

COHOMOLOGICAL FINITENESS PROPERTIES OF THE BRIN–THOMPSON–HIGMAN GROUPS $2V$ AND $3V$

DESSISLAVA H. KOCHLOUKOVA¹, CONCHITA MARTÍNEZ-PÉREZ²
AND BRITA E. A. NUCINKIS^{3*}

¹*Department of Mathematics, University of Campinas,
13083-859 Campinas, São Paulo, Brazil (desi@unicamp.br)*

²*Departamento de Matemáticas, Universidad de Zaragoza,
50009 Zaragoza, Spain (conmar@unizar.es)*

³*School of Mathematics, University of Southampton, University Road,
Southampton SO17 1BJ, UK*

(Received 6 October 2011)

Abstract We show that Brin’s generalizations $2V$ and $3V$ of the Thompson–Higman group V are of type FP_∞ . Our methods also give a new proof that both groups are finitely presented.

Keywords: cohomological finiteness conditions; generalized Thompson’s groups

2010 *Mathematics subject classification:* Primary 20J05

1. Introduction

In this paper, we study the cohomological finiteness conditions of certain generalizations of Thompson’s group V , which is a simple, finitely presented group of homeomorphisms of the Cantor set C . The finiteness conditions we consider are the homotopical finiteness property F_∞ for a group, which was first defined by Wall, and its homological version FP_∞ , which was studied in detail in [3]. We say that a group G is of type F_∞ if it admits a $K(G, 1)$ with finite k -skeleton in all dimensions k . A group is of type FP_∞ if the trivial $\mathbb{Z}G$ -module \mathbb{Z} has a resolution with finitely generated projective $\mathbb{Z}G$ -modules. A group is of type F_∞ if and only if it is of type FP_∞ and is finitely presented. There are, however, examples of groups of type FP_∞ that are not finitely presented [2].

In [7], Brown showed that Thompson’s groups F , T and V , as well as some generalizations such as Higman’s groups $V_{n,r}$ (see [11]), are of type F_∞ . Brown achieved this by expressing these groups as groups of algebra-automorphisms, letting them act on a poset determined by the algebra and then showing that the geometric realization of this poset yields the required finiteness properties.

* Present address: Department of Mathematics, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK (brita.nucinkis@rhul.ac.uk).

In [5], Brin defined a group sV generalizing V for every natural number $s \geq 2$. Analogously to V , these groups are defined as subgroups of the homeomorphism group of a finite Cartesian product of the Cantor set. For each s , the group sV is simple, finitely presented and contains a copy of every finite group [6, 10]. It was also shown in [4] that, for $s \neq t$, sV is not isomorphic to tV .

Our main result is the following.

Main Theorem. Brin's groups $2V$ and $3V$ are of type F_∞ .

The proof of the main theorem is split into two parts: Theorems 4.17 and 5.6. We partially follow the proof of [7], which states that V has type F_∞ . Our proof is more intricate, as the fact that some particular complex K_Y is t -connected if Y is sufficiently large requires more work than in Brown's proof. As in [7], we view sV as a group of algebra automorphisms and consider a poset \mathfrak{A} on which sV acts. This action has the following properties.

- (i) Vertex stabilizers are finite.
- (ii) The complex $|\mathfrak{A}|$ is contractible.
- (iii) There is a filtration $\{|\mathfrak{A}_n|\}_{n \geq 1}$ of sV -subcomplexes of $|\mathfrak{A}|$ such that each complex $|\mathfrak{A}_n|$ is finite modulo sV .
- (iv) For $s = 2$ and $s = 3$ the connectivity of the pair of complexes $(|\mathfrak{A}_{n+1}|, |\mathfrak{A}_n|)$ tends to infinity as $n \rightarrow \infty$.

We then apply Brown's criterion [7, Corollary 3.3] to conclude that $2V$ and $3V$ are finitely presented and of type F_∞ . The key result towards the proof of our main theorem for $s = 2$ is Theorem 4.6. Finally, in the last section, we prove Theorem 5.3 as a variation of Theorem 4.6 and show that the method above can be applied for $s = 3$.

2. Construction of the algebra and the group

In this section, we define the generalized Higman algebra, also called Cantor algebra, in a general setting. We then define sV as a group of automorphisms of this algebra.

Consider a finite set $\{1, \dots, s\}$. We call its elements colours. Also, consider a finite set of integers $\{n_1, \dots, n_s\}$, $n_i > 1$. We call each n_i the arity of the colour i . We begin by defining an Ω -algebra U . For details, the reader is referred to [9] (see also [8]). We say U is an Ω -algebra if, for each colour i , the following operations are defined in U .

- (i) One n_i -ary operation λ_i :

$$\lambda_i: U^{n_i} \rightarrow U.$$

We call these operations ascending operations, or contractions.

- (ii) n_i 1-ary operations $\alpha_i^1, \dots, \alpha_i^{n_i}$:

$$\alpha_i^j: U \rightarrow U.$$

We call these operations 1-ary descending operations.

Throughout this paper all operations act on the right. By definition, $\Omega = \{\lambda_i, \alpha_i^j\}_{i,j}$. In what follows, it will be convenient to consider the following map, which we also call an operation. For each colour i , and any $v \in U$, we define

$$v\alpha_i := (v\alpha_i^1, v\alpha_i^2, \dots, v\alpha_i^{n_i}).$$

Therefore, α_i is a map

$$\alpha_i: U \rightarrow U^{n_i}.$$

We call these maps descending operations, or expansions. Unless otherwise stated, whenever we use the term ‘descending operation’, we refer to one of the α_i .

For any subset Y of U , a simple expansion of colour i of Y consists of substituting some element $y \in Y$ by the n_i elements of the tuple $y\alpha_i$. And a simple contraction of colour i of Y is the set obtained by substituting a certain collection of n_i distinct elements of Y , say $\{a_1, \dots, a_{n_i}\}$, by $(a_1, \dots, a_{n_i})\lambda_i$. We also use ‘operation’ to refer to the effect of a simple expansion (respectively, contraction) on a set.

A morphism between Ω -algebras is a map commuting with all operations in Ω . Let \mathfrak{B}_0 be a category of Ω -algebras. An object $U_0(X) \in \mathfrak{B}_0$ is a free object in \mathfrak{B}_0 , with X as a *free basis*, if for any $S \in \mathfrak{B}_0$ any mapping

$$\theta: X \rightarrow S$$

can be extended in a unique way to a morphism

$$U_0(X) \rightarrow S.$$

We also say that $U_0(X)$ is free on X in the category \mathfrak{B}_0 . Following [9, III.2], we construct the free object on any set X in the category of all Ω -algebras as follows. Take the set of finite sequences of elements of the disjoint union $\Omega \cup X$, with the Ω -algebra structure defined by juxtaposition. Then, $U_0(X)$ is the sub Ω -algebra generated by X .

Definition 2.1. The free object constructed above is called the Ω -word algebra and is denoted $W_\Omega(X)$. An *admissible* subset is any $Y \subset W_\Omega(X)$ that can be obtained from X by a finite number of operations α_i and λ_j , i.e. by a finite number of simple contractions or expansions.

Now we consider the variety of Ω -algebras satisfying a certain set of identities.

Definition 2.2. Let Σ_1 be the following set of laws in a countable (possibly finite) alphabet X .

- (i) For any $u \in W_\Omega(X)$ and any colour i ,

$$u\alpha_i\lambda_i = u.$$

- (ii) For any colour i and any n_i -tuple $(u_1, \dots, u_{n_i}) \in W_\Omega(X)^{n_i}$,

$$(u_1, \dots, u_{n_i})\lambda_i\alpha_i = (u_1, \dots, u_{n_i}).$$

The variety \mathfrak{V}_1 of Ω -algebras that satisfy the identities in Σ_1 obviously contains non-trivial algebras. Hence, it is a non-trivial variety. Therefore, by [9, IV 3.3], it contains free algebras on any set X . Let $U_1(X)$ be the free Ω -algebra on X in \mathfrak{V}_1 . Moreover, by the proof of [9, IV 3.1],

$$U_1(X) = W_\Omega(X)/\mathfrak{q}_1,$$

where \mathfrak{q}_1 is the fully invariant congruence generated by Σ_1 , i.e. the smallest equivalence set in $W_\Omega(X) \times W_\Omega(X)$ containing Σ_1 , which admits any endomorphism of $W_\Omega(X)$ and is Ω -closed (see [9, IV § 1]). In fact, there exists an epimorphism

$$\theta_1: W_\Omega(X) \rightarrow U_1(X)$$

and \mathfrak{q}_1 corresponds precisely to $\text{Ker}(\theta_1)$.

Definition 2.3. Let $U \in \mathfrak{V}_1$ and let Y be a subset of U . A set Z obtained from Y by a finite number of simple expansions is called a descendant of Y . In this case, we define

$$Y \leq Z.$$

Conversely, Y is called an ascendant of Z and can be obtained after a finite number of simple contractions.

In what follows, we will consider Ω -algebras satisfying some additional identities, as described below.

Definition 2.4. Let Σ be the set of identities

$$\Sigma = \Sigma_1 \cup \{r_{ij} \mid 1 \leq i < j \leq s\},$$

where r_{ij} consists of certain identifications between sets of simple expansions of $w\alpha_i$ and $w\alpha_j$ for any $w \in W_\Omega(X)$ that do not depend on w .

Let X be a set and let $U(X) = U_1(X)/\mathfrak{q}$, where \mathfrak{q} is the fully invariant congruence generated by Σ . There exists an epimorphism

$$\begin{aligned} \theta_2: U_1(X) &\rightarrow U(X) \\ a_1 &\mapsto \bar{a}_1. \end{aligned}$$

Let $\theta: W_\Omega(X) \rightarrow U(X)$ be the composition of θ_1 with θ_2 . We say that a subset Y of $U_1(X)$ or of $U(X)$ is *admissible* if it is the image by θ_1 or θ of an admissible subset of $W_\Omega(X)$. We call the set of identities Σ *valid* if the following condition holds: for any admissible set $Y \subseteq U_1(X)$ we have that $|Y| = |\bar{Y}|$, i.e. θ_2 is injective on admissible subsets.

Let \mathfrak{V} be the variety of all Ω -algebras that satisfy the identities in a valid Σ . Note that \mathfrak{V} contains non-trivial Ω -algebras, so it has free objects on every set X . In fact, the algebra $U(X)$ above is a free object on X .

Lemma 2.5. *Any admissible subset is a free basis in $W = U(X)$.*

Proof. This can be proven using the same argument as in [11]. Let X be a free basis of W , let $i \in \{1, \dots, s\}$ be any colour of arity n_i and let

$$Y = (X \setminus \{x\}) \cup \{x\alpha_i^j \mid 1 \leq j \leq n_i\}.$$

We will show that Y is a free basis of W . Recall that \mathfrak{V} is the variety of Ω -algebras satisfying the identities Σ . Then, given any $S \in \mathfrak{V}$ and any mapping $\theta: Y \rightarrow S$, there is a unique way to obtain a map $\theta^*: X \rightarrow S$ such that $\theta^*(\tilde{x}) = \theta(\tilde{x})$ for $\tilde{x} \in X \setminus \{x\}$ and $\theta^*(x) = (\theta(x\alpha_i^1), \dots, \theta(x\alpha_i^{n_i}))\lambda_i$. As there exists a unique $\hat{\theta}: W \rightarrow S$ extending θ^* , the same happens with the original θ .

Analogously, considering n_i distinct elements x_1, \dots, x_{n_i} of X , one proves that the admissible subset

$$Y = (X \setminus \{x_1, \dots, x_{n_i}\}) \cup \{(x_1, \dots, x_{n_i})\lambda_i\}$$

is a free basis of W . □

Definition 2.6. Consider the set of s colours $\{1, \dots, s\}$, all of which have arity 2, together with the relations

$$\Sigma := \Sigma_1 \cup \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \leq i \neq j \leq s; l, t = 1, 2\}.$$

We call the Ω -algebra $W = U(\{x_0\})$, defined by the Σ above, the generalized Higman algebra on s colours.

Remark 2.7 (geometric interpretation of the generalized Higman algebra). Consider the unit cube \mathfrak{C} of \mathbb{R}^s . Fix a bijection between the set of colours $\{1, \dots, s\}$ and the set of hyperplanes that are parallel to the faces of \mathfrak{C} . To each operation α_i we associate a halving using a hyperplane parallel to the hyperplane corresponding to i . In this case, we say that we halve in direction i . Then, to each side of this halving we associate one of the components of α_i : α_i^1 and α_i^2 . This association will stay fixed. For a sequence of 1-ary descending operations $u = \alpha_{i_1}^{r_1} \cdots \alpha_{i_t}^{r_t}$, with $r_j \in \{1, 2\}$, we perform the following operations in \mathfrak{C} . First, halve it in direction i_1 and take the r_1 -half. Repeat the process with the operation $\alpha_{i_2}^{r_2}$ for this half. At the end, we get a subset (subparallelepiped) of \mathfrak{C} . For simplicity, we call the subparallelepipeds s -subcubes, or simply s -cubes. Note that, at any stage, if $i \neq j$, the effect of $\alpha_i^{r_i} \alpha_j^{r_j}$ equals the effect of $\alpha_j^{r_j} \alpha_i^{r_i}$.

The family of s -subcubes of the s -cube \mathfrak{C} , which can be obtained in this way, corresponds to the set $x(D)$ of descendants of x in the generalized Higman algebra $U(\{x_0\})$, where x is an element belonging to some admissible subset. Analogously, we may identify any admissible subset A with a collection of $|A|$ s -cubes. In particular, the set of descendants of A corresponds to the set of those subsets in the collection of $|A|$ s -cubes that are obtained in the prescribed way.

Remark 2.8. In Figure 2, we use two different types of carets to visualize the two colours in the generalized Higman algebra on two colours, each of arity 2.

The first type of caret corresponds to vertical cutting and the second one to horizontal cutting. We view an admissible set that is a descendant of an element x as the set of

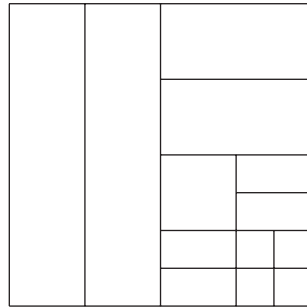


Figure 1. Subdividing the square.



Figure 2. A vertical and a horizontal cutting.

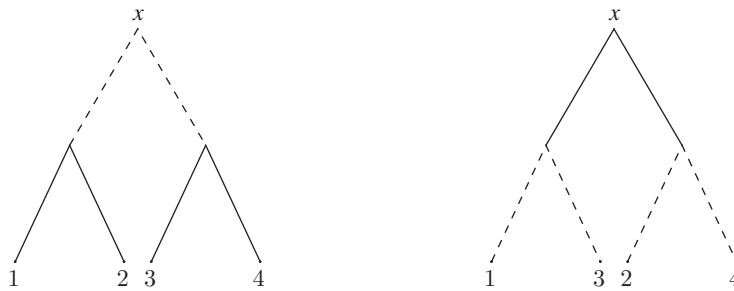


Figure 3. Two trees representing the same admissible set.

leaves of a rooted tree with root x . The rooted tree is constructed by gluing one of the two types of carets when passing to descendants. The two rooted trees in Figure 3 represent the same admissible set. In Figure 1 we present an example of successive subdivisions of the square.

Considering the geometric interpretation of the generalized Higman algebra, both of the rooted trees in Figure 3 represent the subdivision of the square in Figure 4.

Lemma 2.9. *The generalized Higman algebra $W = U(\{x_0\})$ is valid.*

Proof. To begin, we claim that for any pair of admissible subsets Y and $Z \subseteq U_1(\{x_0\})$, such that Z is obtained from Y after a simple expansion, we have that $|\bar{Z}| = |\bar{Y}| + 1$. Recall that \bar{Z} and \bar{Y} are the images of Z and Y in $U(\{x_0\})$. Any admissible set in $U_1(\{x_0\})$ is a descendant of an admissible set with only one element, say y . So for $x = \bar{y}$ we have that $\bar{Z}, \bar{Y} \in x(D)$, where $x(D)$ is as defined in Remark 2.7. Using the geometric interpretation of $x(D)$ as a subdivision of an s -cube, we verify the claim.

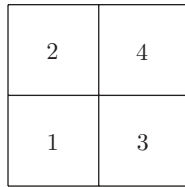


Figure 4. The subdivision of the square corresponding to the tree diagrams in Figure 3.

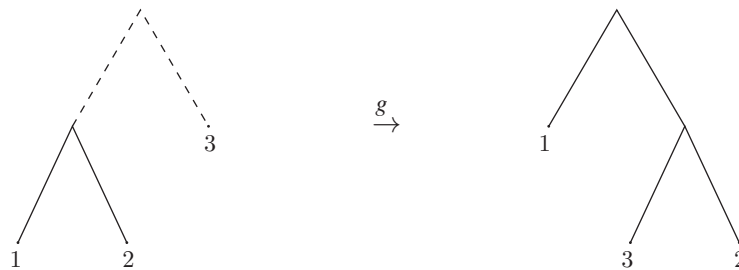


Figure 5. An element of $2V$.

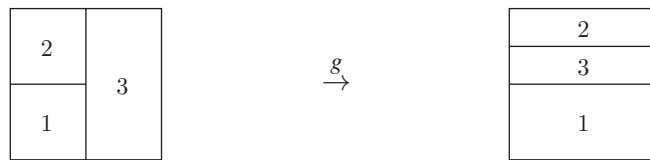


Figure 6. The same element given by subdivisions of the square.

Conversely, if Z is a simple contraction of Y , then Y is a simple expansion of Z . Thus, $|\bar{Y}| = |\bar{Z}| + 1$.

Finally, an induction on the number of simple contractions and expansions needed to obtain an admissible subset $\bar{Y} \subseteq U(\{x_0\})$ from $\{x_0\}$ yields the result. \square

Definition 2.10. The Brin–Thompson–Higman group on $W_0 = U(X)$, which we denote $G(W_0)$, is the group of algebra automorphisms of W_0 that are induced by a bijection $Z \rightarrow Y$ for any two admissible sets Z and Y of the same cardinality. If W is the generalized Higman algebra $U(\{x_0\})$, then $G(W)$ is the Brin group on s colours and is denoted sV .

Figure 5 illustrates an element g of $2V$ sending each leaf to the leaf with the same label.

Remark 2.11. Looking at the geometric interpretation of the generalized Higman algebra, [5, §2.3] implies that this is exactly the definition of Brin’s generalization $2V$ of V as a group of self-homeomorphisms of $C \times C$, where C denotes the Cantor set. The element g in Figure 5 corresponds to Figure 6.

The equivalence of definitions for higher-dimensional sV follows from [5, §4.1]. If there exists only one colour, then V is exactly the Higman–Thompson group as defined in [7].

3. The poset of admissible subsets

In this section, we consider the Brin–Higman algebra on s colours with basis $\{x\}$. We write U for $U(\{x\})$.

Definition 3.1. The set of admissible subsets is a poset with the order defined by $A < B$ if B is a descendant of A . We denote this poset by \mathfrak{A} , and by $|\mathfrak{A}|$ its geometric realization. Note that any descendant and any ascendant of an admissible subset is also admissible.

Given admissible subsets Y and Z of U , we say that they have a unique least upper bound T if $Y \leq T$ and $Z \leq T$, and whenever $Y \leq S$ and $Z \leq S$, then $T \leq S$. Analogously, we define the notion of greatest lower bound.

Lemma 3.2. *Let A, Y and Z be admissible subsets with $A \leq Y$ and $A \leq Z$. Then, there is a unique least upper bound of Y and Z .*

Proof. Consider the geometric representation of the set of descendants of A as subdivisions of s -dimensional cubes (in fact s -dimensional parallelepipeds, but we call them cubes for simplicity) labelled by the elements of A ; see Remark 2.7. Then, the result of performing both sets of subdivisions corresponding to Y and Z yields an upper bound T . Clearly, for any other upper bound S of Y and Z we have $T \leq S$. \square

Lemma 3.3. *Let Y, Y_1 and Z be admissible subsets, with*

$$Y \geq Y_1 \leq Z.$$

Then, there exists some admissible subset Z_1 , with

$$Y \leq Z_1 \geq Z.$$

Proof. Observe that Y and Z are both descendants of Y_1 . Then, by Lemma 3.2 there exists an upper bound Z_1 of Y and Z . So we have $Y \leq Z_1 \geq Z$. \square

Proposition 3.4. *Any two admissible subsets have some upper bound.*

Proof. Let Y and Z be two admissible subsets. By definition, we can obtain Z from Y by a finite number of expansions or contractions. Therefore, we may set

$$Y \geq Y_1 \leq Y_2 \geq Y_3 \leq \cdots \geq Y_r \leq Z.$$

By Lemma 3.3 we get

$$Y \leq Z_1 \geq Y_2 \geq Y_3 \leq \cdots,$$

and we may shorten the previous chain by omitting Y_2 to get a chain

$$Y \leq Z_1 \geq Y_3 \leq \cdots.$$

Thus, after finitely many steps we get

$$Y \leq T \geq Z \quad \text{or} \quad Y \geq T \leq Z$$

for some T . In the second case, we apply Lemma 3.3. \square

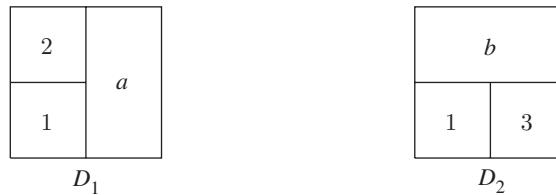


Figure 7. Simple contractions of the basis given in Figure 3.

Proposition 3.4 has the following consequence: for any admissible subset A , any element $g \in G(sV)$ can be represented by its action in the set of descendants of A , i.e. there is some $A \leq Z$ with $A \leq Zg$. To see this, choose Z to be some upper bound of A and Ag^{-1} . Then, $A \leq Z$ and $Ag^{-1} \leq Z$, so $A \leq Zg$.

Lemma 3.5. $|\mathfrak{A}|$ is contractible.

Proof. It is a consequence of Proposition 3.4, as the poset \mathfrak{A} is directed. □

Remark 3.6. Observe that, as in the case of V considered in [7], the stabilizer of any admissible set Y in sV is finite, as it consists precisely of the permutations of the elements of Y .

We consider the filtration of $|\mathfrak{A}|$ given by

$$\mathfrak{A}_n := \{Y \in \mathfrak{A} \mid |Y| \leq n\}.$$

Lemma 3.7. Each $|\mathfrak{A}_n|/sV$ is finite.

Proof. For any Y and $Z \in \mathfrak{A}_n$, with $|Y| = |Z|$, we may consider the element $g \in sV$ given by $yg = y\sigma$, where $\sigma: Y \rightarrow Z$ is a fixed bijection. Thus, sV acts transitively on the admissible sets of the same size. □

Contrary to what happens with upper bounds, it is, in general, not true that any two admissible subsets have some lower bound, as the following example shows.

Example 3.8. Consider the case of the Brin–Higman algebra with two colours and basis $A = \{x\}$. Let B be the basis represented in Figure 3. Using the same notation as in the figure, we label the elements of B as follows: $1 := x\alpha_2^1\alpha_1^1$, $2 := x\alpha_2^2\alpha_1^1$, $3 := x\alpha_2^1\alpha_1^2$, $4 := x\alpha_2^2\alpha_1^2$. Consider the following bases:

$$\begin{aligned} D_1 &:= \{1, 2, a\}, & \text{with } a &:= (3, 4)\lambda_2, \\ D_2 &:= \{1, 3, b\}, & \text{with } b &:= (2, 4)\lambda_1, \\ D_3 &:= \{1, 4, c\}, & \text{with } c &:= (3, 2)\lambda_2. \end{aligned}$$

We have that $D_1, D_2, D_3 \leq B$, and all are simple contractions. Furthermore, we may represent D_1 and D_2 , but not D_3 , as partitions of a square representing x , as in Figure 7.

As $\{x\} \leq D_1, D_2$, these two bases have a common lower bound. However, D_1 and D_3 do not, following from basically the same argument as in [7, Lemma 4.18]. Compare this with

the classical Higman algebra: the case of only one colour. In this case, [7, Lemma 4.18] yields that two simple contractions of a given basis have a common lower bound if and only if the contracted vertices are disjoint.

The existence of greatest lower bounds in some particular cases will be crucial in the subsequent sections. To overcome this problem, we assume that our contractions are descendants of the same A and consider greatest lower bounds above A . For simplicity we use the following notation.

Definition 3.9. Let Λ be a finite set of admissible sets and let A_1 and A_2 be admissible sets. We write that

$$A_1 \leq \Lambda \quad \text{if for every } B \in \Lambda \text{ we have } A_1 \leq B$$

and

$$A \leq A_2 \quad \text{if for every } B \in \Lambda \text{ we have } B \leq A_2.$$

Definition 3.10. Let A be an admissible set and let $\Omega = \{Y_0, \dots, Y_t\}$ be a finite set of admissible sets, with $A \leq \Omega$. Assume that there exists an admissible set M such that $A \leq M \leq \Omega$ and for any other admissible set B , with $A \leq B \leq \Omega$, we have $B \leq M$. Then, we call M a greatest lower bound of Ω above A and define $M = \text{glb}_A(\Omega)$.

Definition 3.11. Let $A \leq Y$ be admissible sets and let $r \geq 0$ be an integer. We say that A involves contractions of r elements of Y , or involves r elements of Y for short, if $|Y \setminus A| = r$; we also say that $Y \setminus A$ are the elements of Y contracted in A . Two contractions $A_1, A_2 \leq Y$ are said to be disjoint if the respective sets of elements of Y contracted in A_1 and A_2 are disjoint.

In the particular case of disjoint contractions of a certain admissible Y , the existence of greatest lower bounds follows easily.

Lemma 3.12. Let $\Omega = \{M_0, \dots, M_t\}$ be a set of pairwise disjoint contractions of Y . Then,

$$\emptyset \neq \bigcap_i \{L \mid L \leq M_i\}$$

has a maximal element M , which we call a global greatest lower bound for Ω and denote by $\text{glb}(\Omega)$. In particular, for any $A \leq \Omega$, M is a $\text{glb}_A(\Omega)$. Moreover,

$$|\text{elements of } Y \text{ involved in } M| = \sum_{0 \leq i \leq t} |\text{elements of } Y \text{ involved in } M_i|.$$

Proof. We obtain M by successively performing the contractions M_i . □

Lemma 3.13. Let A be an admissible set and let $\Omega = \{Y_0, \dots, Y_t\}$ be a finite set of admissible sets such that $A \leq \Omega$. Then, for an admissible subset M we have that $M = \text{glb}_A(\Omega)$ if and only if $A \leq M \leq \Omega$ and there is no expansion N with $M < N$ and $N \leq \Omega$.

Proof. First, assume that $M = \text{glb}_A(\Omega)$. If $M < N \leq \Omega$, then $A \leq N \leq \Omega$ and, therefore, $N \leq M$, which is a contradiction.

Conversely, we prove that if there is no N , as before, then M is a greatest lower bound above A . Assume that there exists some admissible set B such that $A \leq B \leq \Omega$. Recall that by Lemma 3.2 there exists a unique smallest upper bound C of B and M above A . Then,

$$A \leq \{B, M\} \leq C \leq \Omega.$$

If $M < C$, we have a contradiction and, therefore, $M = C$, and thus $B \leq M$. □

Lemma 3.14. *Let A be an admissible set and let $\Omega = \{Y_0, \dots, Y_t\}$ be a finite set of admissible sets such that $A \leq \Omega$. Then, there exists $M = \text{glb}_A(\Omega)$.*

Proof. Observe that the following set is finite and non-empty:

$$\mathfrak{S} = \{N \text{ admissible} \mid A \leq N \leq \Omega\}.$$

This means that we may choose an element $M \in \mathfrak{S}$ maximal with respect to the ordering. By Lemma 3.13, $M = \text{glb}_A(\Omega)$. □

For use in subsequent sections, we now record the following obvious consequence of the definition of greatest lower bounds and Lemma 3.13.

Lemma 3.15. *Let A be an admissible set and let $\Omega = \{Y_0, \dots, Y_t\}$ be a finite set of admissible sets such that $A \leq \Omega$. Consider $A \leq B$ and a subset $\Lambda \subseteq \Omega$ such that $B \leq \Lambda$. Then,*

$$\text{glb}_A \Omega \leq \text{glb}_A \Lambda = \text{glb}_B \Lambda.$$

4. Connectivity of $|K_Y|$ and proof of the main result for $s = 2$

Let Y be any admissible subset of the Brin–Higman algebra on s colours. We set

$$K_Y := K_{<Y} = \{Z \mid Z \text{ is admissible with } Z < Y\}.$$

Note that K_Y is a poset. We also consider its geometric realization, which we denote $|K_Y|$.

Our next objective is to prove that, in the case of two colours and $|Y|$ big enough, this complex $|K_Y|$ is t -connected. To do this, we argue as follows. Firstly, we show that the considered complex can be ‘pushed down’ in the sense that its t -connectedness is equivalent to the connectedness of a certain subcomplex Σ_{4t} , defined in § 4.1. Then, we use an argument similar to Brown’s argument in [7] to prove that Σ_{4t} is t -connected for $|Y|$ big enough and to deduce, in the last subsection, that $2V$ is of type F_∞ .

In the first subsection we begin with some general observations, valid for an arbitrary number of colours s .

Figure 8. Y is a descendant of the admissible set A .

4.1. Some general observations

Definition 4.1. Denote by C_r the following subposet of K_Y ,

$$C_r := \{A \in K_Y \mid A < Y \text{ and } A \text{ involves at most } r \text{ elements of } Y\},$$

and denote by Σ_r the following subcomplex of $|K_Y|$,

$$\Sigma_r := \{\sigma : A_t < A_{t-1} < \cdots < A_1 < A_0 \mid \sigma \in |K_Y|, A_t \in C_r\}.$$

We denote by Σ_r^t the t -skeleton of Σ_r .

To construct the pushing procedure we need to control the number of elements involved in the greatest lower bounds of certain sets of simple contractions of Y . To do that, we use the notion of length, which we define next.

Definition 4.2. Consider $A \in K_Y$. For any $i \in Y$, there exists a unique $m \in A$ such that the s -cube labelled m contains the s -cube labelled i . Then, i is obtained by a certain number of successive subdivisions of m . We call that number the length of i as descendant of A and denote it by $l(A, i)$. We say that two elements $i, j \in Y$ are glueable in A if there exists some simple contraction $Z < Y$ (of any colour) contracting i, j exactly such that $A \leq Z$. Note that, in that case, $l(A, i) = l(A, j)$.

We also say that $i \in Y$ is locally maximal with respect to A if for any other $j \in Y$ obtained from the same $m \in A$ we have that $l(A, i) \geq l(A, j)$. Clearly, in that case, any other vertex that is glueable to i in A is also locally maximal.

For example, consider the admissible subset A (Figure 8) in the case of two colours and its descendant Y . Here, we have that $l(A, 5) = 2$ and that 6 and 5 are glueable. So are 1 and 2. Moreover, all the elements except 4 are locally maximal with respect to A .

Lemma 4.3. Let $A \leq B < Y$ be admissible subsets. If $i \in Y$ is locally maximal with respect to A , then it is also locally maximal with respect to B .

Proof. Let $m_A \in A$, $m_B \in B$ be the elements in the respective set from which i is obtained. It suffices to note that any $j \in Y$ obtained from m_B is also obtained from m_A . \square

If $A \leq Y$ and we use the geometric description of Y as partitions of s -cubes, then the length of $i \in Y$ is related to the size of the subcube labelled i . If two vertices i, j are glueable, then the cubes labelled i and j have exactly the same sizes and are neighbours. This implies that, for fixed i , there are at most $2s$ vertices that are glueable to i . The next result implies that this bound is in fact $2(s - 1)$.

Lemma 4.4. *Let $A \leq \{Y_0, Y_1\} < Y$, where Y_1 and Y_2 are different, not disjoint, simple contractions of Y of colours a and b . Label with $\{1, 2\}$ the vertices contracted in Y_0 and with $\{2, 3\}$ those contracted in Y_1 . Then, the vertices labelled 1 and 3 are different, and $a \neq b$.*

Proof. We use the geometric realization of sV . Assume that $a = b$. As $Y_0 \neq Y_1$ this would mean that the s -cubes labelled 1 and 3 were situated at opposite sides of the s -cube labelled 2. This, however, is impossible since α_a^1 and α_a^2 do not commute. In particular, if one side of an s -cube can be deleted in a contraction, then the opposite side cannot be deleted. Therefore, $a \neq b$ and the s -cubes labelled 1 and 3 are on the sides of the s -cube labelled 2, corresponding to different directions. In particular, the s -cubes labelled 1 and 3 are different. \square

In the following definition, we consider a special graph Γ_A , which will be quite useful in the subsequent subsections.

Definition 4.5. Let $A \leq Y$ be a contraction and consider the coloured graph Γ_A , whose vertices are the vertices of Y and with an edge of colour a between vertices i, j if there exists a simple contraction Z , with $A \leq Z < Y$, that contracts i, j with colour a . Note that whenever $A \leq B \leq Y$, then $\Gamma_B \subseteq \Gamma_A$ and the graph Γ_Y consists of the vertices of Y with no edges. Also, any family of simple contractions $\Omega = \{Y_0, \dots, Y_t\}$ of Y such that $A \leq \Omega$ yields a subgraph of Γ_A formed by the edges associated to the Y_i s. We say that the family is connected if this subgraph is connected. Observe that if Ω is connected, then all the contractions $Y_i \in \Omega$ have the same length in A . In particular, if the vertices involved in Y_i are locally maximal with respect to A , then so are the vertices involved in any other Y_j .

4.2. Construction of the pushing procedure

From now on, we assume we have only two colours. Also, recall that both are of arity 2. In this subsection, we prove the following result.

Theorem 4.6. *There exists an order-reversing poset map*

$$M: \{\text{poset of simplices of } |K_Y|\} \rightarrow K_Y$$

such that for any t -simplex $\sigma: A_t < A_{t-1} < \dots < A_0$ we have that

$$A_t \leq M(\sigma) \in C_{4t}.$$

In the next lemma, we describe certain connected components of the graph Γ_A . Recall that for $M \in K_Y$ the vertices involved in contraction in M , or just involved in M for short, are the elements of $Y \setminus M$.

Lemma 4.7. *Let $A \leq \{Y_0, Y_1\} < Y$, where Y_0 and Y_1 are different, not disjoint, simple contractions of Y such that the vertices involved in them are locally maximal with respect to some B , with $A \leq B \leq \{Y_0, Y_1\}$. Then, the connected component of Γ_A containing them is a square and, for $M = \text{glb}_A(\{Y_0, Y_1\})$, the vertices involved in M are precisely those in the square. In particular, $M \in C_4$.*

Proof. Label with $\{1, 2\}$ the vertices involved in Y_0 , and with $\{2, 3\}$ those involved in Y_1 . Note that $B \leq M \leq Y_0, Y_1$, so the vertices 1, 2, 3 are also locally maximal with respect to M . Moreover, 1, 2, 3 are obtained from the same element $m \in M$. We show that the only possibility is Figure 4, where m is the square subdivided into four small squares.

Consider one of the possible chains of subdivisions of m yielding 1, 2, 3 and let α_b be the first subdivision of the chain. If 1, 2, 3 were all in the same half, i.e. all descendants of the same $m\alpha_b^r$ for a fixed $r \in \{1, 2\}$, then a geometric argument would also prove that $M_1 = \{m\alpha_b^1, m\alpha_b^2\} \cup (M \setminus m) \leq Y_1, Y_2$, which is impossible by the definition of greatest lower bounds. Hence, we may assume that 1, 2 are partitions of $m\alpha_b^1$ and 3 is a partition of $m\alpha_b^2$. Moreover, by the commutativity relations, there are no more subdivisions corresponding to colour b in the path of subdivisions needed to obtain 1, 2, 3 from m . The fact that $M \leq Y_1$ implies that the first subdivision α_b can be inverted, i.e. it must be possible to perform the successive subdivisions in such a way that the second step consists of subdividing both halves $m\alpha_b^1$ and $m\alpha_b^2$ in direction a . But, again, the commutativity relations imply that we may assume that this second subdivision using colour a (i.e. subdivision in direction a) yields precisely the line between the rectangles 1 and 2, and that the rectangles 1, 2, 3 correspond precisely to three of the rectangles $m\alpha_b^i\alpha_a^j$ for $i, j = 1, 2$. It would be possible to subdivide the fourth rectangle, but the hypothesis that the length $l(M, 1)$ is maximal implies that this is not the case. So the fourth is also a rectangle of the same size, which we label 4, and therefore the rooted tree yielding 1, 2, 3 from m is any of the trees of Figure 3. Clearly, the associated graph in Γ_A is a square. \square

Observe that the previous lemma implies that for the contractions Z_0 of $\{3, 4\}$ of colour a and Z_1 of $\{1, 4\}$ of colour b we also have $A \leq M \leq \{Z_0, Z_1\}$. Moreover, $M = \text{glb}_A(Y_0, Y_1, Z_0) = \text{glb}_A(Y_0, Y_1, Z_0, Z_1)$.

Example 4.8. If we have more than two colours, the obvious corresponding version of Lemma 4.7, that two non-disjoint simple contractions are contained in a square in Γ_A , will be false. Consider the following example. Suppose we have three colours a, b, c , and let $A = \{m\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7\}$, with

$$\begin{aligned} 1 &= m\alpha_b^2\alpha_a^2\alpha_c^1, & 2 &= m\alpha_b^1\alpha_a^2\alpha_c^1, & 3 &= m\alpha_b^1\alpha_a^1\alpha_c^1, & 4 &= m\alpha_b^1\alpha_a^1\alpha_c^2, \\ 5 &= m\alpha_b^1\alpha_a^2\alpha_c^2, & 6 &= m\alpha_b^2\alpha_a^2\alpha_c^2, & 7 &= m\alpha_b^2\alpha_a^1. \end{aligned}$$

Consider the tree diagram in Figure 9, where dotted lines represent halving in direction a , dashed lines halving in direction b and normal lines halving in direction c .

If we wanted all nodes of the same length, we would only have to subdivide 7 further, for example into $m\alpha_b^2\alpha_a^1\alpha_c^1$ and $m\alpha_b^2\alpha_a^1\alpha_c^2$. Let Y_0 be the simple contraction of Y of colour b involving $\{1, 2\}$ and let Y_1 be the simple contraction of Y of colour a involving $\{2, 3\}$. Note that $A \leq Y_0, Y_1$ and any contraction of both Y_0 and Y_1 has to involve contraction of either seven elements in the first case or eight elements in the second. One easily checks that (in both cases) there is no square in Γ_A containing Y_0 and Y_1 . The maximal connected component of the graph Γ_A (in both cases) is what will be called an open book

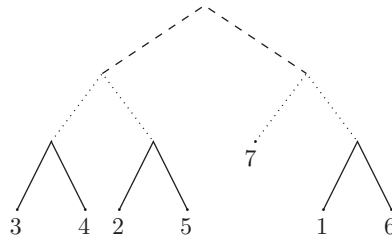


Figure 9. The tree diagram for Example 4.8.

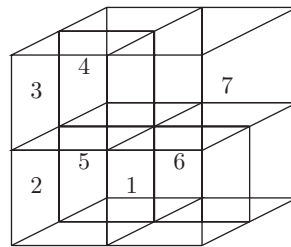


Figure 10. Example 4.8 as subdivision of the cube.

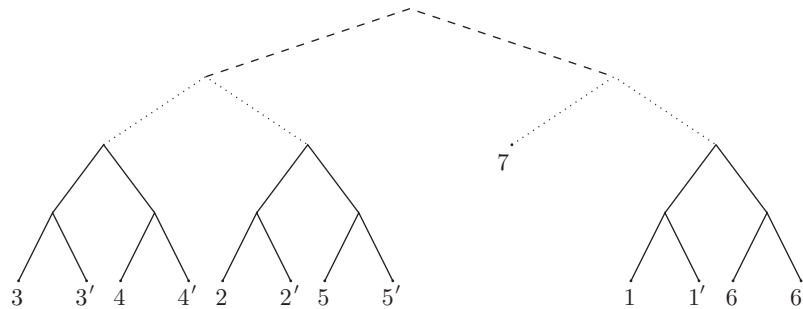


Figure 11. Enlarging Example 4.8.

in § 5, where we consider the case of three colours in detail. We may also represent the elements of Y as subdivisions of a cube labelled m ; Figure 10 illustrates the case when Y has seven elements.

Moreover, if we enlarge in a suitable way we can easily build examples in which the common contraction of Y_0, Y_1 has to involve arbitrarily many elements of Y . For example, by looking at the associated tree diagram, we could insert another subdivision in direction c , as in Figure 11, to obtain a Y' with 13 vertices.

As before, let Y_0 and Y_1 be simple contractions involving $\{1, 2\}$ (with colour b) and $\{2, 3\}$ (with colour a), respectively. Here, any contraction of both Y_0 and Y_1 would involve 13 elements. The effect of this in the representation of Figure 10 would be to halve each of the cubes 1, 2, 3, 4, 5 and 6 (with a plane parallel to the plane between 3 and 4) to yield the new cubes 1, 1', etc.

Proposition 4.9. *Let $A \leq \Omega = \{Y_0, \dots, Y_t\}$, where $t \geq 1$ and Y_i are simple contractions of Y . Assume further that there exist admissible sets $A \leq A_t \leq A_{t-1} \leq \dots \leq A_0$ such that for each i we have $A_i \leq Y_i$ and the elements involved in Y_i are locally maximal with respect to A_i . Then, for $M = \text{glb}_A(\Omega)$,*

$$M \in C_{4t}.$$

Proof. We may subdivide Ω into its connected components:

$$\Omega = \bigcup_{i=1}^r \Omega_i.$$

For any $i \in \{1, \dots, r\}$ there exists $j_i \in \{0, 1, \dots, t\}$ such that $A_{j_i} \leq Y_{l_i}$ for any $Y_{l_i} \in \Omega_i$, with the elements of Y contracted in Y_{l_i} locally maximal with respect to A_{j_i} (recall that Ω_i is connected). Set $M_i = \text{glb}_A(\Omega_i)$.

If Ω_i contains at least two different contractions, Lemma 4.7 tells us that its connected component in Γ_A is a square. In particular, Ω_i is contained in the set of four contractions representing the four sides of the square. Moreover, by the observation after Lemma 4.7, $M_i \in C_4$.

On the other hand, if all the elements of Ω_i are equal to some Z , then $M_i = Z \in C_2$. Clearly, all M_i are pairwise disjoint, so if we set $M = \text{glb}_A(\{M_1, \dots, M_r\})$, then $M = \text{glb}_A(\Omega)$ and Lemma 3.12 implies for $r \leq t$ that

$$|\text{vertices contracted in } M| \leq \sum_{i=1}^r |\text{vertices contracted in } M_i| \leq 4r \leq 4t.$$

If $r = t + 1$, then the elements of Ω are pairwise disjoint and, by Lemma 3.12, $M \in C_{2t+2} \subseteq C_{4t}$. \square

Now we are ready to prove Theorem 4.6.

Proof of Theorem 4.6. Fix any map

$$M: K_Y \rightarrow \{\text{simple contractions of } Y\}$$

such that, for any $A \in K_Y$, if i is any of the elements contracted in $M(A)$, then i is locally maximal with respect to A . We extend the above map M to a map

$$M: \{\text{poset of simplices of } K_Y\} \rightarrow K_Y$$

by setting, for any t -simplex $\sigma: A_t < A_{t-1} < \dots < A_0$,

$$M(\sigma) := \text{glb}_{A_t}(M(A_t), \dots, M(A_1), M(A_0)).$$

Proposition 4.9 and Lemma 3.15 imply that M is a well-defined order-reversing poset map and that

$$A_t \leq M(\sigma) \in C_{4t}.$$

\square

4.3. Construction of the null-homotopy

Remark 4.10. Denote by X^t the t -skeleton of a simplicial complex X . A simplicial complex X is t -connected if it is 0-connected, i.e. path-connected, and its t th homotopy group vanishes. As $\pi_t(X, x_0) = [S^t, s_0; X, x_0]$, this means that every continuous pointed map

$$\mu: (S^t, s_0) \xrightarrow{\nu} (X^t, x_0) \xrightarrow{i_t} (X, x_0)$$

is null-homotopic, i.e. homotopic to the constant map in (X, x_0) . Note that if i_t is null-homotopic, then the composition $\mu = i_t \circ \nu$ will also be null-homotopic. We aim to show that i_t is null-homotopic for $|Y|$ big enough and $X = |K_Y|$.

Due to the following general result, the poset map M constructed in Theorem 4.6 will be useful.

Lemma 4.11. *Let \mathfrak{P} be a poset and consider an order-reversing poset map*

$$M: \{\text{poset of simplices of } \mathfrak{P}\} \rightarrow \mathfrak{P}$$

such that, for any $\sigma: A_t < \dots < A_0$, $A_t \leq M(\sigma)$ in \mathfrak{P} . Then, M induces a map

$$f_t: |\mathfrak{P}|^t \rightarrow |\mathfrak{P}|,$$

which is homotopy equivalent in $|\mathfrak{P}|$ to the inclusion $i_t: |\mathfrak{P}|^t \rightarrow |\mathfrak{P}|$, and such that $f_t(\sigma)$ is contained in the realization of the subposet of those $B \in \mathfrak{P}$ such that $M(\sigma) \leq B$.

Proof. Consider the map

$$h: \{\text{poset of simplices of } \mathfrak{P}\} \rightarrow \mathfrak{P}$$

such that $h(\sigma) = A_t$. Then, as $h(\sigma) \leq M(\sigma)$ by a classical result in posets [1, 6.4.5], we have that $M \simeq h$. This means that $|h| \simeq |M|$. Denote the inclusion by $j: \mathfrak{P} \rightarrow \{\text{poset of simplices of } \mathfrak{P}\}$; we then have $h \circ j = 1_{\mathfrak{P}}$. Therefore, $|1_{\mathfrak{P}}| \simeq |M \circ j|$. Considering the composition

$$f_t: |\mathfrak{P}|^t \xrightarrow{i_t} |\mathfrak{P}| \xrightarrow{|j|} |\{\text{poset of simplices of } \mathfrak{P}\}| \xrightarrow{|M|} |\mathfrak{P}|,$$

we deduce that $f_t = |M| \circ |j| \circ i_t \simeq i_t$. Finally, note that $|j|$ takes any simplex σ to the geometric realization of the poset of those simplices δ such that $\delta \subseteq \sigma$. Thus, $f_t(\sigma)$ is contained in the realization of the subposet of those $B \in \mathfrak{P}$ such that $M(\sigma) \leq B$. \square

As a corollary of Definition 4.1, Theorem 4.6 and Lemma 4.11 we obtain the following result.

Proposition 4.12. *For any t there exists a map*

$$f_t: |K_Y|^t \rightarrow |K_Y|$$

that is homotopy equivalent to the inclusion $i_t: |K_Y|^t \rightarrow |K_Y|$, and such that $f_t(\sigma) \subseteq \Sigma_{4t}^t$.

Lemma 4.13. For any fixed r, t there exists a function $\nu_r(t)$ such that, if $|Y| \geq \nu_r(t)$, the inclusion of Σ_r^t in $|K_Y|$ is null-homotopic.

Proof. We adapt Brown's argument in [7, 4.20] to our context. For $|Y|$ big enough we will construct, by induction on t , a null-homotopy

$$F_t: \Sigma_r^t \times I \rightarrow |K_Y|$$

such that $F_t(\cdot, 0)$ is the identity map and $F_t(\cdot, 1)$ is the constant map sending everything to the point $a \in K_Y$. More precisely, we do the following. We show that there exist functions $\nu_r(t), \mu_r(t)$ such that, for $|Y| \geq \nu_r(t)$, there exists a homotopy F_t , as before, such that for any t -simplex $\sigma \in \Sigma_r^t$, $F_t(\sigma \times I) \subseteq \hat{\Sigma}_{\mu_r(t)}$, where $\hat{\Sigma}_s$ is the set of subcomplexes T of Σ_s such that the union of all elements of Y that are contracted in the vertex of some simplex of T has at most s elements.

The case $t = 0$

We choose any simple contraction a of Y . Hence, it involves two vertices, i.e. elements of Y . Let A be a point of Σ_r^0 , i.e. A is a contraction of Y involving at most r vertices. Now, if $|Y| \geq r + 4$, we may choose a set of two vertices disjoint to both those contracted in A and those contracted in a . Let b_0 be a simple contraction of any colour of Y corresponding to these two vertices. Then,

$$A \geq \text{gglb}(A, b_0) \leq b_0 \geq \text{gglb}(b_0, a) \leq a$$

is a path linking A with a in $\hat{\Sigma}_{r+4}^0$. Therefore, we verify the claim, with

$$\begin{aligned} \nu_r(0) &= r + 4, \\ \mu_r(0) &= r + 4. \end{aligned}$$

Induction step

We assume that there exists a null-homotopy $F_{t-1}: \Sigma_r^{t-1} \times I \rightarrow K_Y$. We want to extend F_{t-1} to F_t . Let $\sigma: A_t < A_{t-1} < \dots < A_0$ be a t -simplex in Σ_r^t . For any face τ of σ of dimension $t - 1$ we have that $F_{t-1}(\tau \times I) \subseteq \hat{\Sigma}_{\mu_r(t-1)}$. This means that if we define

$$\delta\sigma = \bigcup_{i=1}^{t+1} \tau_i,$$

then

$$\Delta := F_{t-1}(\delta\sigma \times I) = \bigcup F_{t-1}(\tau_i \times I) \subseteq \hat{\Sigma}_{(t+1)\mu_r(t-1)}.$$

Now, if $|Y| \geq 2 + (t + 1)\mu_r(t - 1)$, there are at least two vertices of Y not involved in any contraction in $F_{t-1}(\delta\sigma \times I)$. Let b be a simple contraction of any colour of Y contracting these two vertices.

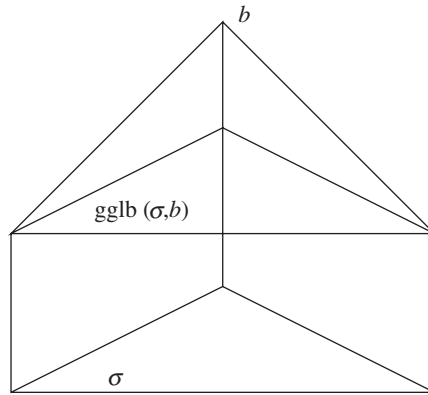


Figure 12. The cylinder and cone of item (1).

We claim that the homotopy F_{t-1} can be extended to $F_t: \sigma \times I \rightarrow |K_Y|$, with

$$F_t(\sigma \times I) \subseteq \hat{\Sigma}_{2+(t+1)\mu_r(t-1)}.$$

As b is a contraction of Y disjoint to all contractions A of Y such that $A \in F_{t-1}(\delta\sigma \times I)$, we may consider the global greatest lower bound of b and A , which we denote $\text{gglb}(A, b)$. Note that this is just the result of contracting in A those elements that are contracted in b . Analogously, we denote by $\text{gglb}(\Delta, b)$ the subcomplex given by $\text{gglb}(A, b)$ for all $A \in \Delta$. The same notation is also used for simplices in Δ . Also, note that, for all $A \in \Delta$, $\text{gglb}(A, b) \leq b$ and we can always form the cone with base $\text{gglb}(\Delta, b)$ and vertex b .

We claim that the homotopy $F_t(\sigma \times I)$ can be built up by gluing:

- (i) the cylinder given by Δ and $\text{gglb}(\Delta, b)$,
- (ii) the cone formed by $\text{gglb}(\Delta, b)$ and b .

Note that for any l -simplex $\tau: A_l < A_{l-1} < \dots < A_0$ lying in Δ the $l + 1$ -simplices

$$\text{gglb}(A_l, b) < \text{gglb}(A_{l-1}, b) < \dots < \text{gglb}(A_i, b) < A_i < A_{i-1} < \dots < A_0$$

for $i = l, \dots, 0$ fill up the cylinder formed by τ and $\text{gglb}(\tau, b)$ (recall that $\text{gglb}(\tau, b)$ is given by $\text{gglb}(A_l, b) < \text{gglb}(A_{l-1}, b) < \dots < \text{gglb}(A_0, b)$).

Furthermore, the cone formed by $\text{gglb}(\tau, b)$ and b is also filled up via the $t + 1$ -simplex

$$\text{gglb}(A_l, b) < \text{gglb}(A_{l-1}, b) < \dots < \text{gglb}(A_0, b) < b.$$

We now explain how the above constructions yield the extension of the homotopy.

- (1) Consider the cylinder with base the simplex σ and top the simplex $\text{gglb}(\sigma, b)$ and glue to the cylinder the cone with base $\text{gglb}(\sigma, b)$ and apex b .

Let $\sigma \cup \tilde{\Sigma}$ be the boundary of Figure 12. Then, σ is homotopic to $\tilde{\Sigma}$ via a homotopy, see Figure 12, fixing $\partial\sigma$.

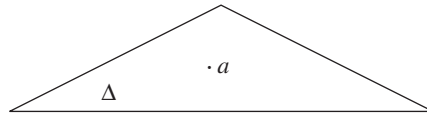
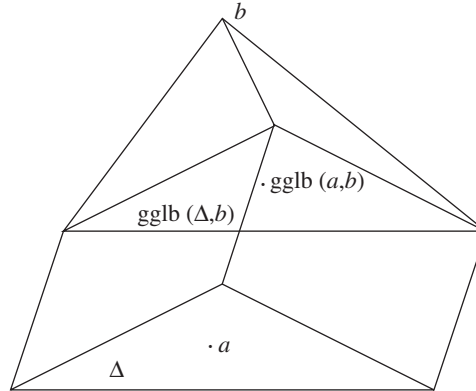
Figure 13. $\Delta\sigma$ squeezes to a .

Figure 14. The cylinder and the cone of item (3).

- (2) Figure 13 illustrates the homotopy F_{t-1} squeezing $\partial\sigma$ to the point a .
- (3) Consider the cylinder with bottom Δ and top $\text{gglb}(\Delta, b)$ and glue to it the cone with bottom $\text{gglb}(\Delta, b)$ and vertex b .

Note that $\Delta \cup \tilde{\Sigma}$ is the boundary of Figure 14. Thus, $\tilde{\Sigma}$ and Δ are homotopy equivalent via a homotopy (see Figure 14), fixing $\partial\Delta = \partial\sigma$. Set $\mu_r(t) = 2 + (t+1)\mu_r(t-1)$. Then, by (1) and (3), σ and Δ are homotopy equivalent via a homotopy, inside $\hat{\Sigma}_{\mu_r(t)}$, which fixes $\partial\sigma$. This completes the proof of the fact that F_{t-1} is extendable to a homotopy F_t (inside $\hat{\Sigma}_{\mu_r(t)}$) that contracts σ to the point a . Therefore, the inductive step is proved for

$$\nu_r(t) = \mu_r(t) = 2 + (t+1)\mu_r(t-1).$$

□

Theorem 4.14. *There exists a function $\alpha(t)$ such that, if $|Y| \geq \alpha(t)$, the inclusion of $|K_Y|^t$ in $|K_Y|$ is null-homotopic.*

Proof. Consider the homotopy equivalent maps $i_t, f_t: |K_Y|^t \rightarrow |K_Y|$ given in Proposition 4.12. Since the image of f_t is contained in Σ_{4t}^t , f_t factors through the inclusion of Σ_{4t}^t in K_Y . But, we have just proven that this last inclusion is null-homotopic whenever $|Y| \geq \nu_{4t}(t)$, and therefore, in that case, f_t and i_t are also null-homotopic. Therefore, it suffices to set $\alpha(t) := \nu_{4t}(t)$. □

Corollary 4.15. *There exists a function $\alpha(t)$ such that if $|Y| \geq \alpha(t)$, K_Y is t -connected.*

4.4. Finiteness properties of $2V$

Now we are ready to prove that the group $2V$ is of type FP_∞ . To do that, we will verify the conditions of [7, Corollary 3.3] with respect to the complex $|\mathfrak{A}|$ defined in Definition 3.1. As before, consider the filtration of $|\mathfrak{A}|$ given by

$$\mathfrak{A}_n := \{Y \in \mathfrak{A} \mid |Y| \leq n\}.$$

Lemmas 3.5 and 3.7 and Remark 3.6 imply that all that remains is to prove the following theorem.

Theorem 4.16. *The connectivity of the pair of complexes $(|\mathfrak{A}_{n+1}|, |\mathfrak{A}_n|)$ tends to infinity as $n \rightarrow \infty$.*

Proof. We use the same argument as in [7, 4.17], i.e. note that $|\mathfrak{A}_{n+1}|$ is obtained from $|\mathfrak{A}_n|$ by gluing cones with base K_Y and top Y for every $Y \in \mathfrak{A}_{n+1} \setminus \mathfrak{A}_n$. By Corollary 4.15, if $n + 1 \geq \alpha(t)$, we have that K_Y is t -connected; hence, $(|\mathfrak{A}_{n+1}|, |\mathfrak{A}_n|)$ is t -connected. \square

Theorem 4.17. *The Brin group on two colours, each of arity 2, i.e. $2V$, is of type F_∞ .*

Proof. By Lemmas 3.5 and 3.7, Remark 3.6 and Theorem 4.16 we may apply [7, Corollary 3.3]. \square

Remark 4.18. As a by-product, we get by [7, Corollary 3.3] a new proof of the fact that $2V$ is finitely presented. This was first proved in [6], where an explicit finite presentation was constructed.

5. The case $s = 3$

In this section, we consider Brin’s group sV for $s = 3$. Our objective is to show that $3V$ is of type F_∞ by adapting the construction of the function M of Lemma 4.11 to the case $s = 3$. In particular, we show that Theorem 4.6 holds with $M \in C_{8t}$. This immediately leads to a modification of Proposition 4.12: that $f_t(\sigma) \in \Sigma_{8t}^t$. The rest of the proof will be analogous to the previous case.

As before, we fix a Y and prove that K_Y is t -connected if $|Y|$ is sufficiently large. For $A < Y$ we consider the coloured graph Γ_A as in Definition 4.5. This time, the graph is embedded in three-dimensional real space and the three possible colours $\{a, b, c\}$ correspond to the axes of the standard coordinate system of \mathbb{R}^3 . For any subgraph $\Delta \subseteq \Gamma_A$ we set

$$\text{glb}_A(\Delta) := \text{glb}_A\{\text{simple contractions associated with the edges of } \Delta\}.$$

Consider a connected component Δ of Γ_A . The vertices of Δ correspond, via the geometric realization of $3V$, to subparallelepipeds of the unit cube I , all of the same shape and size. For simplicity, we draw them as cubes and call them subcubes. Let i be an element (i.e. a vertex) of Δ . By some abuse of notation we also label by i the subcube corresponding to the element i of Δ .

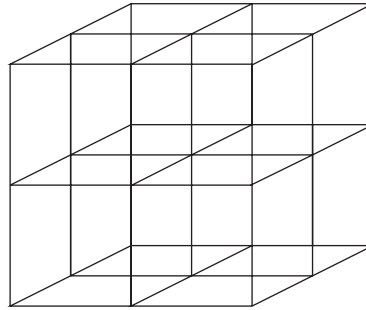
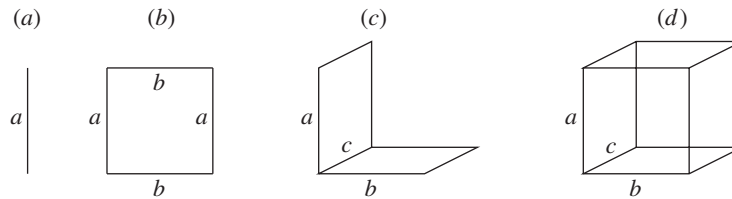


Figure 15. A stack of eight cubes.

Figure 16. The possible $*$ -connected components of Γ_A .

We claim that the vertices of Δ are inside a stack of eight subcubes; see Figure 15. Obviously, one of these subcubes corresponds to i . Observe that we do not claim that all the subcubes in the stack correspond to elements of Y , only that Δ is a set consisting of some of the subcubes in the stack. To see that the claim holds, let i be $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \times [\gamma_1, \gamma_2]$. The interval $A_0 = [\alpha_1, \alpha_2]$ comes from a set of binary subdivisions of $[0, 1]$. The left descendant of an interval $[x, y]$ is $[x, (x+y)/2]$, and the right descendant of $[x, y]$ is $[(x+y)/2, y]$. Then, A_0 is a descendant of some interval J_A that is subdivided into A_0 and A_1 in the binary subdivision. Recall (see, for example, Lemma 4.4) that each cube in a connected set can only have one neighbour of each colour/direction. Define B_1 and C_1 analogously. Then, the cubes in the stack containing Δ are precisely the cubes $A_i \times B_j \times C_k$, where $i, j, k \in \{0, 1\}$.

For a connected component Δ of Γ_A we define the enveloping stack of Δ to be the smallest set $U(\Delta)$ of some subcubes from the eight-cube stack, defined above, such that $U(\Delta)$ contains all $i \in \Delta$, and the union of the elements of $U(\Delta)$ is a cube.

Note that if one of the vertices of Δ is locally maximal with respect to some $C < Y$ such that $A \leq C$, then every vertex of Δ is locally maximal with respect to C . This leads to the following definition.

Definition 5.1. A connected component Δ of Γ_A is called $*$ -connected if there exists some admissible set C such that $A \leq C < Y$ and every vertex of Δ is locally maximal with respect to C .

Figure 16 exhibits possible $*$ -connected components of the graph Γ_A for $A < Y$. Note that parallel edges are labelled by the same colour.

We call the graphs in Figure 16 an edge, a square, an open book and a cube, respectively.

Lemma 5.2. *Let Δ be a $*$ -connected component of Γ_A . Then, up to changing the colours, Δ is one of the graphs in Figure 16. Moreover, if Δ is not an open book, then for $M = \text{glb}_A(\Delta)$ the vertices involved in M lie inside Δ . In particular, $M \in C_8$.*

Proof. We argue as in Lemma 4.7. We consider the element $m \in M$ that yields Δ , i.e. the vertices of Δ are obtained from m by the halving operations. Observe that $M = \{m\} \cup (M \cap Y)$. Consider the geometric realization of M . Then, m is a subcube of the unitary cube and the enveloping stack $U(\Delta)$ lies inside m . Since $M < Y$ we may choose some simple expansion $M < M_1 \leq Y$ of colour a , say. The expansion $M < M_1$ corresponds to halving the cube m by a hyperplane of direction (i.e. colour) a . Furthermore, this halving also yields a halving of the enveloping stack $U(\Delta)$. In other words, not all the vertices of Δ are in the same half of m , as that would mean that $M = M_1$. Moreover, as Δ is connected, this halving can be inverted, by using the commutativity relations, to give a simple contraction of Y of direction a . If $M_1 = Y$, then Δ is an edge and $M \in C_2$.

Note that, since the halving operation of m in direction a halves $U(\Delta)$, we have an edge e in Δ with label a and vertices i, j . In particular, the elements i and j represent neighbouring cubes in $U(\Delta)$, one contained in $m\alpha_a^1$ and the other in $m\alpha_a^2$. Since $e \in \Gamma_A$ there is a contraction of Y contracting precisely i and j . This implies that in the process of obtaining Y from M via halving operations on m , there is another chain of halving operations starting with halving in a direction different from a , say b . Hence, by the commutativity relations, there exists M_2 with $M_1 < M_2 \leq Y$ such that M_2 consists of halving both $m\alpha_a^1$ and $m\alpha_a^2$ in direction b . Clearly, this allows inversion and, therefore, the above procedure for a can also be applied to b . After performing these two subdivisions we get a stack S of four cubes. Moreover, we may assume that there exist vertices of Δ lying in at least three of those four cubes. Otherwise Δ would be either disconnected or $M \neq \text{glb}_A(\Delta)$. Also, note that, to obtain Δ , only halving of those four cubes in a direction c different from directions a and b is possible. So it remains to consider the three possibilities below. Recall that we are assuming that Δ is $*$ -connected.

- (1) If none of the cubes is halved, then $M_2 = Y$, Δ is a square and $M \in C_4$.
- (2) Suppose all four cubes are halved at least once. Then, the rooted tree representing the way that Δ is obtained from m starts as the first tree in Figure 17. In this case, we may use the commutativity relations to get a rooted tree with halving in direction c at the beginning. Therefore, the assumptions that Δ is connected and that $M = \text{glb}_A(\Delta)$ imply that, in fact, there is only one halving in direction c . In particular, the rooted tree is precisely the first tree in Figure 17. Thus, Δ is a cube, m yields the whole stack of eight cubes, $M \in C_8$ and M involves precisely the vertices of Δ .
- (3) Finally, assume that only three of the four cubes are halved at least once in direction c . Then, we may assume that the rooted tree representing the halving operations

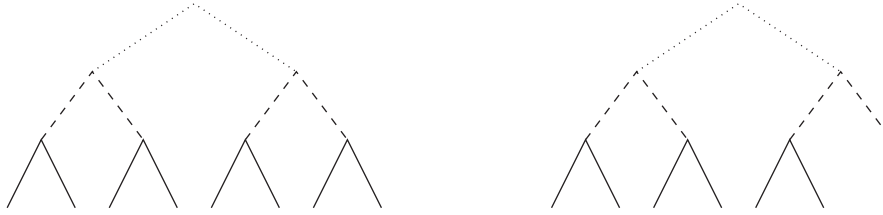


Figure 17. The three-coloured trees of items (2) and (3).

done on m begins exactly as the second tree in Figure 17. Note that at this point, and as a consequence of the geometric interpretation, we know that Δ is a subgraph of the open book B containing the three edges labelled c . Also, B lies inside the eight-cube stack associated with Δ . Furthermore, the elements of B correspond to elements of Y . We show that Δ is exactly the open book B . Since Δ is connected, it suffices to show that any two neighbouring cubes in the open book B can be contracted in Y . Consider the admissible set M_a , obtained as follows. First, halve m in direction a and assume that the second half of m , i.e. $m\alpha_a^2$, contains the only one of the cubes not cut in direction c . Then, perform in $m\alpha_a^2$ all halvings needed to reach those elements of Y that are descendants of $m\alpha_a^2$. The first half of m , $m\alpha_a^1$, is not cut anymore. Then, $M \leq M_a$, where $M_a = \{m\alpha_a^1\} \cup (M_a \cap Y)$.

Observe that, in the first half of m , there are only two colours in the path needed to obtain the elements of $\Delta \cap \Gamma_{M_a}$ from M_a . As $\Delta \cap \Gamma_{M_a}$ is $*$ -connected in Γ_{M_a} , we may apply Lemma 4.7 and deduce that the square of the open book B , with edges labelled by b and c , is in Δ . The same argument with b substituted by a implies that the square of the open book B , with edges labelled by c and a , is in Δ . Thus, Δ is the open book B . \square

We are now ready to prove the analogue to Theorem 4.6 with $M \in C_{8t}$.

Theorem 5.3. *Let $s = 3$. There exists an order-reversing poset map*

$$M: \{\text{poset of simplices of } |K_Y|\} \rightarrow K_Y$$

such that for any t -simplex $\sigma: A_t < A_{t-1} < \dots < A_0$ we have that

$$A_t \leq M(\sigma) \in C_{8t}.$$

Proof. We split the proof into three steps. Fix a linear ordering on the colours a, b, c .

Step 1 (the definition of M on vertices of K_Y). For each admissible $A < Y$ we define a designated edge $M(A)$ as follows.

Consider the graph Γ_A . We define $M(A)$ as an edge of Γ_A such that if $\Gamma_A = \Gamma_B$ for some $B < Y$, then $M(A) = M(B)$. If there exists some open book between the $*$ -connected components of Γ_A , we define $M(A)$ to be the middle edge of the open book with middle edge of smallest possible colour amongst those open books that are $*$ -connected components of Γ_A (see Figure 18).

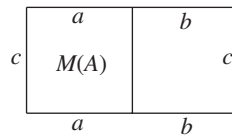


Figure 18. The open book extended.

If Γ_A does not have an open book as a $*$ -connected component, but contains a $*$ -connected component, which is a separate edge e , i.e. Figure 16 (a), we define $M(A) = e$. Again, there might be more than one such edge e and we choose e of the smallest possible colour.

If Γ_A does not contain $*$ -connected components, which are open books or separate edges, we choose $M(A)$ to be an edge of the smallest possible colour of a $*$ -connected component of Γ_A .

From now on, we write Δ_A for the $*$ -connected component of Γ_A such that $M(A) \in \Delta_A$. We can further assume that if $\Delta_A = \Delta_B$, then $M(A) = M(B)$.

Step 2. Let $A = A_r < A_{r-1} < \dots < A_0$ be contractions of Y such that all $M(A_i)$ belong to Δ_A . Recall that each $M(A_i)$ is a simple contraction of Y . Let $\Omega = \{M(A_r), \dots, M(A_0)\}$ and set $N = \text{glb}_A(\Omega)$. We aim to show that $N \in C_8$ and that the vertices of Y involved in N are inside Δ_A .

Observe that Δ_A is $*$ -connected. So, it must be one of the graphs of Figure 16. If it is an edge, a square or a cube, then our claim that $N \in C_8$ follows from Lemma 5.2. So, we may assume that Δ_A is an open book. We have that

$$\Delta_A = \Delta_A \cap \Gamma_{A_r} \supseteq \dots \supseteq \Delta_A \cap \Gamma_{A_0}.$$

The definition of M yields that if $\Delta_A = \Delta_A \cap \Gamma_{A_r} = \dots = \Delta_A \cap \Gamma_{A_0}$, then $M(A_r) = \dots = M(A_0)$. In this case, $N = M(A_r) \in C_2$. So, we may assume that there exists some $0 \leq i < r$ such that

$$\Delta_A = \Delta_A \cap \Gamma_{A_r} = \dots = \Delta_A \cap \Gamma_{A_{i+1}} \supsetneq \Delta_A \cap \Gamma_{A_i}.$$

Define $B = A_i$. We have that

$$\Delta_B \subseteq \Delta_A \cap \Gamma_B \subsetneq \Delta_A.$$

Moreover, by the definition of M , $M(A) = M(A_r) = \dots = M(A_{i+1})$ is the middle edge of the open book Δ_A .

We claim that $\Delta_A \cap \Gamma_B$ is a subgraph of one of the two graphs in Figure 19.

Observe that $\Delta_A \cap \Gamma_B$ is not connected. Indeed, in the process of obtaining B from A there was a cutting of a cube containing $U(\Delta_A)$, which halved $U(\Delta_A)$. The structure of Δ_A as an open book with three parallel edges c implies that such a halving cannot be in direction c . The case when the direction of this halving is a corresponds to Γ_1 (see Figure 19), i.e. $\Delta_A \cap \Gamma_B \subseteq \Gamma_1$, and the case when the direction is b corresponds to Γ_2 , i.e. $\Delta_A \cap \Gamma_B \subseteq \Gamma_2$. Alternatively, consider the second tree in Figure 17. The

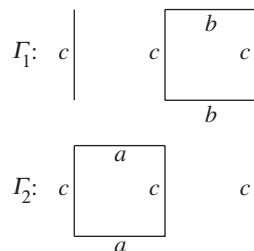


Figure 19. Possible graphs.

commutativity relations do not allow us to move c to the top, whereas having a or b at the top yields a disconnected graph. A similar argument shows that there exists an expansion $A_{i+1} < \tilde{B}$ such that $\Delta_A \cap \Gamma_{\tilde{B}} = \Gamma_k$, where we have fixed one $k \in \{1, 2\}$ such that $\Delta_A \cap \Gamma_B \subseteq \Gamma_k$.

For any $0 \leq j \leq i$ we also have that $M(A_j) \in \Delta_{A_j} \subseteq \Delta_A \cap \Gamma_B$. Then, since $\Delta_A \cap \Gamma_B \subseteq \Gamma_k$, we have that $\Omega \subset (\Delta_A \cap \Gamma_B) \cup \{M(A)\} \subseteq \Gamma_k = \Delta_A \cap \Gamma_{\tilde{B}} \subseteq \Gamma_{\tilde{B}}$. Hence, $A < \tilde{B} \leq \Omega$, and so

$$\text{glb}_{\tilde{B}}(\Gamma_k) \leq \text{glb}_{\tilde{B}}(\Omega) = N.$$

Now, split $\Gamma_k = D_1 \cup D_2$ into its connected components, where D_1 is the edge and D_2 is the square. Note that D_1 and D_2 are $*$ -connected components of $\Gamma_{\tilde{B}}$; hence, Lemma 5.2 implies that $\text{glb}_{\tilde{B}}(D_i)$ involves (i.e. contracts) 2^i vertices (i.e. elements) of Y . Then, by Lemma 3.12 $\text{glb}_{\tilde{B}}(D_1 \cup D_2)$ contracts $2 + 4 = 6$ vertices of Y . Hence, $N \in C_6 \subseteq C_8$.

Step 3 (the definition of M on a simplex of K_Y). Let $\sigma: A_t < A_{t-1} < \dots < A_0$ be a simplex of K_Y and let $t \geq 1$. Thus, $\Gamma_{A_0} \leq \dots \leq \Gamma_{A_{t-1}} \leq \Gamma_{A_t}$ and we have already defined $M(A_i)$ as an edge of Γ_{A_i} for all i . Let $\Omega = \{M(A_t), M(A_{t-1}), \dots, M(A_0)\}$, which is a set of edges of Γ_{A_t} .

Consider the following partition of Ω .

Set $\alpha_1 = t$ and

$$\Omega_1 = \Omega \cap \Delta_{A_{\alpha_1}}.$$

Assume that Ω_{r-1} is defined. If

$$\bigcup_{i=1}^{r-1} \Omega_i \neq \Omega,$$

choose the largest $j \in \{0, \dots, t\}$ such that

$$M(A_j) \in \Omega \setminus \left(\bigcup_{i=1}^{r-1} \Omega_i \right).$$

Rename A_j as A_{α_r} and set $\Omega_r = \Omega \cap \Delta_{A_{\alpha_r}}$. Hence, at each step we have a subchain (i.e. subsimplex) of σ satisfying the conditions of Step 2.

At some point, we will have that

$$\Omega = \bigcup_{i=1}^k \Omega_i.$$

Let

$$N_i := \text{glb}_{A_{\alpha_i}}(\Omega_i).$$

By Step 2, $N_i \in C_8$ and the vertices of Y involved in N_i are contained in $\Delta_{A_{\alpha_i}}$. Now, we claim that these N_i are pairwise disjoint contractions of Y . To see this, let $i \neq j$. We may assume that $A_{\alpha_i} \leq A_{\alpha_j}$ and, therefore, $\Gamma_{A_{\alpha_i}} \supseteq \Gamma_{A_{\alpha_j}}$. As $\Delta_{A_{\alpha_i}}$ is a $*$ -connected component in $\Gamma_{A_{\alpha_i}}$, we deduce that either $\Delta_{A_{\alpha_i}}$ and $\Delta_{A_{\alpha_j}}$ are disjoint (and in this case N_i and N_j are also disjoint) or $\Delta_{A_{\alpha_j}} \subseteq \Delta_{A_{\alpha_i}}$. But the second case is impossible by the construction of the partition above.

Next we define

$$M(\sigma) = \text{glb}_A(\Omega).$$

Clearly,

$$M(\sigma) = \text{glb}_A(\{N_1, \dots, N_k\})$$

and, if $k \leq t$, then

$$M(\sigma) \in C_{8k} \subseteq C_{8t}.$$

Finally, if $k = t + 1$, then all Ω_i contain precisely one edge, so for all i we have that $N_i = M(A_i)$, and so $M(\sigma) \in C_{2(t+1)} \subseteq C_{8t}$. □

As a corollary, we get the following modified version of Proposition 4.12.

Corollary 5.4. *For any t there exists a map*

$$f_t: |K_Y|^t \rightarrow |K_Y|,$$

which is homotopy equivalent to the inclusion $i_t: |K_Y|^t \rightarrow |K_Y|$ such that $f_t(\sigma) \subseteq \Sigma_{8t}^t$.

From now on, we can proceed analogously to the case $s = 2$. As a first step, we have a three-dimensional analogue to Theorem 4.14.

Corollary 5.5. *Let $s = 3$. There exists a function $\alpha(t)$ such that if $|Y| \geq \alpha(t)$, the inclusion of $|K_Y|^t$ in $|K_Y|$ is null-homotopic.*

Proof. Follow the proofs of Theorem 4.14 and Lemma 4.13, substituting Proposition 4.12 with Corollary 5.4. □

Theorem 5.6. *The Brin group $3V$ on three colours of arity 2 is of type F_∞ .*

Proof. The proof follows the proof of Theorem 4.17. The main point is the construction of the poset map M of Theorem 5.3. Applying Corollary 5.5, the rest follows as before. □

Acknowledgements. This work was partly supported by the Royal Society (Grant TG080182) and LMS Scheme 4 (Grant 4814). D.H.K. was partly supported by ‘Bolsa de Produtividade em Pesquisa’ from CNPq, Brazil. C.M.-P. was supported by Gobierno de Aragon and MTM2010-19938-C03-03.

References

1. D. J. BENSON, *Representations and cohomology, Volume 2: Cohomology of groups and modules* (2nd edn), Cambridge Studies in Advanced Mathematics, Volume 31 (Cambridge University Press, 1998).
2. M. BESTVINA AND B. BRADY, Morse theory and finiteness properties of groups, *Invent. Math.* **129** (1997), 445–470.
3. R. BIERI, *Homological dimension of discrete groups* (2nd edn), Queen Mary College Mathematical Notes (Queen Mary College, London, 1981).
4. C. BLEAK AND D. LANOE, A family of non-isomorphism results, *Geom. Dedicata* **146** (2010), 21–26.
5. M. G. BRIN, Higher dimensional Thompson groups, *Geom. Dedicata* **108** (2004), 163–192.
6. M. G. BRIN, Presentations of higher-dimensional Thompson groups, *J. Alg.* **284** (2005), 520–558.
7. K. S. BROWN, Finiteness properties of groups, *J. Pure Appl. Alg.* **44** (1987), 45–75.
8. S. N. BURRIS AND H. P. SANKAPPANAVAR, *A course in universal algebra*, Graduate Texts in Mathematics, Volume 78 (Springer, 1981).
9. P. M. COHN, *Universal algebra*, Mathematics and Its Applications, Volume 6 (D. Reidel, Dordrecht, 1981).
10. J. HENNIG AND F. MATUCCI, Presentations for the higher-dimensional Thompson's groups nV , *Pac. J. Math.* **257**(1) (2012), 53–74.
11. G. HIGMAN, *Finitely presented infinite simple groups*, Notes on Pure Mathematics, Volume 8 (Australian National University, Canberra, 1974).