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ON THE DIOPHANTINE EQUATION $x^2 + 5^a\,13^b = y^n$

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Abstract. In this note, we find all the solutions of the Diophantine equation $x^2 + 5^a\,13^b = y^n$ in positive integers $x, y, a, b, n \geq 3$ with x and y coprime.

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1. Introduction. The history of the Diophantine equation

$$x^2 + C = y^n, \quad x \geq 1, \quad y \geq 1, \quad n \geq 3 \tag{1.1}$$

is very rich. In 1850, Lebesgue [**14**] proved that the above equation has no solutions when $C = 1$. In 1965, Chao Ko [**11**] proved that the only solution of the above equation with $C = -1$ is $x = 3,\ y = 2$. J. H. E. Cohn [**10**] solved the above equation for several values of the parameter C in the range $1 \leq C \leq 100$. A couple of the remaining values of C in the above range were covered by Mignotte and De Weger in [**18**], and the remaining ones in the recent paper [**9**]. In [**20**], all solutions of the equation $x^2 + C = 2y^n$ with $n \geq 3$, coprime integers x and y and $C = B^2$ with $B \in \{3, 4, \ldots, 501\}$ were found.

Recently, several authors became interested in the case when only the prime factors of C are specified. For example, the case when $C = p^k$ with a fixed prime number p, was dealt with in [**1**] and [**13**] for $p = 2$, in [**2**], [**3**] and [**15**] for $p = 3$, and in [**4**] and [**6**] for $p = 5$. Partial results for a general prime p appear in [**5**] and [**12**]. All the solutions when $C = 2^a\,3^b$ were found in [**16**]. See also the recent survey [**7**] for more results of this type. Not included in this survey is a result by the second and the third authors concerning the solutions of the above equation for the case $C = 2^a\,5^b$ (see [**17**]), as well as Pink's study [**19**] of the case $C = 2^a\,3^b\,5^c\,7^d$.

Here, we continue this study with the equation

$$x^2 + 5^a\,13^b = y^n, \quad x \geq 1, \quad y \geq 1, \quad \gcd(x, y) = 1, \quad n \geq 3, \quad a \geq 0, \quad b \geq 0. \tag{1.2}$$

Our main result is the following.

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THEOREM 1.1. *The equation* (1.2) *has no solution except for*:
$n = 3$ $(x, y, a, b) = (70, 17, 0, 1),\ (142, 29, 2, 2)$;
$n = 4$ $(x, y, a, b) = (4, 3, 1, 1)$.

2. The case $n = 4$.

Here, we have the following result.

LEMMA 2.1. *If $n = 4$, then the only solution to equation* (1.2) *is*

$$(x, y, a, b) = (4, 3, 1, 1). \tag{2.1}$$

Proof. Equation (1.2) can be written as

$$\left(\frac{x}{z^2}\right)^2 + A = \left(\frac{y}{z}\right)^4, \tag{2.2}$$

where A is fourth-power free and defined implicitly by $5^a\,13^b = Az^4$. One can see that $A = 5^{a_1}\,13^{b_1}$ with $a_1,\ b_1 \in \{0, 1, 2, 3\}$. Hence, the problem consists in determining the $\{5, 13\}$-integer points on the totality of the 16 elliptic curves

$$V^2 = U^4 - 5^{a_1}\,13^{b_1}, \tag{2.3}$$

with $a_1,\ b_1 \in \{0, 1, 2, 3\}$.

Recall that if $\mathcal{S}$ is a finite set of prime numbers, then an $\mathcal{S}$-integer is rational number a/b with coprime integers a and $b > 0$, where the prime factors of b are in $\mathcal{S}$. We use MAGMA to determine the $\{5, 13\}$-integral points on the above elliptic curves. We find

$$(U, V, a_1, b_1) = (1, 0, 0, 0), (3, 4, 1, 1), (13, 156, 2, 2).$$

With the conditions on x, y and the definition of U, V, one can see that the only corresponding solution is $(x, y, a, b) = (4, 3, 1, 1)$. This concludes the proof. □

If (x, y, a, b, n) is a solution of the Diophantine equation (1.2) and d is any proper divisor of n, then $(x, y^d, a, b, n/d)$ is also a solution of the same equation. Since $n \geq 3$ and we have already dealt with the case $n = 4$, it follows that it suffices to look at the solutions n for which $p \mid n$ for some odd prime p. In this case, we may certainly replace n by p, and thus assume for the rest of the paper that n is an odd prime.

3. The case $n \geq 5$.

LEMMA 3.1. *The Diophantine equation* (1.2) *has no solution with $n \geq 5$ prime.*

Proof. We write the Diophantine equation (1.2) as $x^2 + dz^2 = y^p$, where $d = 1, 5, 13, 65$ according to the parities of the exponents a and b. Here, $z = 5^\alpha\,13^\beta$ for some nonnegative integers α and β. Let $\mathbb{K} = \mathbb{Q}[i\sqrt{d}]$. We factor the above equation in $\mathbb{K}$ getting

$$(x + i\sqrt{d}\,z)(x - i\sqrt{d}\,z) = y^p. \tag{3.1}$$

Since $5^a 13^b \equiv 1 \pmod 4$, it follows by considerations modulo 4 in equation (1.2) that x is even. Since x and y are coprime, a standard argument shows that the ideals generated by $x + i\sqrt{d}z$ and $x - i\sqrt{d}z$ are coprime in $\mathbb{K}$. Hence, the ideal $x + i\sqrt{d}z$ is a pth power

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of some ideal in $\mathcal{O}_{\mathbb{K}}$. The class number of $\mathbb{K}$ belongs to $\{1, 2, 8\}$. In particular, it is coprime to p. Thus, by a standard argument, it follows that $x + i\sqrt{d}z$ is associated to a pth power in $\mathcal{O}_{\mathbb{K}}$. Since the group of units in $\mathbb{K}$ is of order 2 or 4 (coprime to p), it follows that we may assume that

$$x + i\sqrt{d}z = \gamma^p \tag{3.2}$$

holds with some algebraic integer $\gamma \in \mathcal{O}_{\mathbb{K}}$. Finally, since the discriminant of $\mathbb{K}$ is $-4d$, it follows that $\{1, i\sqrt{d}\}$ is a base for $\mathcal{O}_{\mathbb{K}}$. In conclusion, we can write $\gamma = u + i\sqrt{d}v$. Conjugating equation 3.2 and subtracting the two relations, we get

$$2i\sqrt{d}\, 5^\alpha\, 13^\beta = \gamma^p - \bar{\gamma}^p. \tag{3.3}$$

The right hand side of the above equation is a multiple of $2i\sqrt{d}v = \gamma - \bar{\gamma}$. We deduct that $v \mid 5^\alpha\, 13^\beta$, and that

$$\frac{5^\alpha\, 13^\beta}{v} = \frac{\gamma^p - \bar{\gamma}^p}{\gamma - \bar{\gamma}} \in \mathbb{Z}. \tag{3.4}$$

Let $\{L_m\}_{m\geq 0}$ be the sequence of general term $L_m = \frac{\gamma^m - \bar{\gamma}^m}{\gamma - \bar{\gamma}}$, for all $n \geq 0$. This is called a *Lucas sequence* and it consists of integers. For any nonzero integer k, we write $P(k)$ for the largest prime factor of k. Equation (3.6) leads to the conclusion that

$$P(L_p) = P\left(\frac{5^\alpha 13^\beta}{v}\right). \tag{3.5}$$

Recall that the Primitive Divisor Theorem for Lucas sequences implies that if $p \geq 5$, then L_p has a *primitive* prime factor except for finitely many pairs $(\gamma, \bar{\gamma})$ and all of them appear in Table 1 in [**8**]. These exceptional Lucas numbers are called *defective*. A primitive prime factor q has the properties (among others), that $q \nmid -4dv^2 = (\gamma - \bar{\gamma})^2$, and $q \equiv \pm 1 \pmod{p}$. More precisely, $q \equiv e \pmod{p}$, where $e = (\frac{-4d}{q})$. Here, and in what follows, $(\frac{a}{q})$ stands for the Legendre symbol of a with respect to the odd prime q.

Since $\mathbb{K} = \mathbb{Q}[i\sqrt{d}]$ with $d \in \{1, 5, 13, 65\}$, a quick inspection of Table 1 in [**8**] reveals that our number L_p cannot be defective. Thus, L_p must have a primitive divisor q. Clearly, $q \in \{5, 13\}$ and $q \equiv \pm 1 \pmod{p}$, where $p \geq 5$. Hence, the only possibility is $q = 13$, and we conclude that $p \mid 12,\ 14$. The only possibility is $p = 7$, and since $13 \equiv -1 \pmod{7}$, we must have that $(\frac{-4d}{13}) = -1$. Since $d \in \{1, 5, 13, 65\}$, we conclude that $d = 5$. Using now 3.3 with $p = 7$, we obtain

$$v(7u^6 - 175u^4v^2 + 525u^2v^2 - 125v^6) = 5^\alpha\, 13^\beta. \tag{3.6}$$

Since u and v are coprime, we have the possibilities

$$v = \pm 5^\alpha\, 13^\beta, \quad v = \pm 13^\beta, \quad v = \pm 5^\alpha, \quad v = \pm 1. \tag{3.7}$$

The first two cases lead to the conclusion that $P(L_p) = P(5^\alpha\, 13^\beta / v) \leq 5$, which is impossible since it leads again to the conclusion that L_p has no primitive divisors, so we look at the last two possibilities.

Case 1: $v = \pm 5^{\alpha}$.

In this case, the Diophantine equation (3.6) is

$$7u^6 - 175u^4v^2 + 525u^2v^2 - 125v^6 = \pm 13^{\beta}. \tag{3.8}$$

Dividing both sides of the above equation by v^6, we obtain the elliptic equations

$$7X^3 - 175X^2 + 525X - 125 = D_1 Y^2, \tag{3.9}$$

where

$$X = \frac{u^2}{v^2}, \quad Y = \frac{13^{\beta_1}}{v^3}, \quad \beta_1 = \lfloor \beta/2 \rfloor, \quad D_1 = \pm 1, \pm 13.$$

• In the case $D_1 = \pm 1$ (changing X to $-X$ when $D_1 = -1$), we have to find the $\{5\}$-integer points on the elliptic curves

$$7X^3 + \eta 175X^2 + 525X + \eta 125 = Y^2, \qquad \eta \in \{-1, 1\}. \tag{3.10}$$

We multiply both sides of equation (3.10) by 7^2 to obtain

$$U^3 + \eta 175U^2 + 3675U + \eta 6125 = \pm V^2, \tag{3.11}$$

where $(U, V) = (\eta 7X, 7Y)$ are $\{5\}$-integer points on the above elliptic curve. We use MAGMA to determine all these points. We find only $(U, V) = (21, 56)$, for $\eta = 1$. This gives us $(X, Y) = (3, 8)$ which does not lead to a solution of (1.2).

• When $D = \pm 13$, we multiply 3.9 by $7^2\, 13^3$ and obtain the elliptic curve

$$U^3 + \eta 2275U^2 + 621075U + \eta 13456625 = V^2, \qquad \eta \in \{-1, 1\}, \tag{3.12}$$

where

$$U = \eta 91X, \quad V = 1183Y,$$

for which we need again its $\{5\}$-integer points. In the same way, for $\eta = -1$, we find $(U, V) = (91, 9464), (679, 42392)$ so $(X, Y) = (1, 8), (97/13, 6056/169)$. This is inconsistent with the definition of X and Y.

Case 2: $v = \pm 1$.

Here, we obtain the following Thue-Mahler equations

$$7u^6 - 175u^4 + 525u^2 - 125 = 5^{\alpha}\, 13^{\beta}. \tag{3.13}$$

By the same method, we can rewrite the above equation as

$$7X^3 - 175X^2 + 525X - 125 = D_1 Y^2, \tag{3.14}$$

where

$$X = u^2, \quad Y = 5^{\alpha_1}\, 13^{\beta_1}, \quad \alpha_1 = \lfloor \alpha/2 \rfloor, \quad \beta_1 = \lfloor \beta/2 \rfloor, \quad D_1 = \pm 1, \pm 5, \pm 13, \pm 65.$$

When $D_1 = \pm 1, \ \pm 13$, we get again the two curves shown at (3.10) and (3.12), respectively, except that now we need only their integer points.

• When $D_1 = \pm 5$, we then multiply both sides of equation (3.14) by $7^2\, 13^3$ and get the two elliptic curves

$$U^3 + \eta 2275 U^2 + 621075 U + \eta 13456625 = V^2, \qquad \eta \in \{-1, 1\}, \tag{3.15}$$

where $U = \eta 91 X$, $V = 1183 Y$, and we need their integer points. Here also we use MAGMA to find, for $\eta = -1$, the integral point $(U, V) = (91, 9464)$ so $(X, Y) = (u^2, 5^{\alpha_1} 13^{\beta_1}) = (1, 8)$, which has does not lead to integer solutions α_1 and β_1.

• Finally, for the case $D = \pm 65$, we multiply both sides of equation (3.14) by $7^2\, 5^3\, 13^3$ to obtain

$$U^3 + \eta 11375 U^2 + 15526875 U + \eta 1682078125 = V^2, \qquad \eta \in \{-1, 1\}, \tag{3.16}$$

where $U = 455X$, $V = 29575Y$, whose integer points we need to compute. We determine two such integral points for $\eta = 1$ and nine of them for $\eta = -1$ using MAGMA. None of them leads to a solution of (1.2). This completes the proof of the lemma.

It now remains to deal with the case $n = 3$. □

4. The case $n = 3$.

Lemma 4.1. *When $n = 3$, then the only solutions to equation* (1.2) *are*

$$(x, y, a, b) = (70, 17, 0, 1), \ (142, 29, 2, 2). \tag{4.1}$$

Proof. Equation (1.2) can be rewritten as

$$\left(\frac{x}{z^3}\right)^2 + A = \left(\frac{y}{z^2}\right)^3, \tag{4.2}$$

where A is cube-free and defined implicitly by $5^a\, 13^b = Az^6$. One can see that $A = 5^{a_1}\, 13^{b_1}$ with $a_1,\ b_1 \in \{0, 1, 2, 3, 4, 5\}$. We thus get

$$V^2 = U^3 - 5^{a_1} \cdot 13^{b_1}, \tag{4.3}$$

with $a_1,\ b_1 \in \{0, 1, 2, 3, 4, 5\}$, and we need to determine all the $\{5, 13\}$-points on the above 36 elliptic curves. Here, we use again MAGMA to determine all the $\{5, 13\}$-integral points on the above elliptic curves. We find

$$\begin{aligned}(U, V, a_1, b_1) = {} & (1, 0, 0, 0), (17, 70, 0, 1), (13, 0, 0, 3), (5, 10, 2, 0), (65, 520, 2, 2),\\ & (29, 142, 2, 2), (169, 2028, 2, 4), (5, 0, 3, 0), (65, 0, 3, 3),\\ & (365, 5850, 4, 2), (10289, 1126892, 4, 3).\end{aligned}$$

As the numbers x and y are coprime positive integers, the above solutions lead to only two solutions for the original equation, namely $(x, y, a, b) = (70, 17, 0, 1),\ (142, 29, 2, 2)$. This concludes the proof. □

5. Comments on the limitation of the method. The method used in this paper to deal with the case $C = 5^a\, 13^b$ will work for other values of $C = p_1^{a_1} \ldots p_k^{a_k}$, where $p_1, \ldots, p_k$ are fixed primes provided that three conditions are satisfied. Write $C = dz^2$, where d is squarefree and let $\mathbb{K} = \mathbb{Q}[i\sqrt{d}]$. Note that d can take at most 2^k values according to the parities of the exponents a_i for $i = 1, \ldots, k$.

The first necessary condition is that any solution (x, y, d, z, n) of $x^2 + dz^2 = y^n$ with $n \geq 3$ and coprime integers x and y leads to a factorization $(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n$ in $\mathcal{O}_{\mathbb{K}}$, where the two factors appearing in the left hand side are coprime. This is always the case when y is odd, but it is not the case when y is even. In particular, if either $2 \mid C$ or $C \not\equiv 7 \pmod 8$, then this condition will be satisfied. In our example, $k = 2$, $p_1 = 5$, $p_2 = 13$, so the condition $C \not\equiv 7 \pmod 8$ is satisfied. This condition is not satisfied, say, for the equation $x^2 + 3^a \cdot 5^b = y^n$ when a and b are both odd.

The next necessary condition is that the class number of $\mathbb{K}$ is not divisible by a prime $p \geq 5$. For example, when $k = 1$, $p_1 = 47$ and $C = 47^a$ with a odd, then $\mathbb{K} = \mathbb{Q}[i\sqrt{47}]$ has class number 5. In this case, our general approach fails when $n = 5$, so the particular equation $x^2 + 47^a = y^5$ should be solved by different means. Writing $a = 10\alpha + a_1$, where α is a nonnegative integer and $a_1 \in \{0, 1, \ldots, 9\}$, we get

$$X^2 + 47^{a_1} = Y^5,$$

where $X = x/47^{5\alpha}$, $Y = y/47^{2\alpha}$, so we need to determine all $\{47\}$-integer points on 10 curves of genus 2, and this is a harder problem.

Finally, for the last necessary condition, note that assuming that $n = p \geq 5$ is a prime, then the only allowable values for p resulting upon applying the theory of primitive divisors of Lucas numbers for which the associated Lucas number L_p is not defective are the ones such that $p \mid p_i \pm 1$ for some $i = 1, \ldots, k$. In turn, by a method similar to the one used in this paper, this leads to an equation of the form $F(U, W) = L$, where both W and L are $\mathcal{S}$-units for $\mathcal{S} = \{p_1, \ldots, p_k\}$ and F is a homogeneous polynomial of degree $(p-1)/2$. Thus, the last necessary condition is that we can find all the solutions of these last equations. In case $p = 7$, F is of degree 3, so writing $L = D_1V^2$, where D_1 is squarefree, it follows that all the solutions to the above equations can be seen as $\mathcal{S}$-integer points on a collection of at most 2^{k+1} elliptic curves, which are, in fact, all quadratic twists of the same one (here, a factor of 2 accounts for the sign of D_1, and 2^k for the number of positive square free values of $|D_1|$), and this is easy. When $p > 7$, this is no longer the case. Of course, the resulting equations are Thue-Mahler equations even when $p > 7$, but finding all their solutions is no longer accomplished in such a quick way as in the case when $p = 7$.

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