Can. J. Math., Vol. XXXI, No. 3, 1979, pp. 458–464

HAMILTONIAN CUBIC GRAPHS AND CENTRALIZERS OF INVOLUTIONS

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1. Introduction. In 1948, R. Frucht [**5**] proved that, *given a finite group G, there are infinitely many connected cubic graphs X such that the automorphism group* Aut X *is isomorphic to G.* In a letter, Professor Frucht has proposed the problem, whether in addition X can be required to be *hamiltonian.* One of the aims of the present note is to answer this question affirmatively.

THEOREM 1.1. *Given a finite group G there are infinitely many finite hamiltonian cubic graphs Y such that* Aut $Y \cong G$.

In fact, we prove the following:

THEOREM 1.2. *Given a finite cubic graph X having no component isomorphic to* K_4*, there exists a hamiltonian cubic graph Y such that* Aut $Y \cong$ Aut X *and* $|V(Y)| = 6|V(X)|$.

This implies 1.1 by the theorem of Frucht [**5**] mentioned above.

One might ask the question, analogous to 1.1, for endomorphism monoids rather than automorphism groups. It is, however, not true that any finite monoid would be isomorphic to the endomorphism monoid of a cubic graph [**1**]. Nevertheless, we mention a positive result, analogous to 1.2, without proof:

THEOREM 1.3. *There exists a full embedding of the category of finite cubic graphs into the category of finite hamiltonian cubic graphs.*

The proof goes via a more involved construction than the one given in this note, but the underlying idea is the same.

It turns out that the same construction yields an answer to quite different kinds of problems. Papers by G. Birkhoff and others [**2**, **6**, **8**, **7**, **4**] have shown that various classes of algebraic structures (2-unary algebras, commutative semigroups, integral domains, distributive lattices, etc.) permit a prescribed automorphism group. This is not the case for the class of groups, e.g., the automorphism group of a group must not be a cyclic group of odd order (cf. [**11**] and [**9**]). From the point of view of universal algebra, the structures next to groups are groups endowed with additional constants (0-ary operations). The automorphism group of such an algebra is the stabilizer of the constants in the original automorphism group.

Received April 6, 1977 and in revised form June 19, 1978. This paper is dedicated to Professor Roberto Frucht on his 73rd birthday.

Problem. Given a group G, does there exist a group H and a finite number of members of H, $\alpha_1, \ldots, \alpha_k$ say, such that the stabilizer of $\{\alpha_1, \ldots, \alpha_k\}$ in Aut H is isomorphic to G?

We give an affirmative answer to this question for finite G (Theorem 1.4) but the problem remains open for infinite groups.

THEOREM 1.4. *Given a finite group G there are infinitely many finite groups H such that*

$$(\text{Aut } H)_{\alpha,\beta,\gamma} = (\text{Aut } H)_{\alpha,\delta} \cong G$$

for some $\alpha, \beta, \gamma, \delta \in H$, *where* $|\alpha| = |\beta| = |\gamma| = 2, \quad |\delta| = 4$.

(The subscript indicates stabilizer subgroup; $|\zeta|$ denotes the order of $\zeta \in G$.)

It is easy to see that this problem is related to the centralizers of small subsets of a group. This lead us to the following problem: which groups are isomorphic to the centralizer of some small set of involutions in a (finite) simple group? The question seems quite natural in view of the importance of the centralizer of an involution in a finite simple group (Brauer-Fowler, [**3**]). Theorem 1.5 shows that any finite group occurs as the centralizer of *three involutions* in an infinity of finite simple groups. We do not know the answer for two involutions, nor for infinite simple groups

THEOREM 1.5. *Given a finite group G there is an n_0 such that for each $n \geqq n_0$, A_n contains three involutions α, β, γ for which*

$$C_{S_n}(\alpha, \beta, \gamma) = C_{A_n}(\alpha, \beta, \gamma) \cong G.$$

S_n and A_n denote the symmetric and alternating groups, respectively.

2. Preliminaries. First, let us fix some conventions. By a graph we mean an undirected graph (except if otherwise stated), without loops and multiple edges. $V(X)$ and $E(X)$ denote the set of vertices and edges of the graph X, respectively. A *k-valent* graph has vertices of degree k only; a 3-valent graph will be called *cubic*. A *3-edge-coloring* of a graph X is a partition of $E(X)$ into 3 classes such that adjacent edges belong to different classes. For the rest of graph terminology the reader is referred to any textbook on graph theory.

A permutation group H, acting on the set Ω, is said to act *semiregularly* if the stabilizer H_x of any point $x \in \Omega$ consists of the identity only. An *involution* is a group element of order 2 (in most cases, a permutation of order 2). $C_G(x, y, \ldots)$ denotes the centralizer of $x, y, \ldots \in G$ in the group G.

By a *matching* on a set S we mean any fixed-point-free involution, acting on S.

For X a graph put

$$R(X) = \{(v, e) \colon \quad v \in V(X), \quad e \in E(X), \quad v \text{ and } e \text{ are incident}\}.$$

By a *crossing rule* in a 4-valent graph X we mean a matching f on $R(X)$ such that $f(v, e) = (v', e')$ implies $v = v'$. A closed walk $v_0 - e_1 - v_1 - \ldots - e_k - v_k = v_0$ $(v_i \in V(X),\ e_i \in E(X))$ in a 4-valent graph X is said to *respect the crossing rule* f, if $f(v_i, e_i) \neq (v_i, e_{i+1})$ for any $i \pmod k$.

The crossing rule f is *invariant*, if for any $(v, e) \in R(X)$ and $\alpha \in \operatorname{Aut} X$, $f(v, e) = (v, e')$ implies $f(\alpha v, \alpha e) = (\alpha v, \alpha e')$.

LEMMA 2.1. *Let X be a connected 4-valent graph and f a crossing rule in X. Then X has a closed eulerian trail which respects f.*

Proof. This is a consequence of a result of Kotzig [**10**], Theorem 1. For a given vertex v the crossing rule f can be localized to a matching f_v on E_v (the set of edges of X incident with v) by setting $f(v, e) = (v, f_v e)$ for each $e \in E_v$. f_v partitions E_v into two blocks $\{e, f_v e\}$ of size 2 (the "forbidden transitions" in Kotzig's terminology), and the result follows.

Remark 2.2. The usual algorithms for finding an eulerian trail can be easily modified to yield directly a trail respecting f.

COROLLARY 2.3. *Let X be a connected 4-valent graph. Replace each vertex v of X by a 4-circuit and attach the edges, incident to v, to the vertices of this 4-circuit in an arbitrary order. Then the resulting cubic graph Y is hamiltonian.*

Proof. The vertex-set of Y may be identified with $R(X)$ in a natural way. Define a crossing rule f by $f(v, e) = (v, e')$ if and only if (v, e) and (v, e') (as vertices of Y) are opposite vertices of the 4-circuit in Y which corresponds to v. Now, we apply Lemma 2.1 to find a closed eulerian trail $v_0 - e_1 - v_1 - \ldots - e_m - v_m = v_0$ in X, respecting f. Then clearly

$$(v_0, e_1), (v_1, e_1), (v_1, e_2), \ldots, (v_{m-1}, e_m), (v_m, e_m), (v_m, e_1) = (v_0, e_1)$$

is a hamiltonian cycle of Y.

3. The proof of theorem 1.2. Let $L(X)$ denote the line-graph of X. We shall need the following result of Whitney:

LEMMA 3.1. (Whitney [**12**]). *If X and Y are graphs every component of which has at least five vertices, then any isomorphism $L(X) \cong L(Y)$ is induced by an isomorphism $X \cong Y$.*

LEMMA 3.2. *If X is a cubic graph, then $L(X)$ is a 4-valent graph which admits an invariant crossing rule.*

Proof. Let e_1, e_2, e_3 be the edges of X, incident with a vertex v. Any member of $R(L(X))$ has the form $(e_1, \{e_1, e_2\})$. Define the crossing rule f by

$$f(e_1, \{e_1, e_2\}) = (e_1, \{e_1, e_3\}).$$

Clearly, f is invariant under Aut X. We have to prove that it is invariant under Aut $(L(X))$. This is easily seen if $X = K_4$ $(L(K_4)$ is the octahedron graph), and it follows from 3.1 if X is connected and has more than 4 vertices. Finally, it holds for any X since it holds for its connected components.

LEMMA 3.3. *Let X be a 4-valent graph, admitting an invariant crossing rule f. Define the graph $Y = M_f(X)$ by setting $V(Y) = R(X)$, and $\{(v_1, e_1), (v_2, e_2)\} \in E(Y)$ if and only if $(v_i, e_i) \in R(X)$, $i = 1, 2$, and either*

$$e_1 = e_2, v_1 \neq v_2, \quad or$$

$$e_1 \neq e_2, \quad v_1 = v_2 \quad and \quad f(v_1, e_1) \neq (v_2, e_2).$$

The graph Y is cubic, hamiltonian, has $4|V(X)|$ vertices, and satisfies $\text{Aut } Y \cong \text{Aut } X$.

Proof. It is clear that Y is cubic and that it has $4|V(X)|$ vertices. It is hamiltonian by 2.3, our construction being a particular case of that of 2.3. Clearly, Aut X induces a subgroup A of Aut Y. The fact that A actually coincides with Aut Y follows from the observation that all 4-circuits of Y correspond to vertices of X, and hence X can be uniquely reconstructed from Y.

Now, 1.2 follows: Let X be as in 1.2. Let f denote an invariant crossing rule in $X' = L(X)$ (according to 3.2). Apply 3.3 to obtain $Y = M_f(X')$. X' is 4-valent, Y is cubic and hamiltonian by 3.3. Aut $Y \cong$ Aut $X' \cong$ Aut X by 3.3 and 3.1. Since $2|V(X')| = 3|V(X)|$, we obtain $|V(Y)| = 6|V(X)|$ by 3.3.

4. The proof of theorems 1.4 and 1.5.

LEMMA 4.1. *If X is a cubic graph and* Aut X *acts semiregularly on $E(X)$, then* Aut $(L(X))$ *acts semiregularly on $V(L(X))$;* Aut $(L(X)) \cong$ Aut X*; and there exists a function g: $R(L(X)) \to \{1, 2, 3, 4\}$ such that*

(*i*) $g(e, l) = g(\alpha e, \alpha l)$ *for any* $e \in V(L(X))$, $l \in E(L(X))$, $\alpha \in \text{Aut } (L(X))$;

(*ii*) $g(e, l_1) \neq g(e, l_2)$ *for any two edges* $l_1 \neq l_2$ *of* $L(X)$ *having* $e \in V(L(X))$ *in common.*

Proof. Aut K_4 does not act semiregularly on $E(K_4)$, hence every component of X has more than 4 vertices. This implies, by 3.1, that Aut $(L(X))$ is induced by Aut X. Now let T be a set of representatives of the orbits of Aut $(L(X))$ on $V(L(X)) = E(X)$. For $t \in T$ assign the values 1, 2, 3, 4 to the pairs (t, l) $(l \in E(L(X))$, l and t incident) in an arbitrary order. Then, extend g to the other members of $R(L(X))$ by the rule $g(\alpha t, \alpha l) = g(t, l)$ $(t \in T)$. This definition is unique by the semiregularity of Aut $(L(X))$, and clearly g satisfies (*i*) and (*ii*).

LEMMA 4.2. *Let X be a 4-valent graph and g a function g: $R(X) \to \{1, 2, 3, 4\}$ such that*

(*i*) $g(v, e) = g(\alpha v, \alpha e)$ *for any* $(v, e) \in R(X)$, $\alpha \in \text{Aut } X$;
(*ii*) $g(v, e_1) \neq g(v, e_2)$ *for any* $e_1 \neq e_2$ $((v, e_i) \in R(X))$.

Let f denote the crossing rule on X satisfying $g(f(v, e)) = g(v, e) \bmod 2$ *for any* $(v, e) \in R(X)$. *Then* $Y = M_f(X)$ *has an invariant 3-edge-coloring, i.e., every color-class is invariant under* Aut Y. *Moreover*, $\text{Aut } Y \cong \text{Aut } X$, *and* $|V(Y)| = 4|V(X)|$.

Note that a crossing rule f of the required kind exists and is unique. $M_f(X)$ is defined in Lemma 3.3.

Proof. By 3.3, $\text{Aut } Y \cong \text{Aut } X$, and every automorphism of Y is induced by an automorphism of X. We divide $E(Y)$ into 3 classes:

$$A = \{\{(v_1, e), (v_2, e)\}: \ v_1, v_2 \in V(X), \quad e = \{v_1, v_2\} \in E(X)\};$$
$$B_1 = \{\{(v, e_1), (v, e_2)\}: \ \{g(v, e_1), g(v, e_2)\} = \{1, 2\} \text{ or } \{3, 4\}\};$$
$$B_2 = \{\{(v, e_1), (v, e_2)\}: \ \{g(v, e_1), g(v, e_2)\} = \{2, 3\} \text{ or } \{4, 1\}\}.$$

$\{A, B_1, B_2\}$ is a 3-edge-coloring of Y. A is obviously invariant under Aut X, and so are B_1 and B_2 by the invariance of g.

Remark 4.3. We observe that $B_1 \cup B_2$ is the disjoint union of 4-cycles which can be endowed with an orientation which is invariant under Aut Y.

Indeed, each 4-cycle corresponds to the four edges incident with a vertex of X, and a cyclic order of these edges is induced by g.

Let $\Omega = \{1, \ldots, n\}$ and π a permutation of Ω. The *permutation graph* P_π is a *directed graph* having Ω as its vertex set and $E(P_\pi) = \{(x, \pi x): \ x \in \Omega, \pi x \neq x\}$ as its set of edges. (P_π is undirected if and only if $\pi^2 = \text{id}$.)

PROPOSITION 4.4. *For* $\pi_1, \ldots, \pi_k \in S_n$ *we have*

$$C_{S_n}(\pi_1, \ldots, \pi_k) = \bigcap_1^k \text{Aut } (P_{\pi_i}).$$

Proof. The proof is straightforward.

PROPOSITION 4.5. *For* $k \geqq 6$ *there exists a graph Z satisfying* (*a*) $|V(Z)| = k$; (*b*) $|\text{Aut } Z| = 1$; (*c*) *Z is 3-edge-colorable such that each color-class consists of an even number of edges.*

Proof. Consider the $(k - 2)$-path $1, 2, \ldots, k - 1$. Join the point k to point 3; and for k even, join k additionally to point 2. Let Z be the graph so obtained. Obviously, (*a*) and (*b*) hold. The number of edges of Z is always even. Finding an appropriate 3-edge-coloring is an easy exercise.

Now we turn to the proof of 1.5. Let G be a given finite group and X_0 a connected cubic graph such that $\text{Aut } X_0 \cong G$ and Aut X_0 acts semiregularly both on the vertices and on the edges of X_0 (Frucht [**5**]). Let $X = L(X_0)$. So, Aut X is isomorphic to G and it acts semiregularly both on the vertices and on the edges of X by Whitney's Lemma 3.1. Use 4.1 to obtain g: $R(X) \to \{1, 2, 3, 4\}$ satisfying 4.2 (*i*) and (*ii*). Then, apply 4.2 to obtain the cubic graph Y. Let A, B C denote the invariant edge-color-classes obtained by 4.2.

Let $n_0 = 4|V(X)| + 6$ and $n \geqq n_0$. Furthermore, let $k = n - 4|V(X)|$. We have $k \geqq 6$, hence Proposition 4.5 applies. Let Z be the graph obtained and A', B', C' its edge-color-classes according to 4.5 (*c*).

The disjoint union of Y and Z has n vertices. We may assume $V(Y) \cup V(Z) = \{1, \ldots, n\}$. Aut $(Y \cup Z)$ consists of even permutations only, since Z is fixed under the action of Aut $(Y \cup Z)$, the action of Aut Y is semiregular and the number of its orbits is even (actually, it is divisible by 4, cf. 4.2). Moreover, the edge-sets $A \cup A'$, $B \cup B'$, $C \cup C'$ are invariant under Aut $(Y \cup Z)$. These sets form a 3-edge-coloring of $Y \cup Z$, hence they are permutation graphs P_α, P_β, P_γ resp., with α, β, γ involutions. By the above, applying Proposition 4.4, we have

$$C_{S_n}(\alpha, \beta, \gamma) = \text{Aut } (Y \cup Z) \leqq A_n.$$

Moreover, $\alpha, \beta, \gamma \in A_n$ since each color-class contains an even number of edges. (This is so by 4.5 (*c*) and since 4 divides $|V(Y)|$.)

Hence, observing that Aut $(Y \cup Z) \cong$ Aut $Y \cong G$, the proof of Theorem 1.5 is complete.

For 1.4, let $n = 4|V(X)|$ (by Frucht [**5**] there are infinitely many ways to choose X_0 and hence X). We shall assume that $|V(X)|$ is even. (A repeated application of our procedure, starting with Y instead of X_0, surely yields even $|V(X)|$.) We repeat the above arguments, in a simpler version, omitting Z. Now the color-classes A, $B = B_1$, $C = B_2$ (in the notation of the proof of Lemma 4.2) correspond to the involutions $\alpha, \beta, \gamma \in A_n$. Let δ be a permutation of order 4, keeping the 4-circuits of Y invariant, and increasing $g(v, e)$ by 1 (mod 4) at each vertex $(v, e) \in V(Y)$. Clearly, P_δ is a directed graph, whose symmetrisation has $B_1 \cup B_2$ as edges (cf. Remark 4.3). Moreover, Aut $P_\delta \geqq$ Aut Y, whence

$$C_{S_n}(\alpha, \delta) = \text{Aut } (P_\delta) \cap \text{Aut } (P_\alpha) = \text{Aut } Y.$$

δ belongs to A_n since $8|n$ ($|V(X)|$ being even).

Set $H = A_n$. As $n \neq 6$, Aut H can be identified with S_n (centralizer corresponding to stabilizer). We have

$$(\text{Aut } H)_{\alpha,\beta,\gamma} = C_{S_n}(\alpha, \beta, \gamma) = \text{Aut } Y = C_{S_n}(\alpha, \delta) = (\text{Aut } H)_{\alpha,\delta}.$$

The proof of Theorem 1.4 is complete.

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