq \Delta(H) - 2$. Now by Theorem 7 for every edge $e = uv$, where $u \in V(H_{\Delta_f})$, we have

$$3 \geq d_{H_{\Delta_f}}(u) \geq \Delta(H) - d_H(v) + 1.$$

Hence, $d_H(v) \geq \Delta(H) - 2$, which implies that $d_H(v) = \Delta(H) - 2$.

Also, for every $u \in V(H_{\Delta_f})$, $e_H(H[u], H \setminus H_{\Delta_f}) \geq \Delta(H) - 3$, which implies that $|H \setminus H_{\Delta_f}| \geq \Delta(H) - 3$. Now, since $|H_{\Delta_f}| = 4$, we have $|H| \geq \Delta(H) + 1$. On the other hand, since there are 4 vertices of degree $\Delta(H)$ in H, one vertex of degree $\Delta(H) - 2$ and $2n + 1 - 5$ vertices of degree at most $\Delta(H) - 1$, we conclude that

$$n\Delta(H) + 1 \leq |E(H)| \leq \frac{4\Delta(H) + \Delta(H) - 2 + (2n + 1 - 5)(\Delta(H) - 1)}{2}.$$

So $|H| = \Delta(H) + 1$. Since the equality holds in the above inequality, we conclude that there are $2n + 1 - 5$ vertices of degree $\Delta(H) - 1$ in H, and so for every $x \in V(H) \setminus \big(V(H_{\Delta_f}) \cup \{v\}\big)$, $d_H(x) = \Delta(H) - 1$. If G contains a vertex not in H, then since G is 2-edge connected, $e_G(H, G \setminus H) \geq 2$ which implies that $|G_{\Delta_f}| \geq 5$, a contradiction. So we can assume that $V(G) = V(H)$. If there exists an edge $e \in E(G) \setminus E(H)$, then two end points of e is in $V(H) \setminus V(H_{\Delta_f})$, and so $|G_{\Delta_f}| \geq 5$, a contradiction. Thus $G = H$ and hence G is f-critical and the proof is complete. ■

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Remark 2 The assumption of 2-edge connectivity in Lemma 5 is not superfluous. Let G be the graph in Figure 3 and define $f(x) = 1$, for every $x \in V(G)$. Clearly, $G_{\Delta_f} = K_4$ and G and $H = G \setminus \{uv\}$ are f-class 2 (note that since $f(x) = 1$, for every $x \in V(G)$, the f-coloring and proper edge coloring coincide). Thus G is not f-critical.

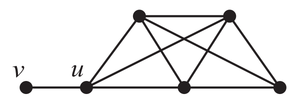

Figure 3

We close the paper with the following theorem.

Theorem 18 *Let G be a 2-edge connected graph with* $|G_{\Delta_f}| = 4$*.*

(i) *If G has an even order, then G is f-class* 1*.*

(ii) *If G has an odd order, then G is f-class* 2 *if and only if G is f-overfull.*

Proof (i) Assume to the contrary that G is f-class 2. Then by Lemma 5 G is f-critical, and so by Lemma 4 $f(v) = 1$ for every $v \in V(G)$. Now by Theorem 13 G is f-class 1, a contradiction.

(ii) First note that if G is f-overfull, then by Theorem 6 we are done. Now suppose that G is f-class 2. Then by Lemma 5, G is f-critical, and so by Lemma 4 $f(v) = 1$, for every $v \in V(G)$. Thus $f(V) = |G|$. Now by Theorem 14 $|E(G)| \geq \frac{|G|-1}{2}\Delta(G) + 1 > \lfloor \frac{|G|}{2} \rfloor \Delta(G)$. Therefore, G is overfull, and since $f(v) = 1$, for every $v \in V(G)$, G is f-overfull, and the proof is complete. ■

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