

RESEARCH ARTICLE

Conjugacy growth in the higher Heisenberg groups

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Received: 29 October 2021; **Revised:** 1 July 2022; **Accepted:** 9 July 2022; **First published online:** 23 January 2023

Keywords: conjugacy growth, nilpotent groups, Heisenberg groups, twisted conjugacy

2020 Mathematics Subject Classification: *Primary* - 20F65; *Secondary* - 20E45, 20F18

Abstract

We calculate asymptotic estimates for the conjugacy growth function of finitely generated class 2 nilpotent groups whose derived subgroups are infinite cyclic, including the so-called higher Heisenberg groups. We prove that these asymptotics are stable when passing to commensurable groups, by understanding their twisted conjugacy growth. We also use these estimates to prove that, in certain cases, the conjugacy growth series cannot be a holonomic function.

1. Introduction

The *conjugacy growth function*, $c_{G,S}(n)$, of a group G with respect to a finite generating set S counts the number of conjugacy classes that intersect the n -ball in the Cayley graph, just as the standard growth function counts the number of elements contained in the n -ball. The study of conjugacy growth was originally motivated by counting geodesics on Riemannian manifolds, for example in work of Margulis [26], and since then has received significant attention in its own right, for example [6, 7, 24].

It is conjectured by Guba and Sapir [22] that for ‘ordinary’ groups, exponential standard growth implies exponential conjugacy growth. This has been verified in the case of hyperbolic, soluble and linear groups amongst others. On the other hand, Osin [28] has constructed groups of exponential standard growth where *every* non-identity element is conjugate, and so the conjecture emphatically fails in these cases. Furthermore, Osin and Hull [24] have shown that conjugacy growth fails to be a commensurability invariant in general. It is worth noting however that these counter-examples are not finitely presented.

In contrast, this paper investigates the polynomial end of the growth spectrum, namely virtually nilpotent groups (as suggested by a question of Guba and Sapir [22]). We generalise work of Babenko [2] to derive estimates for the conjugacy growth of a class of virtually nilpotent groups whose standard growth was studied by Stoll [33] and which contains the so-called higher Heisenberg groups. Some of this work appears in the author’s PhD thesis [15].

Section 2 introduces the necessary notions and proves some basic results. Section 3 summarises Stoll’s classification of class 2 nilpotent groups with infinite cyclic derived subgroup, in which it is shown that such groups can be constructed from finitely many copies of the first Heisenberg group (the free nilpotent group of class 2 on two generators). The number of copies is called the *Heisenberg rank* of the group. We also derive an explicit description of the automorphisms of such groups. In Section 4, we investigate the conjugacy growth of this class of groups. We prove the following theorem, which follows Theorem 4.1 of [2], but for a more general class of groups, and a more restricted range of metrics. Our proof uses more elementary arguments.

Theorem 4.5. *Let G be a finitely generated class 2 nilpotent group with infinite cyclic derived subgroup, with Heisenberg rank r . Then there exists $s \in \mathbb{N}$ such that*

$$c_G(n) \sim \begin{cases} n^{s+2} \log n & r = 1 \\ n^{s+2r} & r > 1. \end{cases}$$

In Section 5, we prove the following necessary and sufficient condition for conjugacy growth to be preserved in finite extensions, which depends on understanding the *twisted conjugacy growth*, c_H^ϕ , of Definition 5.5.

Proposition 5.6. *Let H be any finitely generated group. Then $c_H^\phi(n) \preceq c_H(n)$ for every finite-order automorphism $\phi \in \text{Aut}(H)$ if and only if every finite extension G of H satisfies $c_G(n) \sim c_H(n)$.*

This allows us to use the description of automorphisms to demonstrate that passing to a finite index supergroup does not alter the conjugacy growth function (up to the usual equivalence).

Theorem 5.3. *Suppose that H is a class 2 nilpotent group with infinite cyclic derived subgroup. If a group G contains H as a finite index subgroup, then G and H have equivalent conjugacy growth functions.*

This result naturally suggests the question of quasi-isometric invariance.

Question 5.8. *In which classes of finitely generated groups is conjugacy growth a quasi-isometry invariant?*

As noted above, there are certainly groups which do not have this property. On the other hand, it is not hard to see that conjugacy growth is a quasi-isometry invariant of virtually abelian groups (see Proposition 2.12). We conjecture that this behaviour extends to virtually nilpotent groups in general. More specifically, we make the following conjecture, in the spirit of Theorem 5.3.

Conjecture 5.10. *The conjugacy growth function of a finitely generated nilpotent group depends only on the structure of its lower central series.*

A closely related measure of conjugacy growth is the formal power series whose coefficients are the values of the conjugacy growth function. Ciobanu, Hermiller, Holt and Rees [9] and Antolín and Ciobanu [1] confirmed a conjecture of Rivin [31], that the conjugacy growth series of a hyperbolic group is a rational function if and only if the group is virtually cyclic (or finite). Continuing in a similar vein, the author proved the rationality of the series for all virtually abelian groups [14], and Gekhtman and Yang proved transcendence for relatively hyperbolic groups and certain acylindrically hyperbolic groups [19]. All of the above results apply to any choice of finite generating set and lead to the following conjecture.

Conjecture 1.1 (Conjecture 7.2 of [8]). *The conjugacy growth series of any finitely presented group that is not virtually abelian is transcendental.*

This is further supported by generating set specific calculations for, amongst others, the soluble Baumslag–Solitar groups [8], certain wreath products [27] and graph products [10]. The asymptotic estimates derived in this paper have implications for some conjugacy growth series (see Corollary 5.4), providing further evidence for this conjecture.

2. Preliminaries

We begin with the basic definitions and results. In this paper, \mathbb{N} will denote the non-negative integers and \mathbb{N}_+ the positive integers. We occasionally use the notation $f(n) = \mathcal{O}(g(n))$ to indicate that there exists a constant $C \geq 1$ with $f(n) \leq Cg(n)$ for large enough n , and the notation $f(n) = o(g(n))$ to indicate that $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$.

2.1. Growth and conjugacy growth

Definition 2.1. Let G be a finitely generated group, and S a choice of finite generating set. We write $\{S \cup S^{-1}\}^*$ to denote the language of all words over the alphabet consisting of the elements of S and their inverses (i.e. the free monoid on $S \cup S^{-1}$).

(1) The word length of $g \in G$ with respect to S is

$$|g|_S = \min\{|w| \mid w \in \{S \cup S^{-1}\}^*, w =_G g\}.$$

(2) The (cumulative) standard growth function of G with respect to S is

$$\beta_{G,S}(n) = \#\{g \in G \mid |g|_S \leq n\}.$$

Definition 2.2. Let G be a finitely generated group and S be a choice of finite generating set. Denote by \mathcal{C}_G the set of conjugacy classes of G .

(1) For $\kappa \in \mathcal{C}_G$, define the length of κ with respect to S as:

$$|\kappa|_S = \min\{|g|_S \mid g \in \kappa\}.$$

(2) The (cumulative) conjugacy growth function of G with respect to S is

$$c_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S \leq n\}$$

Geometrically, the standard growth function counts the elements in the ball of radius n in the Cayley graph of G with respect to S , and the conjugacy growth function counts the number of distinct conjugacy classes intersecting the same ball. One could also study the ‘strict’ standard and conjugacy growth functions, by considering the sphere instead of the ball, but for most purposes their qualitative behaviour is the same.

We will use the usual notion of equivalence of growth functions.

Definition 2.3. Let $f, g: \mathbb{N} \rightarrow \mathbb{N}$ be two functions. We write $f \preceq g$ if there exists $\lambda > 1$ such that

$$f(n) \leq \lambda g(\lambda n) + \lambda$$

for all $n \in \mathbb{N}$. If $f \preceq g$ and $g \preceq f$, then we write $f \sim g$ and say that the functions are equivalent. Note that this defines an equivalence relation. We write $f < g$ if $f \preceq g$ but it is not the case that $f \sim g$.

It is a classical result that the equivalence class of the standard growth function of a finitely generated group does not depend on the choice of generating set. It is less well known that the same is true of the conjugacy growth function.

Proposition 2.4. Let S and T be two finite generating sets for a group G . Then

- (1) $\beta_{G,S} \sim \beta_{G,T}$, and
- (2) $c_{G,S} \sim c_{G,T}$.

Proof. The proof of the first part is standard, see for example [25]. The second statement is proved similarly. □

Another key fact that we will use is that standard growth is a quasi-isometry invariant.

Proposition 2.5. *Let G and H be quasi-isometric groups. Then $\beta_G \sim \beta_H$.*

This is not the case for conjugacy growth.

Theorem 7.2 of [24]. *There exists a finitely generated group G and a finite index subgroup $H \leq G$ such that H has two conjugacy classes, while G is of exponential conjugacy growth.*

We can study the growth of a subset of G with respect to the same metric. This is referred to as *relative growth*. Formally, we have the following definition.

Definition 2.6. *The relative growth function of a subset U of a group G with respect to a finite generating set S is the following function:*

$$\beta_U(n) = \#\{g \in U \mid |g|_S \leq n\}.$$

In the following lemma, we see that the relative growth of a finite index subgroup is equivalent to that of its cosets.

Lemma 2.7. *Let H be a finite index subgroup of G and consider the coset tH for some $t \in G$. Then the relative growth of tH is equivalent to the growth of G .*

Proof. Since tH is simply a translation of H , H and tH are quasi-isometric. Since G and H are also quasi-isometric, Proposition 2.5 implies that $\beta_G \sim \beta_H \sim \beta_{tH}$. □

We will need the following result of Breuillard and Cornulier.

Lemma 2.8 (Lemma 3.1 (2) of [7]). *Let H be a finite index subgroup of G . Then $c_H \asymp c_G$.*

The next Lemma shows that conjugacy growth behaves well with respect to direct products. Here, and in the rest of the paper, we use the standard notation $[g]$ to denote the conjugacy class of a group element g .

Lemma 2.9. *Let G and H be groups generated by finite sets S and T , respectively. Then $c_{G \times H, S \cup T} \sim c_{G,S} \cdot c_{H,T}$.*

Proof. First, note that for $(g, h) \in G \times H$ we have

$$[(g, h)] = \{(g, h)^{(x,y)} \mid (x, y) \in G \times H\} = \{(g^x, h^y) \mid (x, y) \in G \times H\} = [g] \times [h].$$

Now if $|[g, h]|_{S \cup T} \leq n$ then there exists $(x, y) \in G \times H$ with $(x, y)(g, h)(x, y)^{-1} = u_1 u_2 \cdots u_l$ for some $u_i \in S \cup T$, $l \leq n$. Rearranging, we find elements $s_1, \dots, s_{l_1} \in S$ and $t_1, \dots, t_{l_2} \in T$, with $l_1 + l_2 = l$, so that

$$(xgx^{-1}, yhy^{-1}) = s_1 \cdots s_{l_1} t_1 \cdots t_{l_2}.$$

Therefore, $|xgx^{-1}|_S \leq l_1 \leq n$ and $|yhy^{-1}|_T \leq l_2 \leq n$, and so $|[g]|_S \leq n$ and $|[h]|_T \leq n$. Thus, $c_{G \times H}(n) \leq c_G(n) \cdot c_H(n)$.

Conversely, suppose $|[g]_S| \leq n$ and $|[h]_S| \leq n$. Then, there are elements $\gamma \in G$ and $\delta \in H$ such that $|\gamma g \gamma^{-1}|_S \leq n$ and $|\delta h \delta^{-1}|_T \leq n$, so $|(\gamma g \gamma^{-1}, \delta h \delta^{-1})|_{S \cup T} \leq 2n$. But $(\gamma g \gamma^{-1}, \delta h \delta^{-1}) = (\gamma, \delta)(g, h)(\gamma, \delta)^{-1}$ and so $|[(g, h)]|_{S \cup T} \leq 2n$. Thus, $c_G(n) \cdot c_H(n) \leq c_{G \times H}(2n)$, giving the required result. \square

2.2. Nilpotent groups

We recall the definition of a nilpotent group in order to fix some notation. For elements g, h of some group G , denote their commutator $[g, h] = ghg^{-1}h^{-1}$. For a pair of subgroups U, V of G , let $[U, V] = \langle [u, v] \mid u \in U, v \in V \rangle$. For any group G , let $G^{(0)} = G$ and inductively define the i -fold commutator subgroup $G^{(i)} = [G^{(i-1)}, G]$. Recall that a group G is *nilpotent of class c* if and only if $G^{(c)} = \{1\}$ and $G^{(c-1)} \neq \{1\}$. In particular, the nilpotent groups of class 1 are precisely the abelian groups. We write $\text{Ab}(G) = G/G^{(1)}$ for the abelianisation of G .

Definition 2.10. *Let G be a finitely generated group. Then, each quotient $G^{(i)}/G^{(i+1)}$ is a finitely generated abelian group. Denote the torsion-free rank of G^i/G^{i+1} by $r_i \in \mathbb{N}$, so that $G^{(i)}/G^{(i+1)} \cong \mathbb{Z}^{r_i} \times T$ for some finite abelian group T .*

Theorem 2.11 (Bass-Guivarc’h [3]). *Let G be a finitely generated nilpotent group. Then the standard growth function $\beta_G(n)$ is equivalent to the polynomial n^d where*

$$d = \sum_{i=0}^{c-1} (i + 1)r_i.$$

Gromov [21] famously proved the converse that any group of polynomial (standard) growth is virtually nilpotent.

The asymptotic behaviour of conjugacy growth in virtually *abelian* groups is well understood, as we see in the following Proposition.

Proposition 2.12. *The (cumulative) standard and conjugacy growth functions of a virtually abelian group G are equivalent.*

Proof. Let H be an abelian subgroup of G of finite index. Since H is nilpotent, Theorem 2.11 gives $d \in \mathbb{N}$ with $\beta_H(n) \sim n^d$, and hence $\beta_G(n) \sim n^d$ by Proposition 2.5. Thus, $c_G(n) \preceq n^d$ (since conjugacy growth is clearly bounded above by standard growth). On the other hand, Lemma 2.8 gives $c_G(n) \succeq c_H(n) \sim \beta_H(n) \sim n^d$ (since a conjugacy class in an abelian group is simply an element). Therefore, $c_G(n) \sim n^d \sim \beta_G(n)$. \square

As an immediate corollary, we see that conjugacy growth is a quasi-isometry invariant within the class of virtually abelian groups.

2.3. Generating functions

We will also be interested in the formal power series associated with the standard and conjugacy growth functions. We write $\mathbb{Q}[[z]]$ for the ring of formal power series over a variable z with coefficients in \mathbb{Q} , and $\mathbb{Q}[z]$ for the ring of polynomials over z with rational coefficients.

Definition 2.13. *Let G be a group with a finite generating set S . Then the standard growth series is*

$$B_{G,S}(z) = \sum_{n=0}^{\infty} \beta_{G,S}(n)z^n \in \mathbb{Q}[[z]].$$

Similarly, the conjugacy growth series is

$$C_{G,S}(z) = \sum_{n=0}^{\infty} c_{G,S}(n)z^n \in \mathbb{Q}[[z]].$$

Here, z is a complex variable.

When referring to growth functions and series, we will often suppress the subscripts when the groups and/or generating sets are clear from context.

Definition 2.14. A series $\Gamma(z) \in \mathbb{Q}[[z]]$ is said to be

- (1) rational if it is an element of the field of fractions of $\mathbb{Q}[z]$, denoted $\mathbb{Q}(z)$ – in other words, there are polynomials $p, q \in \mathbb{Q}[z]$ such that $\Gamma = \frac{p}{q}$;
- (2) algebraic if it is algebraic over $\mathbb{Q}[z]$ – in other words, it is the root of some polynomial expression with coefficient from the ring of polynomials $\mathbb{Q}[z]$;
- (3) transcendental if it is not algebraic;
- (4) holonomic if it is the solution to a linear finite-order differential equation with coefficients from the ring of polynomials $\mathbb{Q}[z]$; therefore, the non-holonomic series form a proper subset of the transcendental series.

We will refer to this classification as the algebraic complexity of $\Gamma(z)$.

Generating functions are a well-studied topic. Duchin has written a very readable introduction [12] to generating functions and growth in groups. For a more rigorous treatment, we use [20] or [32]. There are some slightly subtle connections between the algebraic complexity of a power series and the asymptotics of the coefficients. For example, we will use the following result of Stoll.

Proposition 2.15 (Proposition 3.3 of [33]). Let $\Gamma(z) = \sum_{n \geq 0} \gamma(n)z^n \in \mathbb{Q}[[z]]$, and suppose that $\lim_{n \rightarrow \infty} \frac{\gamma(n)}{n^d} = a$. Then if a is an irrational (resp. transcendental) number then $\Gamma(z)$ is irrational (resp. transcendental) as a series.

The next result relates to generating functions whose coefficients are in the polynomial range.

Theorem 2.16. Let $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ be strictly between polynomials, that is $n^d < \gamma(n) < n^{d+1}$ for some $d \in \mathbb{N}$. Then the series $\sum_{n \geq 0} \gamma(n)z^n$ is not holonomic.

To prove this, we will need another definition.

Definition 2.17. A function $f : \mathbb{N} \rightarrow \mathbb{C}$ is called eventually quasi-polynomial if there exist some positive integer period N , threshold $T \geq 0$, and polynomials f_0, f_1, \dots, f_{N-1} so that for all $n \geq T$, $f(n) = f_i(n)$ whenever $n \equiv i \pmod N$.

The following is an immediate consequence of Proposition 4.4.1 of [32].

Proposition 2.18. Let $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ be in the polynomial range, that is, $\gamma(n) \leq Cn^d$ for some $C > 1, d \in \mathbb{N}$. Then, $\sum_{n \geq 0} \gamma(n)z^n$ is rational if and only if $\gamma(n)$ is eventually quasi-polynomial.

Corollary 2.19. Suppose $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ is non-decreasing and in the polynomial range as above. If $\sum_{n \geq 0} \gamma(n)z^n$ is rational then $\gamma \sim n^d$ for some $d \in \mathbb{N}$.

Proof. Proposition 2.18 implies that γ is eventually quasi-polynomial, say with polynomials $\gamma_0, \dots, \gamma_N$ as in the definition. The degree of each γ_i is at least the degree of γ_{i-1} (since large enough n would otherwise violate the non-decreasing assumption). But since these polynomials cycle, the degree of γ_0 must be at least the degree of γ_N . So they all have the same degree, say $d \in \mathbb{N}$. Thus, γ cycles between finitely many polynomials, all equivalent to n^d , and so $\gamma(n) \sim n^d$. \square

We also need the following two results (see [18]).

Lemma 2.20 (Pólya–Carlson). *If $\Gamma(z)$ is a power series with integer coefficients that converges on the open unit disc, then $\Gamma(z)$ is either rational or admits the unit circle as a natural boundary.*

Lemma 2.21. *Holonomic functions necessarily have only finitely many singularities.*

Proof of Theorem 2.16. By the Cauchy–Hadamard theorem, the series $\sum \gamma(n)z^n$ converges inside the unit disc, and so Lemma 2.20 applies. Thus, if the series were holonomic, and hence had only finitely many singularities, it would be rational. But by Corollary 2.19, this contradicts the hypothesis. Thus, the series cannot be holonomic. \square

We finish our discussion of generating functions by noting the following results from the literature.

Theorem 2.22 ([4]). *Let G be a finitely generated virtually abelian group. Then the standard growth series of G is rational, with respect to any choice of finite generating set.*

Theorem 2.23 ([14]). *Let G be a finitely generated virtually abelian group. Then the conjugacy growth series of G is rational, with respect to any choice of finite generating set.*

2.4. GCD sums

To count conjugacy classes, we will need various facts about greatest common divisors of tuples of integers, starting with the following lemma of Fernández and Fernández.

Lemma 2.24 (Section 3 of [17]). *For $n \geq 1$, let $X_1^{(n)}, X_2^{(n)}, \dots$ be a sequence of independent random variables, uniformly distributed in $\{1, 2, \dots, n\}$. Then the expected value of the greatest common divisor of the first s of these random variables behaves as follows:*

$$\mathbb{E}(\gcd(X_1^{(n)}, X_2^{(n)}, \dots, X_s^{(n)})) = \begin{cases} \frac{1}{\zeta(2)} \log n + C + \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right) & s = 2 \\ \frac{\zeta(s-1)}{\zeta(s)} + \mathcal{O}\left(\frac{\log n}{n}\right) & s \geq 3 \end{cases}$$

where $C \geq 0$ is a constant.

For our purposes, we will phrase this in terms of the sum of the greatest common divisors of tuples of integers whose absolute values are at most n .

Definition 2.25. *We define two different n -balls in the free abelian group \mathbb{Z}^s .*

(1) Let $B_{\square}^{(s)}(n) = \{(x_1, \dots, x_s) \in \mathbb{Z}^s \mid |x_i| \leq n \text{ for each } 1 \leq i \leq s\}$. That is, the n -ball in \mathbb{Z}^s with respect to the ‘cubical’ generating set:

$$\{(\varepsilon_1, \dots, \varepsilon_s) \mid \varepsilon_i \in \{0, 1, -1\}\}.$$

(2) Let $B_{\ell_1}^{(s)}(n) = \{(x_1, \dots, x_s) \in \mathbb{Z}^s \mid \sum |x_i| \leq n\}$. That is, the n -ball in \mathbb{Z}^s with respect to the generating set consisting of standard basis vectors.

We will omit the superscript s when it is clear which dimension we are working with.

Then, Lemma 2.24 can be reinterpreted as follows.

Corollary 2.26. Let $\mathbf{x} = (x_1, \dots, x_s) \in \mathbb{Z}^s$. Then,

$$\sum_{\mathbf{x} \in B_{\square}^{(2)}(n)} \gcd(\mathbf{x}) = \frac{R_2}{\zeta(2)} n^2 \log n + \mathcal{O}(n^2)$$

where $R_2 \in \mathbb{Q}$, and

$$\sum_{\mathbf{x} \in B_{\square}^{(s)}(n)} \gcd(\mathbf{x}) = R_s \frac{\zeta(s-1)}{\zeta(s)} n^s + \mathcal{O}(n^{s-1} \log n)$$

where $R_s \in \mathbb{Q}$ depends on the dimension s .

Proof. The sum of the values of a function over some fixed finite domain is equal to the expected value of the function over the domain, multiplied by the cardinality of the domain. The standard growth function of \mathbb{Z}^s is equivalent to Dn^s (by Theorem 2.11), where $D \in \mathbb{R}$ depends on the choice of generating set, but is always rational since otherwise Proposition 2.15 would imply that the standard growth series was irrational, contradicting Theorem 2.22. \square

We will also need the following generalisation of Corollary 2.26, showing that offsetting \mathbf{x} by a constant does not affect the asymptotics of the GCD sum.

Corollary 2.27. Fix an element $\mathbf{a} = (a_1, \dots, a_s) \in \mathbb{Z}^s$. Then,

$$\sum_{\mathbf{x} \in B_{\square}^{(2)}(n)} \gcd(\mathbf{x} + \mathbf{a}) = \frac{R_2}{\zeta(2)} n^2 \log n + \mathcal{O}(n^2)$$

and

$$\sum_{\mathbf{x} \in B_{\square}^{(s)}(n)} \gcd(\mathbf{x} + \mathbf{a}) = R_s \frac{\zeta(s-1)}{\zeta(s)} n^s + \mathcal{O}(n^{s-1} \log n)$$

where $R_2, R_s \in \mathbb{Q}$ are the same as in Corollary 2.26.

Proof. Let $a_{\max} = \max(|a_1|, \dots, |a_s|)$. Then, we have

$$B_{\square}^{(s)}(\min(n - a_{\max}, 0)) \subseteq \{\mathbf{x} + \mathbf{a} \mid \mathbf{x} \in B_{\square}^{(s)}(n)\} \subseteq B_{\square}^{(s)}(n + a_{\max})$$

for all n and thus

$$\sum_{\mathbf{x} \in B_{\square}^{(s)}(\min(n - a_{\max}, 0))} \gcd(\mathbf{x}) \leq \sum_{\mathbf{x} \in B_{\square}^{(s)}(n)} \gcd(\mathbf{x} + \mathbf{a}) \leq \sum_{\mathbf{x} \in B_{\square}^{(s)}(n + a_{\max})} \gcd(\mathbf{x}).$$

Applying Corollary 2.26 gives the result, since adding a constant to n does not alter the asymptotic expressions on the right-hand sides. \square

3. A family of class 2 nilpotent groups

In this section, we discuss the nilpotent groups of class 2 whose derived subgroup is infinite cyclic. This includes the following family.

Definition 3.1. *The (higher) Heisenberg groups are class 2 nilpotent groups, with a parameter $r \in \mathbb{N}_+$, given by the following presentation:*

$$H_r = \left\langle a_1, b_1, a_2, b_2, \dots, a_r, b_r \mid \begin{array}{l} [a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1 \ \forall i \neq j \\ [a_i, b_i] = [a_j, b_j] \ \forall i \neq j \\ [[a_i, b_i], a_j] = [[a_i, b_i], b_j] = 1 \ \forall i, j \end{array} \right\rangle.$$

The commutator subgroup $H_r^{(1)}$ is infinite cyclic, generated by the commutator $c = [a_i, b_i]$. These groups play an important role in the story of standard growth, as they provide essentially the only known examples of growth series behaviour which depends on the choice of generating set, as we will see in Stoll’s result, Theorem 3.9. Furthermore, Duchin and Shapiro have shown [13] that the first Heisenberg group (also known as the integer or discrete Heisenberg group) H_1 has rational standard growth series with respect to any choice of finite generating set.

3.1. Stoll’s classification

In [33], Stoll classifies all finitely generated class 2 nilpotent groups with infinite cyclic derived subgroup in terms of the first Heisenberg group H_1 . We summarise this classification below.

Let G_1 and G_2 be groups with central subgroups Z_1 and Z_2 , respectively. Suppose that there exists an abelian group Z and homomorphisms $\varphi_1: Z_1 \rightarrow Z$ and $\varphi_2: Z_2 \rightarrow Z$, and consider the product homomorphism $\varphi: Z_1 \times Z_2 \rightarrow Z$ defined by:

$$\varphi: (z_1, z_2) \mapsto \varphi_1(z_1)\varphi_2(z_2).$$

Furthermore, suppose that φ is surjective.

Definition 3.2. *The group*

$$G = \frac{G_1 \times G_2}{\ker \varphi}$$

is called the centrally amalgamated direct product of G_1 and G_2 with respect to φ . We will write central product for brevity.

Example 3.3. *Let $G_1 \cong G_2 \cong \mathbb{Z}^2$ be given by the presentations:*

$$G_1 = \langle x_1, y_1 \mid [x_1, y_1] \rangle, \quad G_2 = \langle x_2, y_2 \mid [x_2, y_2] \rangle,$$

and let $Z = \langle z \mid z^2 \rangle$. Define homomorphisms from the second direct factor of each G_i to Z as $\varphi_i: \langle y_i \rangle \rightarrow Z$ given by $\varphi: y_i \mapsto z$. Then the centrally amalgamated direct product, G , of G_1 and G_2 with respect to φ is given by the presentation:

$$\langle x_1, x_2, z \mid [x_1, x_2], [x_1, z], [x_2, z], z^2 \rangle \cong \mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}.$$

From now on, we will deal exclusively with the case where each Z_i is infinite cyclic, generated by an element z_i , and similarly Z is infinite cyclic, generated by an element z . Hence, each φ_i is determined by an integer d_i as follows $\varphi_i: z_i \mapsto z^{d_i}$. Given pairs (G_1, z_1) and (G_2, z_2) such that each $\langle z_i \rangle$ is an infinite cyclic central subgroup of G_i , we will write $(G_1, z_1) \otimes_d (G_2, z_2)$ for the central product of G_1 and G_2 ,

amalgamated over the subgroups $\langle z_1 \rangle$ and $\langle z_2 \rangle$ with $\varphi_1(z_1) = z$ and $\varphi_1(z_2) = z^d$. If $d = 1$, we simply write $(G_1, z_1) \otimes (G_2, z_2)$.

Lemma 3.4 (Lemma 7.1 of [33]). *Let G be a finitely generated 2-step nilpotent group with $G^{(1)} \cong \mathbb{Z}$. Then there exists a finitely generated infinite abelian group G_0 , and a tuple $D = (\delta_1, \dots, \delta_{r-1}) \in (\mathbb{N}_+)^{r-1}$, with $\delta_i | \delta_{i+1}$ for each i , such that*

$$G \cong (\dots (((G_0, z) \otimes (H_1, c_1)) \otimes_{\delta_1} (H_1, c_2)) \otimes_{\delta_2} (H_1, c_3)) \dots \otimes_{\delta_{r-1}} (H_1, c_r),$$

where each $H_1 = \langle a_i, b_i \mid [[a_i, b_i], a_i], [[a_i, b_i], b_i] \rangle$ is a copy of the first Heisenberg group, c_i denotes the commutator $[a_i, b_i]$, and z generates an infinite cyclic subgroup of G_0 .

Remark 3.5. *Note that due to the amalgamation in G , $c_1 = z$ and for each i , we have $c_i = c_1^{\delta_i-1}$.*

Definition 3.6. *Let $D = (\delta_1, \dots, \delta_{r-1}) \in (\mathbb{N}_+)^{r-1}$, with $\delta_i | \delta_{i+1}$ for each i . Then we will write*

$$H_D = (\dots (((H_1, c_1) \otimes_{\delta_1} (H_1, c_2)) \otimes_{\delta_2} (H_1, c_3)) \dots \otimes_{\delta_{r-1}} (H_1, c_r).$$

Note that if $I = (1, 1, \dots, 1) \in (\mathbb{N}_+)^{r-1}$, we can express the r th higher Heisenberg group as $H_r = H_I$. Since the abelian subgroup $\Gamma := G_0 / \langle z \rangle$ is central, we have the following immediate corollary of Lemma 3.4.

Corollary 3.7. *Let G be a finitely generated 2-step nilpotent group with $[G, G] \cong \mathbb{Z}$. Then there exists a finitely generated abelian group Γ (possibly finite or trivial) and a tuple $D = (\delta_1, \dots, \delta_{r-1}) \in (\mathbb{N}_+)^{r-1}$, with $\delta_i | \delta_{i+1}$ for each i , such that*

$$G \cong \Gamma \times H_D.$$

Definition 3.8. *As in [33], the Heisenberg rank of $G \cong \Gamma \times H_D$ will refer to the number, r , of copies of H_1 appearing in the construction. Furthermore, we will write s for the torsion-free rank of the finitely generated abelian factor Γ .*

To emphasise the importance of this class of groups, we record the following, which is the main result of [33].

Theorem 3.9. *If G is a class 2 nilpotent group with infinite cyclic derived subgroup and Heisenberg rank at least 2, then it possesses a generating set yielding rational standard growth series, and a generating set yielding transcendental standard growth series.*

We fix some notation that we will use throughout the paper.

Definition 3.10. *Suppose that $\Gamma \times H_D$ is torsion-free and choose a basis $\{z_1, \dots, z_s\}$ for Γ . Then the elements of $\Gamma \times H_D$ are in bijection with words of the form $z_1^{i_1} \dots z_s^{i_s} a_1^{j_1} b_1^{k_1} \dots a_r^{j_r} b_r^{k_r} c^k$ where $i_1, \dots, i_s, j_1, \dots, j_r, k \in \mathbb{Z}$. This is known as the Mal'cev normal form, or Mal'cev coordinates, and a variation of it exists for all finitely generated nilpotent groups (see [11], or the basic commutators of [23]). We will frequently represent elements of $\Gamma \times H_D$ as αc^k where α is an element of the form $z_1^{i_1} \dots z_s^{i_s} a_1^{j_1} b_1^{k_1} \dots a_r^{j_r} b_r^{k_r}$ and $c = c_1 = [a_1, b_1]$.*

For the remainder of the paper, we use the generating set $\{z_1, \dots, z_s, a_1, \dots, b_r\}$ for $\Gamma \times H_D$ and lengths of elements will be taken with respect to this generating set.

Definition 3.11. For $g \in \Gamma \times H_D$, we write \bar{g} for the image of g in the abelianisation $\text{Ab}(\Gamma \times H_D)$. With this notation, we have

$$\text{Ab}(\Gamma \times H_D) = \langle \bar{z}_1, \dots, \bar{z}_s, \bar{a}_1, \bar{b}_1, \dots, \bar{a}_r, \bar{b}_r \rangle \cong \mathbb{Z}^{2r+s}.$$

The preimage of $x \in \text{Ab}(\Gamma \times H_D)$ under the abelianisation map is a coset of the commutator subgroup $\langle c \rangle$. We define the canonical lift of x to be the unique element of the preimage whose c -coordinate is zero (when expressed in Mal'cev normal form) and denote it x^\wedge . Therefore, if $x = \bar{z}_1^{l_1} \cdots \bar{z}_s^{l_s} \bar{a}_1^{i_1} \bar{b}_1^{j_1} \cdots \bar{a}_r^{i_r} \bar{b}_r^{j_r}$ then $x^\wedge = z_1^{l_1} \cdots z_s^{l_s} a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r} \in \Gamma \times H_D$.

We will also write elements of $\text{Ab}(\Gamma \times H_D)$ as vectors in \mathbb{Z}^{2r+s} with respect to the basis $\{\bar{z}_1, \dots, \bar{z}_s, \bar{a}_1, \bar{b}_1, \dots, \bar{a}_r, \bar{b}_r\}$ so that $\bar{z}_1^{l_1} \cdots \bar{z}_r^{l_r} \bar{a}_1^{i_1} \cdots \bar{b}_r^{j_r} = (l_1, \dots, l_s, i_1, \dots, j_r)$.

3.2. Automorphisms of class 2 nilpotent groups

We will need to understand the automorphisms of our family of nilpotent groups. First, note the following easy lemma.

Lemma 3.12. Let G be a group with a characteristic subgroup H . Then the natural homomorphism $G \rightarrow G/H$ induces a homomorphism $\theta: \text{Aut}(G) \rightarrow \text{Aut}(G/H)$.

Definition 3.13. Let Ω_r denote the $2r \times 2r$ block matrix consisting of blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal with zeroes elsewhere. Let $\Omega_{r,s}$ be the $(2r + s) \times (2r + s)$ matrix with Ω_r in the bottom right corner and zeroes elsewhere.

For example,

$$\Omega_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Note that the skew-symmetric bilinear form $\Omega_{r,s}$ is precisely that given by taking commutators of pairs of elements of $\text{Ab}(\Gamma \times H_r)$ with respect to the basis $\{\bar{z}_1, \dots, \bar{z}_s, \bar{a}_1, \bar{b}_1, \dots, \bar{a}_r, \bar{b}_r\}$, that is, $[\cdot, \cdot]: \mathbb{Z}^{2r+s} \times \mathbb{Z}^{2r+s} \rightarrow \mathbb{Z}$. Let $\alpha c^{k_1}, \beta c^{k_2} \in \Gamma \times H_r$. Then, $\overline{\alpha c^{k_1}} = \bar{\alpha}$ and $\overline{\beta c^{k_2}} = \bar{\beta}$ are elements of \mathbb{Z}^{2r+s} and we have

$$[\alpha c^{k_1}, \beta c^{k_2}] = [\alpha, \beta] = e^{\bar{\alpha} \Omega_{r,s} \bar{\beta}^T}.$$

Definition 3.14. Let $\mathbf{M} < \text{GL}_{2r+s} \mathbb{Z}$ be the group of matrices that either preserve or reverse the bilinear form given by $\Omega_{r,s}$ (i.e. the group of matrices M such that $M \Omega_{r,s} M^T = \varepsilon \Omega_{r,s}$ for $\varepsilon \in \{1, -1\}$). Note that if $s = 0$, then $\mathbf{M} \cong \text{Sp}(2r, \mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$.

Definition 3.15. Suppose that $N = \Gamma \times H_D$ is torsion-free and let $M \in \mathbf{M}$. Define a map $\phi_M: N \rightarrow N$ as follows. For each $1 \leq i \leq s$, set $\phi_M(z_i) = (\bar{z}_i M)^\wedge$, that is, the canonical lift of the image of \bar{z}_i under M . Similarly, set $\phi_M(a_i) = (\bar{a}_i M)^\wedge$ and $\phi_M(b_i) = (\bar{b}_i M)^\wedge$ for each $1 \leq i \leq r$. Then extend this map to an endomorphism of N in the usual way by setting $\phi_M(x_1 \cdots x_n) = \phi_M(x_1) \cdots \phi_M(x_n)$ for any word $x_1 \cdots x_n$ in the generators.

Lemma 3.16. The map ϕ_M is a well-defined automorphism of N .

Proof. First note that for any $x \in \mathbb{Z}^{2r+s}$, we have $\overline{x^\wedge} = x$. It is easily checked that the relators of N are mapped to the identity and therefore ϕ_M defines an endomorphism of N . For example, for the relator $[z_i, a_j]$, we have

$$[(\bar{z}_i M)^\wedge, (\bar{a}_j M)^\wedge] = c^{\bar{z}_i M \Omega_{r,s} (\bar{a}_j M)^T} = c^{\varepsilon \bar{z}_i \Omega_{r,s} \bar{a}_j^T} = 1.$$

Furthermore, for any M , $\phi_{M^{-1}}$ is the inverse of ϕ_M , which is therefore an automorphism. It is sufficient to check this on the generators, for example,

$$\phi_{M^{-1}} \circ \phi_M(z_i) = \left(\overline{(\bar{z}_i M)^\wedge M^{-1}} \right)^\wedge = (\bar{z}_i M M^{-1})^\wedge = \bar{z}_i^\wedge = z_i.$$

□

Proposition 3.17. *Suppose that $N = \Gamma \times H_D$ is torsion-free. Then the natural homomorphism $N \rightarrow N/N^{(1)}$ induces (via Lemma 3.12) an epimorphism $\theta : \text{Aut}(N) \rightarrow \mathbf{M}$.*

Proof. We first show that the image of the induced homomorphism θ is contained in \mathbf{M} . Let $f \in \text{Aut}(N)$ and write $\theta(f) = M \in \text{GL}_{2r+s}(\mathbb{Z})$. Since the commutator subgroup $N^{(1)}$ is characteristic, f restricts to an automorphism of $N^{(1)} = \langle c \rangle \cong \mathbb{Z}$, so we have $f : c \mapsto c^\varepsilon$ where $\varepsilon \in \{-1, 1\}$.

Then, for any $\alpha c^{k_1}, \beta c^{k_2} \in N$, we have

$$f([\alpha c^{k_1}, \beta c^{k_2}]) = f([\alpha, \beta]) = [\alpha, \beta]^\varepsilon$$

and hence $[f(\alpha), f(\beta)] = [\alpha, \beta]^\varepsilon$. Therefore,

$$c^{\bar{\alpha} M \Omega_{2r+s} M^T \bar{\beta}^T} = c^{\varepsilon \bar{\alpha} \Omega_{2r+s} \bar{\beta}^T}$$

for all $\bar{\alpha}, \bar{\beta} \in \text{Ab}(N)$ and hence $M \Omega_{2r+s} M^T = \varepsilon \Omega_{2r+s}$, that is, $M \in \mathbf{M}$ as claimed.

To see that \mathbf{M} is contained in the image of $\text{Aut}(N)$, we note that for each $M \in \mathbf{M}$, $\theta(\phi_M) = M$. □

Remark 3.18. *Although we do not need it for the arguments that follow, we note that in the special case of $N = H_r$, we can use a straightforward generalisation of an argument of Osipov [29] to extend Proposition 3.17 to give the following short exact sequence, where $\text{Inn}(H_r) = \mathbb{Z}^{2r}$:*

$$1 \rightarrow \mathbb{Z}^{2r} \rightarrow \text{Aut}(H_r) \rightarrow \text{Sp}_{2r}(\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

In Section 5, we will need a more explicit description of the automorphisms of N .

Proposition 3.19. *Suppose $N = \Gamma \times H_D$ is torsion-free and fix $f \in \text{Aut}(N)$. Write $\theta(f) = M \in \mathbf{M}$ for the image of f as above. Then there exists $\varepsilon \in \{1, -1\}$ and a polynomial function $\gamma : \mathbb{R}^{2r+s} \rightarrow \mathbb{R}$ of degree 2, with $\gamma(0, \dots, 0) = 0$, restricting to a function $\mathbb{Z}^{2r+s} \rightarrow \mathbb{Z}$, such that for each element $z_1^{i_1} \dots z_s^{i_s} a_1^{j_1} b_1^{j_1} \dots a_r^{i_r} b_r^{j_r} c^k \in N$ we have*

$$\begin{aligned} f(z_1^{i_1} \dots z_s^{i_s} a_1^{j_1} b_1^{j_1} \dots a_r^{i_r} b_r^{j_r} c^k) &= \left(\overline{z_1^{i_1} \dots z_s^{i_s} a_1^{j_1} b_1^{j_1} \dots a_r^{i_r} b_r^{j_r} M} \right)^\wedge c^{\varepsilon k + \gamma(l_1, \dots, l_s, i_1, j_1, \dots, i_r, j_r)} \\ &= ((l_1, \dots, l_s, i_1, j_1, \dots, i_r, j_r) M)^\wedge c^{\varepsilon k + \gamma(l_1, \dots, l_s, i_1, j_1, \dots, i_r, j_r)}, \end{aligned}$$

with ε and γ depending only on f .

Proof. Proposition 3.17 gives us a short exact sequence $1 \rightarrow K \rightarrow \text{Aut}(N) \xrightarrow{\theta} \mathbf{M} \rightarrow 1$, for some normal subgroup $K \triangleleft \text{Aut}(N)$ (which contains but may not be equal to $\text{Inn}(N)$). The automorphisms

$\{\phi_M \in \text{Aut}(N) \mid M \in \mathbf{M}\}$ defined above form a transversal for $\text{Aut}(N)/K$, and hence $f = \kappa \circ \phi_M$ for some $\kappa \in K$ and $M \in \mathbf{M}$.

Since $\kappa \in K$, κ maps to the identity in \mathbf{M} . So there exist integers k_i such that

$$\begin{aligned} \kappa : z_1 &\mapsto z_1 c^{k_1} \\ &\vdots \\ z_s &\mapsto z_s c^{k_s} \\ a_1 &\mapsto a_1 c^{k_s+1} \\ b_1 &\mapsto b_1 c^{k_s+2} \\ &\vdots \\ b_r &\mapsto b_r c^{k_s+2r}. \end{aligned}$$

Therefore, applying κ to some $z_1^{l_1} \cdots z_s^{l_s} a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r} c^k \in N$ fixes the powers of the generators z_1, \dots, b_r and adds $p(l_1, \dots, j_r)$ to the power of c , where p is a linear function determined by the integers k_1, \dots, k_{s+2r} .

On the other hand, $\phi_M \in \text{Aut}(N)$ takes each generator to a linear combination of the generators z_1, \dots, b_r , determined by the matrix M . We also have $\phi_M(c) = c^\epsilon$ where $\epsilon \in \{1, -1\}$. Rearranging into the normal form then results in an adjustment to the power of c consisting of a sum of terms of the form $i_\lambda j_\lambda$ with coefficients determined by M . Thus, the image of $z_1^{l_1} \cdots z_s^{l_s} a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r} c^k$ under ϕ_M may be expressed as:

$$\left(\overline{z_1^{l_1} \cdots z_s^{l_s} a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r} M} \right)^\wedge c^{\epsilon k + q(l_1, \dots, j_r)} = ((l_1, \dots, j_r)M)^\wedge c^{\epsilon k + q(l_1, \dots, j_r)}$$

where q is a degree 2 polynomial.

Letting $x = z_1^{l_1} \cdots z_s^{l_s} a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r} c^k$, we have

$$\begin{aligned} f(x) &= \kappa \circ \phi_M(z_1^{l_1} \cdots z_s^{l_s} a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r} c^k) \\ &= \kappa \left(((l_1, \dots, l_s, i_1, j_1, \dots, i_r, j_r)M)^\wedge c^{\epsilon k + q(l_1, \dots, j_r)} \right) \\ &= ((l_1, \dots, l_s, i_1, j_1, \dots, i_r, j_r)M)^\wedge c^{\epsilon k + \gamma(l_1, \dots, j_r)} \end{aligned}$$

where $\gamma(l_1, \dots, j_r) = q(l_1, \dots, j_r) + p((l_1, \dots, j_r)M)$ is a degree 2 polynomial with coefficients determined by f . □

4. Conjugacy growth of higher Heisenberg groups

This section is dedicated to proving Theorem 4.5. First, we show that the parameter D does not affect the conjugacy growth of the group H_D .

Proposition 4.1. *Let $D = (\delta_1, \delta_2, \dots, \delta_{r-1})$ with each $\delta_i \mid \delta_{i+1}$. Then the conjugacy growth of H_D is equivalent to that of H_r , that is, $c_{H_D} \sim c_{H_r}$.*

Proof. We will show that there exist groups Γ_1 and Γ_2 , both isomorphic to H_r , such that $\Gamma_1 \leq H_D \leq \Gamma_2$, with $[H_D : \Gamma_1]$ and $[\Gamma_2 : H_D]$ both finite. Then, Lemma 2.8 will give the result.

From the definition, H_D is generated by elements $a_1, b_2, \dots, a_r, b_r$ with $[a_i, b_i] = c_i = c^{\delta_i-1}$ for each i (and $[a_1, b_1] = c$). Let $\gamma_i = \frac{\delta_{r-1}}{\delta_{i-1}}$ for $i > 1$, and $\gamma_1 = \delta_{r-1}$, and define the subgroup:

$$\Gamma_1 = \langle a_1^{\gamma_1}, b_1, a_2^{\gamma_2}, b_2, \dots, a_r^{\gamma_r}, b_r \rangle \leq H_D.$$

For any i , we have $[a_i^{\gamma_i}, b_i] = c_i^{\gamma_i} = c^{\delta_{i-1}\gamma_i} = c^{\delta_{r-1}}$, and so Γ_1 is isomorphic to H_r . For an element $a_1^{i_1} b_1^{j_1} a_2^{i_2} b_2^{j_2} \dots a_r^{i_r} b_r^{j_r} c^k \in H_D$, we have

$$a_1^{i_1} b_1^{j_1} a_2^{i_2} b_2^{j_2} \dots a_r^{i_r} b_r^{j_r} c^k \in a_1^{m_1} a_2^{m_2} \dots a_r^{m_r} c^n \Gamma_1$$

where each $m_\lambda = i_\lambda \bmod \gamma_\lambda$, and $n = k \bmod \delta_{r-1}$. Therefore, the following is a set of representatives for the cosets H_D / Γ_1 :

$$\{a_1^{m_1} a_2^{m_2} \dots a_r^{m_r} c^k \mid 0 \leq m_\lambda < \gamma_\lambda, 0 \leq k < \delta_{r-1}\},$$

and so $[H_D : \Gamma_1] = \gamma_1 \gamma_2 \dots \gamma_r \delta_{r-1} < \infty$ as required.

Now let $\Gamma_2 \cong H_r$ be generated by $\{d_1, e_1, d_2, e_2, \dots, d_r, e_r\}$ and denote the commutator by $f = [d_i, e_i]$. Define a map $\phi : H_D \rightarrow \Gamma_2$ by its action on the generators:

$$\begin{aligned} a_i &\mapsto d_i^{\delta_i-1} \\ b_i &\mapsto e_i. \end{aligned}$$

It is easily checked that the relators of H_D are sent to the identity by ϕ , and thus it is a well-defined homomorphism. Furthermore, if $\phi(a_1^{i_1} b_1^{j_1} a_2^{i_2} b_2^{j_2} \dots a_r^{i_r} b_r^{j_r} c^k) = 1$, then $d_1^{i_1} e_1^{j_1} d_2^{i_2} e_2^{j_2} \dots d_r^{i_r} e_r^{j_r} f^k = 1$ and so $I_1 = J_1 = \dots = J_r = K = 0$. Thus, ϕ is a monomorphism. Similarly to the previous argument, a set of representatives for the cosets $\Gamma_2 / \phi(H_D)$ is

$$\{d_1^{n_1} \dots d_r^{n_r} \mid 0 \leq n_\lambda < \delta_{\lambda-1}\},$$

and so $[\Gamma_2 : \phi(H_D)] = \delta_1 \delta_2 \dots \delta_{r-1}$. □

Definition 4.2. If $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is any tuple of integers, we will write

$$g(\mathbf{x}) = \gcd(x_1, x_2, \dots, x_n)$$

for the greatest common divisor of the entries of \mathbf{x} .

The next two results describe the structure of the conjugacy classes of H_r .

Lemma 4.3. Let αc^k be an element of H_r in Mal'cev normal form so that $\alpha = a_1^{i_1} b_1^{j_1} a_2^{i_2} b_2^{j_2} \dots a_r^{i_r} b_r^{j_r}$ and $\alpha c^k = \bar{\alpha} \in \mathbb{Z}^{2r}$. Then the conjugacy class represented by $\alpha c^k \in H_r$ is either a singleton or a coset of a cyclic subgroup:

$$[\alpha c^k] = \begin{cases} \{c^k\} & \text{if } \alpha \text{ is the identity} \\ \alpha c^k \langle c^{g(\bar{\alpha})} \rangle & \text{otherwise.} \end{cases} \tag{4.1}$$

Proof. First note that c^k is central, so its conjugacy class is $[c^k] = \{c^k\}$. Now consider non-identity α . Since $[a_t, b_t] = c$ for each $1 \leq t \leq r$, we have

$$a_t \alpha c^k a_t^{-1} = \alpha c^{k+i_t} \text{ and } b_t \alpha c^k b_t^{-1} = \alpha c^{k-j_t}.$$

So we can express the conjugacy class of αc^k as follows:

$$\begin{aligned} [\alpha c^k] &= \{ \alpha c^{k + \sum_{i=1}^r (l_{1i} j_i - l_{2i} i_i)} \mid l_{1i}, l_{2i} \in \mathbb{Z} \} \\ &= \alpha c^k \langle c^{\gcd(j_1, -i_1, j_2, -i_2, \dots, j_r, -i_r)} \rangle \\ &= \alpha c^k \langle c^{g(\bar{\alpha})} \rangle. \end{aligned}$$

□

Lemma 4.4. *Let $\alpha = a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r} \in H_r$ be a non-trivial element in Mal'cev normal form. Then the length of the conjugacy class $[\alpha c^k]$ lies in the range $[|\alpha|, |\alpha| + 2]$, where $|\alpha| = \sum_{\lambda} (|i_{\lambda}| + |j_{\lambda}|)$ is the word length of α .*

Proof. We claim that any element αc^k has length at least $|\alpha|$, so $|\alpha| \leq |[\alpha c^k]|$. To see this, consider any word over the generators $a_1, b_1, \dots, a_r, b_r$ that represents αc^k . We can put this into normal form by collecting the powers of generators into the given order, at the cost of powers of c , using the identity $[a_t, b_t] = c$ for each t . Note that the exponent sum of each generator can never increase. So any word representing αc^k has at least $|i_t|$ instances of $a_t^{\pm 1}$, and so on.

From the structure of conjugacy classes given in Lemma 4.3, each conjugacy class $[\alpha c^k]$ has a representative of the form αc^{-l} where $0 \leq l < g(\bar{\alpha})$ and a representative of the form αc^m where $0 < m \leq g(\bar{\alpha})$.

Assume that each power i_t, j_t is non-negative. Let $I = \{m \in \mathbb{N} \mid i_m \neq 0\}$ and $J = \{n \in \mathbb{N} \mid j_n \neq 0\}$. Then, by definition, $g(\bar{\alpha}) \leq \min\{i_m, j_n \mid m \in I, n \in J\}$.

Suppose that there is some $t \in I \cap J$, that is, both i_t and j_t are non-zero. We have

$$a_t^{i_t-1} b_t a_t^i b_t^{j_t-1} =_G a_t^i [a_t^{-1}, b_t] b_t^{j_t} =_G a_t^i b_t^{j_t} c^{-1}$$

and so

$$a_1^{i_1} b_1^{j_1} \cdots a_t^{i_t-1} b_t a_t^i b_t^{j_t-1} \cdots a_r^{i_r} b_r^{j_r} =_G \alpha c^{-1}, \tag{4.2}$$

and thus we can represent the element αc^{-1} with a word of length $|\alpha|$.

Now suppose that $I \cap J$ is empty. Since $\alpha \neq 1$, there exists $t \in \mathbb{N}$ with either $i_t \neq 0$ or $j_t \neq 0$. If $i_t \neq 0$, we have

$$a_t^{i_t-1} b_t a_t^i b_t^{-1} = a_t^i [a_t^{-1}, b_t] =_G a_t^i c^{-1} \tag{4.3}$$

and if $j_t \neq 0$ we have

$$a_t^{-1} b_t^j a_t b_t^{j_t-1} = [a_t^{-1}, b_t^j] b_t^{j_t-1} =_G b_t^j c^{-1}. \tag{4.4}$$

Now, similarly to equation (4.2), we can represent the element αc^{-1} with a word of length $|\alpha| + 2$, since in equation (4.3) we have inserted an extra b_t and b_t^{-1} , and in equation (4.4) we have inserted an extra a_t and a_t^{-1} . Therefore, in general, we have the bound $|\alpha c^{-l}| \leq |\alpha| + 2$. This in turn implies that $|\alpha| \leq |[\alpha c^k]| = |[\alpha c^{-l}]| \leq |\alpha| + 2$.

If some powers i_t, j_t are negative, we can find words analogous to equation (4.2) that represent either αc^{-1} or αc^m , depending on the combination of signs. Thus, the result holds for all values of α . \square

Theorem 4.5. *Let $G = \Gamma \times H_D$ be a finitely generated class 2 nilpotent group with infinite cyclic derived subgroup, with Heisenberg rank r . Let s be the torsion free rank of Γ . Then,*

$$c_G(n) \sim \begin{cases} n^{s+2} \log n & r = 1 \\ n^{s+2r} & r \geq 2. \end{cases}$$

Proof. By Lemma 2.9 and Theorem 2.11, we have $c_G(n) \sim n^s \cdot c_{H_D}(n) \sim n^s \cdot c_{H_r}(n)$ and so it suffices to show that the conjugacy growth of H_r is equivalent to $n^2 \log n$ in the case $r = 1$ and n^{2r} otherwise. This follows from [2] but we provide a new argument here using more elementary methods.

For a fixed non-identity $\alpha = a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r} \in H_r$, Lemma 4.3 implies that there are exactly $g(\bar{\alpha})$ conjugacy classes in the coset $\alpha \langle c \rangle$, and Lemma 4.4 implies that they have length in the range $[|\alpha|, |\alpha| + 2]$. The conjugacy classes of elements where α is the identity are simply the elements of $\langle c \rangle$. Thus, the conjugacy growth function satisfies the bounds:

$$\beta_{\langle c \rangle}(n) + \sum_{\bar{\alpha} \in B_{\ell_1}(n-2)} g(\bar{\alpha}) \leq c_{H_r}(n) \leq \beta_{\langle c \rangle}(n) + \sum_{\bar{\alpha} \in B_{\ell_1}(n)} g(\bar{\alpha}),$$

where $B_{\ell_1}(n)$ denotes the n -ball in $\mathbb{Z}^{2r} \cong \text{Ab}(H_r)$ with respect to the ℓ_1 norm. It is standard (and not hard to see) that $\beta_{(c)}(n) \sim n^2$ for any H_r , noting for example that $[a_i^n, b_i^n]$ is a geodesic spelling of c^{n^2} , with length $4n$. Thus, from Corollary 2.26 we have

$$c_{H_r}(n) \sim \begin{cases} n^2 \log n & r = 1 \\ n^{2r} & r \geq 2 \end{cases}$$

which finishes the proof. □

Remark 4.6. Comparing Theorem 4.5 to Corollary 4.2 of [2], we see that Babenko proves a stronger result, covering a more general class of metrics and providing the leading coefficient of the growth function, but in a more restricted class of groups. In the case of $r \geq 2$, we have

$$c_{H_r}(n) = \frac{\zeta(2r - 1)}{\zeta(2r)} R_r n^{2r} + o(n^{2r})$$

where R_r is a rational number depending on the metric, and ζ is the Riemann zeta function. Although it would contradict Conjecture 1.1, if the conjugacy growth series of H_r turns out to be rational (respectively algebraic), then Proposition 2.15 would imply that $\frac{\zeta(2r-1)}{\zeta(2r)}$ is a rational (respectively algebraic) number. As far as the author is aware, it is not known whether such a fraction is algebraic, although it seems unlikely.

5. Conjugacy growth of virtually higher Heisenberg groups

In this section, we show that if G is commensurable to a higher Heisenberg group then they have equivalent conjugacy growth functions. We will need the following lemma.

Lemma 5.1. *If U is a subgroup of H_1 , then the Heisenberg rank of U is at most 1.*

Proof. First, we claim that a pair of elements of H_1 commute if and only if their images in $\text{Ab}(H_1)$ are colinear as vectors in \mathbb{Z}^2 . To see this, let $g = a^i b^j c^k$ and $h = a^l b^l c^k$ be elements of H_1 . We have $gh = a^{i+l} b^{j+l} c^{k+k-jl}$ and $hg = a^{l+i} b^{l+j} c^{k+k-Ji}$ and so they commute if and only if $jl = Ji$. This is equivalent to the vectors $\bar{g} = (i, j)$ and $\bar{h} = (l, J)$ being colinear (including the possibility that one or both is the zero vector).

To prove the Lemma, suppose, on the contrary, that U has Heisenberg rank at least 2. This implies that there exist four elements $x_1, y_1, x_2, y_2 \in U \leq H_1$ such that $[x_1, y_1] \neq 1, [x_2, y_2] \neq 1$, and

$$[x_1, x_2] = [x_1, y_2] = [y_1, x_2] = [y_1, y_2] = 1. \tag{5.1}$$

Now the claim above along with (5.1) implies that \bar{x}_1 and \bar{x}_2 are colinear, and \bar{y}_1 and \bar{y}_2 are colinear. Since being colinear is a transitive relation, \bar{x}_1 and \bar{y}_1 are colinear, and hence x_1 and y_1 commute, which is a contradiction. □

Next, we show that conjugacy growth is preserved when passing to finite index subgroups.

Lemma 5.2. *Let G be a class 2 nilpotent group with infinite cyclic derived subgroup. If H is a finite index subgroup of G , then H is also class 2 nilpotent with infinite cyclic derived subgroup. Furthermore, G and H have equivalent conjugacy growth functions.*

Proof. Since H is a subgroup of G it is nilpotent of class at most 2 and since it has finite index, $[H, H]$ has finite index in $[G, G]$ and is therefore infinite cyclic. We have $G = \Gamma_1 \times H_{D_1}$ and $H = \Gamma_2 \times H_{D_2}$, where Γ_1 and Γ_2 are abelian and the groups H_{D_1} and H_{D_2} are as in Definition 3.6. Let r_1 and r_2 denote the corresponding Heisenberg ranks, and let s_i denote the torsion-free rank of Γ_i .

By Proposition 2.5, G and H have equivalent standard growth functions. Therefore by Theorem 2.11, we have $s_1 + 2r_1 + 2 = s_2 + 2r_2 + 2$, that is,

$$s_1 + 2r_1 = s_2 + 2r_2. \tag{5.2}$$

This also follows from Theorem 5.9. If $r_1, r_2 \geq 2$, then Theorem 4.5 implies that $c_G(n) \sim n^{s_1+2r_1}$ and $c_H(n) \sim n^{s_2+2r_2}$ and hence $c_G \sim c_H$ by (5.2). If $r_1 = r_2 = 1$, then $s_1 = s_2$ and Theorem 4.5 again implies that $c_G \sim c_H$.

Suppose $r_1 > 1$ and $r_2 = 1$. Applying Theorem 4.5, we have $c_G(n) \sim n^{s_1+2r_1}$ and $c_H(n) \sim n^{s_2+2} \log n$. From (5.2), $n^{s_1+2r_1} = n^{s_2+2}$, and then $c_G(n) \sim n^{s_2+2}$ and $c_H(n) \sim n^{s_2+2} \log n$ together violate Lemma 2.8.

Finally, suppose that $r_1 = 1$ and $r_2 > 1$. So there exist four elements $x_1, y_1, x_2, y_2 \in H < G = \Gamma_1 \times H_1$ such that $\langle x_1, y_1, x_2, y_2 \rangle \leq G$ has Heisenberg rank 2. Since G is a direct product and Γ_1 is abelian, two elements of G commute if and only if their H_1 components commute. So by passing to just the H_1 components of each element, we find elements x'_1, y'_1, x'_2, y'_2 contained in the H_1 factor of G , that generate a subgroup of Heisenberg rank 2. This contradicts Lemma 5.1. □

Now we show that conjugacy growth is also preserved when passing to finite index supergroups.

Theorem 5.3. *Suppose H is a class 2 nilpotent group with infinite cyclic derived subgroup. If a group G contains H as a finite index subgroup, then G and H have equivalent conjugacy growth functions.*

Before we prove this Theorem, we note the following consequence.

Corollary 5.4. *If G is (virtually) nilpotent with Heisenberg rank equal to 1, then its conjugacy growth series is non-holonomic, with respect to any finite generating set.*

Proof. Theorems 5.3 and 4.5 show that the conjugacy growth of such a group is equivalent to $n^d \log n$ for some positive integer d . But by Theorem 2.16, no such function can have a holonomic power series. □

To prove Theorem 5.3, we will use a general necessary and sufficient condition for conjugacy growth to be preserved under finite extensions. First, we define twisted conjugacy growth.

Definition 5.5. *Let ϕ be a fixed automorphism of a group G . Define the ϕ -twisted conjugacy class of $g \in G$ by*

$$[g]^\phi = \{\phi(h)gh^{-1} \mid h \in G\}$$

and write C_G^ϕ for the set of ϕ -twisted conjugacy classes of G (it is easy to check that ϕ -twisted conjugacy defines an equivalence relation). As in Definitions 2.1 and 2.2, the length of a twisted conjugacy class is defined as:

$$|[g]^\phi|_S = \min\{|h|_S \mid h \in [g]^\phi\}$$

and the corresponding ϕ -twisted conjugacy growth function is

$$c_{G,S}^\phi(n) = \#\{\kappa \in C_G^\phi \mid |\kappa|_S \leq n\}.$$

It is not hard to prove that the analogue of Proposition 2.4 holds in this case. In other words, the equivalence class of ϕ -twisted conjugacy growth does not depend on the choice of generating set and we can therefore drop the S from the notation. The following result is reminiscent of Theorem 3.1 of [5] in that conjugacy in an extension is understood via twisted conjugacy of the base group.

Proposition 5.6. *Let H be any finitely generated group. Then $c_H^\phi(n) \asymp c_H(n)$ for every finite-order automorphism $\phi \in \text{Aut}(H)$, if and only if every finite extension G of H satisfies $c_G(n) \sim c_H(n)$.*

Proof. First, suppose that $c_H^\phi(n) \preccurlyeq c_H(n)$ for every finite order $\phi \in \text{Aut}(H)$. For any finite extension G , Lemma 2.8 states that $c_H(n) \preccurlyeq c_G(n)$, so it remains to show the opposite bound. Fix a choice of transversal, $T \subset G$, for G/H . We consider the cosets tH separately, for each $t \in T$ (since the conjugacy class of any element of tH is contained within tH by normality). If a pair of elements of tH are conjugate by an element of H , then they are in the same conjugacy class (in G). Therefore, the number of H -conjugacy classes in the n -ball is at least the number of G -conjugacy classes. Therefore, only considering conjugation by elements of H will give an upper bound for the G -conjugacy growth of tH . Fixing $t \in T$, and choosing an element $h \in H$, we have the H -conjugacy class:

$$[th]_H = \{xthx^{-1} \mid x \in H\} = \{t\phi_t(x)hx^{-1} \mid x \in H\} = t[h]^\phi,$$

where $\phi_t \in \text{Aut}(H)$ is the finite-order automorphism defined by conjugation: $\phi_t: \gamma \mapsto t^{-1}\gamma t$. So we can understand H -conjugation in tH as ϕ_t -twisted conjugation in H itself.

Since $c_H^\phi(n) \preccurlyeq c_H(n)$ by hypothesis, and the growth of tH is equivalent to the growth of H (as per Lemma 2.7), the contribution to the conjugacy growth of G from the coset tH is at most the conjugacy growth of H . Since there are only finitely many such cosets, we have $c_G(n) \preccurlyeq c_H(n)$ as claimed.

For the converse part of the statement, we prove its contrapositive, namely that if there exists some finite-order automorphism giving twisted conjugacy growth strictly greater than untwisted conjugacy growth, then there is a finite extension of H with inequivalent conjugacy growth. Suppose $\phi \in \text{Aut}(H)$ is such an automorphism. So there exists $k \in \mathbb{N}$ such that ϕ^k is the identity automorphism, and we have $c_H^\phi(n) \succ c_H(n)$. Form the semidirect product $G := H \rtimes_\phi \mathbb{Z}/k\mathbb{Z}$, where the finite cyclic group is generated by t , which acts on H via $t^{-1}ht = \phi(h)$ for all $h \in H$. We have $c_H \preccurlyeq c_G$ from Lemma 2.8, and we need to show this is a strict inequality to prove our result. It is enough to do this for one of the k cosets of H in G (which are again closed under conjugation since $H \triangleleft G$). We consider conjugating an element of the coset tH by elements of H . Let $h \in H$. As above, we have

$$[th]_H = \{xthx^{-1} \mid x \in H\} = \{t\phi(x)hx^{-1} \mid x \in H\} = t[h]^\phi$$

and so, again, H -conjugacy classes of the coset tH are t -translates of ϕ -twisted conjugacy classes of H . Therefore, the number of H -conjugacy classes in tH which intersect the n -ball is equivalent to the twisted conjugacy growth function $c_H^\phi(n)$. We have

$$[th]_G = \{gthg^{-1} \mid g \in G\} = \{t^i xthx^{-1} t^{-i} \mid x \in H, 0 \leq i \leq k-1\} = \bigcup_{i=0}^{k-1} t^i [th]_H t^{-i}$$

and so passing from H -conjugacy classes to full conjugacy can only increase the number of conjugacy classes in the n -ball by at most a constant factor of k , which does not change the equivalence class. Thus, the contribution to conjugacy growth from tH is equivalent to $c_H^\phi(n)$, which is strictly greater than $c_H(n)$ by hypothesis, and we have $c_G(n) \succ c_H(n)$ as required. \square

Now we apply this general criterion to the specific case at hand. We will need the following simple observation.

Lemma 5.7. *Let U be a subgroup of \mathbb{Z}^d , and therefore a free abelian group of rank $u \leq d$. Then the n -ball in \mathbb{Z}^d (with respect to the standard generating set of unit vectors) intersects $\mathcal{O}(n^{d-u})$ distinct cosets of U .*

Proof of Theorem 5.3. To apply Proposition 5.6, we need a normal subgroup. Any torsion in H is contained in the abelian factor and is therefore a finite subgroup, so we initially pass to the torsion-free direct factor. The conjugacy growth of a finite group is clearly eventually constant, so Lemma 2.9 shows that removing a finite direct factor preserves (the equivalence class of) the conjugacy growth function. Then, a standard argument allows us to pass to a normal subgroup of G , contained in H , also of finite index. By Lemma 5.2, this subgroup has equivalent conjugacy growth to the original subgroup.

Therefore, we may assume without loss of generality that H is normal and torsion-free. Now to prove the Theorem, it is enough to show that $c_H^\phi(n) \preceq c_H(n)$ for any $\phi \in \text{Aut}(H)$.

We explicitly describe ϕ -twisted conjugacy in H , for any fixed $\phi \in \text{Aut}(H)$. With $h = z_1^{x_1} \dots z_s^{x_s} a_1^{u_1} b_1^{v_1} \dots a_r^{u_r} b_r^{v_r} c^w \in H$ and $x = z_1^{l_1} \dots z_s^{l_s} a_1^{j_1} b_1^{j_1} \dots a_r^{j_r} b_r^{j_r} c^k \in H$, Proposition 3.19 gives

$$\begin{aligned} \phi(x)hx^{-1} &= \phi(z_1^{l_1} \dots b_r^{j_r} c^k) \cdot z_1^{x_1} \dots b_r^{v_r} c^w \cdot z_1^{-l_1} \dots b_r^{-j_r} c^{-k - \sum_\lambda i_\lambda j_\lambda} \\ &= \left((l_1, \dots, j_r)(M - I) + (x_1, \dots, v_r) \right)^\wedge c^{\varepsilon k - k + w - \sum_\lambda i_\lambda j_\lambda + \sum_\lambda (v_\lambda i_\lambda - u_\lambda j_\lambda) + \gamma(l_1, \dots, j_r)} \\ &= \left((l_1, \dots, j_r)(M - I) + (x_1, \dots, v_r) \right)^\wedge c^{\varepsilon k - k + w + f(l_1, \dots, j_r)} \end{aligned} \tag{5.3}$$

where γ is a polynomial of degree 2 (and therefore so is f), $\varepsilon \in \{-1, 1\}$, and $M \in \mathbf{M}$, and these all depend only on ϕ . We analyse various cases depending on the value of ε and the nature of the matrix $M - I$ (which defines an endomorphism of $\text{Ab}(H) \cong \mathbb{Z}^{2r+s}$).

- (1) If ϕ is the identity automorphism, then ϕ -twisted conjugacy is nothing more than standard conjugacy, and so $c_H^\phi(n) = c_H(n)$ in this case. In the remaining cases, we assume that ϕ is not the identity.
- (2) Suppose $\varepsilon = -1$, that is, ϕ inverts the commutator. If $x = c^k$, then $\phi(x)hx^{-1} = hc^{-2k}$. So just considering conjugators of the form c^k for varying k , we can see that each fixed $(x_1, \dots, v_r) \in \text{Ab}(H)$ corresponds to at most two ϕ -twisted conjugacy classes in H , which both have representatives of length $|(x_1, \dots, v_r)|$, as in Lemma 4.4. Considering all conjugators will not increase the number of conjugacy classes of a given length, so we have $c_H^\phi(n) \preceq \beta_{\text{Ab}(H)}(n) \sim n^{2r+s} \preceq c_H(n)$.
- (3) Now let $\varepsilon = 1$, and so equation (5.3) becomes

$$\phi(x)hx^{-1} = \left((l_1, \dots, j_r)(M - I) + (x_1, \dots, v_r) \right)^\wedge c^{w + f(l_1, \dots, j_r)}.$$

We think of (the Cayley graph of) H embedded in \mathbb{R}^{2r+s+1} via Mal'cev coordinates (see Definition 3.10) and see that the projection of a ϕ -twisted conjugacy class $[h]^\phi$ onto the first $2r + s$ coordinates (i.e. the image under the abelianisation map) is a coset of the image of $M - I$. That is,

$$\overline{[h]^\phi} = \bar{h} + \text{Im}(M - I) \subset \mathbb{Z}^{2r+s}.$$

Write $\text{rk}(M - I)$ for the dimension of the image of $M - I$. Lemma 5.7 implies that the number of distinct images of ϕ -twisted conjugacy classes in the abelianisation \mathbb{Z}^{2r+s} that have length at most n is $\mathcal{O}(n^{2r+s - \text{rk}(M - I)})$. Note that if $h \in H$ has length at most n , then its image in the abelianisation will also have length at most n in \mathbb{Z}^{2r+s} . So the number of distinct ϕ -twisted conjugacy classes in the ball of radius n in H is $\mathcal{O}(n^{2r+s - \text{rk}(M - I) + 2})$, with the extra n^2 coming from the growth of the c axis. We consider subcases depending on the dimension of the image of $M - I$.

- (a) For $2 \leq \text{rk}(M - I) \leq 2r + s$, we have

$$c_H^\phi(n) = \mathcal{O}(n^{2r+s - \text{rk}(M - I) + 2}) \preceq n^{2r+s} \preceq c_H(n)$$

as required.

- (b) Consider the case $\text{rk}(M - I) = 1$. Since the kernel of $M - I$ must be non-trivial in this case, we can fix a non-trivial element $\mathbf{b} \in \ker(M - I)$. Then we have

$$\{\bar{h}c^{w+f(\eta\mathbf{b})} \mid \eta \in \mathbb{Z}\} \subset [\bar{h}c^w]^\phi = [h]^\phi.$$

Now $f(\eta\mathbf{b})$ is a quadratic polynomial in the single variable η with zero constant term. Considering just those η in the range $[-n, n]$ yields at least n distinct elements of $\bar{h}\langle c \rangle \cap [h]^\phi$, all with c -coordinate of length $\mathcal{O}(n^2)$ (since $w = \mathcal{O}(n^2)$, f has degree 2, and \mathbf{b} is constant), and hence contained in the n -ball in H (or more precisely the Cn -ball for some constant C). So amongst the $\mathcal{O}(n^2)$ elements of any $\langle c \rangle$ -coset in the n -ball, there can be at most $\mathcal{O}(n)$

distinct ϕ -twisted conjugacy classes. So in this case, the contribution from the c -axis is reduced from n^2 to n , and the number of distinct ϕ -twisted conjugacy classes in the n -ball is reduced from $\mathcal{O}(n^{2r+s-\text{rk}(M-I)+2})$ to $\mathcal{O}(n^{2r+s-\text{rk}(M-I)+1})$. Since the rank is 1 in this case, we have the claimed upper bound:

$$c_H^\phi(n) = \mathcal{O}(n^{2r+s-1+1}) \preceq n^{2r+s} \preceq c_H(n).$$

- (c) The final case is when $\text{rk}(M - I) = 0$, so $M = I$ and $\phi \in \ker \theta$. In this specific case, we can be more explicit about the function f in the calculation at (5.3). We have

$$\begin{aligned} \phi(x)hx^{-1} &= z_1^{-l_1} \cdots b_r^{j_r} c^{k+p(l_1, \dots, j_r)} hc^{-k} b_r^{-j_r} \cdots z_1^{-l_1} \\ &= hc^{p(l_1, \dots, j_r) + x_1 l_1 + \dots + v_r j_r} \\ &= (x_1, \dots, v_r)^\wedge c^{w+(x_1+\lambda_1)l_1+\dots+(v_r+\lambda_{s+2r})j_r} \end{aligned}$$

where, as in the proof of Proposition 3.19, p is a linear function with coefficients λ_i determined by ϕ .

At each point (x_1, \dots, v_r) in the abelianisation, there are $\text{gcd}(x_1 + \lambda_1, \dots, v_r + \lambda_{s+2r})$ ϕ -twisted conjugacy classes, all of length (asymptotically) equal to the length of $z_1^{x_1} \cdots b_r^{v_r}$. Thus, by Corollary 2.27, we have $c_H^\phi(n) \sim c_H(n)$. \square

5.1. A conjecture

As mentioned above, Hull and Osin [24] exhibit a finitely generated group with exponential conjugacy growth function, possessing a finite index subgroup with only two conjugacy classes, and so conjugacy growth fails to be a commensurability invariant in general. On the other hand, the conjugacy growth of a virtually abelian group is equivalent to its standard growth (see Proposition 2.12), and so conjugacy growth is a quasi-isometry invariant in this very restricted context. Theorem 5.3 tells us that conjugacy growth is a commensurability invariant amongst class 2 nilpotent groups with infinite cyclic derived subgroup.

We ask the following natural question.

Question 5.8. *In which other classes of finitely generated groups is conjugacy growth a quasi-isometry invariant?*

In light of Proposition 5.6, a detailed understanding of (finite-order) automorphisms would be valuable in understanding how conjugacy growth behaves under finite extensions. The following Theorem of Pansu gives insight into the nature of quasi-isometry in nilpotent groups.

Theorem 5.9 ([16, 30]). *For a finitely generated nilpotent group G , the numbers r_i , the torsion-free ranks of the quotients $G^{(i+1)}/G^{(i)}$, are quasi-isometry invariants.*

This Theorem suggests that ‘geometrically speaking’, the numbers r_i define the structure of a finitely generated nilpotent group. In light of this, and Theorem 5.3, we venture the following conjecture, which would imply that conjugacy growth was a quasi-isometry invariant amongst (virtually) nilpotent groups.

Conjecture 5.10. *The conjugacy growth of a finitely generated nilpotent group G depends only on the numbers r_i .*

A natural next step to approach this conjecture would be to extend Theorems 4.5 and 5.3 to include groups whose derived subgroup is *virtually* cyclic, which would imply that conjugacy growth is a quasi-isometry invariant amongst this class of nilpotent groups.

Acknowledgments. The author was partially supported by EPSRC grant EP/R035814/1 and would like to thank Alex Bishop, Turbo Ho, Andrei Jaikin and Matthew Tointon for very early discussions on various aspects of this work. Thanks are also due to the anonymous referee for many useful comments and in particular for suggesting the upgrading of Proposition 5.6 from a sufficient condition to an equivalence.

Conflicts of interest. The author declares none.

References

- [1] Y. Antolín and L. Ciobanu, Formal conjugacy growth in acylindrically hyperbolic groups, *Int. Math. Res. Not. IMRN*, (1) (2017), 121–157.
- [2] I. K. Babenko, Closed geodesics, asymptotic volume and the characteristics of growth of groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **52**(4) (1988), 675–711, 895.
- [3] H. Bass, The degree of polynomial growth of finitely generated nilpotent groups, *Proc. London Math. Soc. (3)* **25** (1972), 603–614.
- [4] M. Benson, Growth series of finite extensions of \mathbf{Z}^n are rational, *Invent. Math.* **73**(2) (1983), 251–269.
- [5] O. Bogopolski, A. Martino and E. Ventura, Orbit decidability and the conjugacy problem for some extensions of groups, *Trans. Am. Math. Soc.* **362**(4) (2010), 2003–2036.
- [6] E. Breuillard, Y. Cornuier, A. Lubotzky and C. Meiri, On conjugacy growth of linear groups, *Math. Proc. Cambridge Philos. Soc.* **154**(2) (2013), 261–277.
- [7] E. Breuillard and Y. de Cornulier, On conjugacy growth for solvable groups, *Illinois J. Math.* **54**(1) (2010), 389–395.
- [8] L. Ciobanu, A. Evetts and M. Ho, The conjugacy growth of the soluble Baumslag-Solitar groups, *New York J. Math.* **26** (2020), 473–495.
- [9] L. Ciobanu, S. Hermiller, D. Holt and S. Rees, Conjugacy languages in groups, *Israel J. Math.* **211**(1) (2016), 311–347.
- [10] L. Ciobanu, S. Hermiller and V. Mercier, Formal conjugacy growth in graph products I, ArXiv e-prints (2021). arXiv:2103.04696.
- [11] A. E. Clement, S. Majewicz and M. Zyman, *The theory of nilpotent groups* (Birkhäuser/Springer, Cham, 2017).
- [12] M. Duchin, Counting in groups: Fine asymptotic geometry, *Not. AMS* **63**(8) (2016), 871–874.
- [13] M. Duchin and M. Shapiro, The Heisenberg group is pan-rational, *Adv. Math.* **346** (2019), 219–263.
- [14] A. Evetts, Rational growth in virtually abelian groups, *Illinois J. Math.* **63**(4) (2019), 513–549.
- [15] A. Evetts, Aspects of growth in finitely generated groups, PhD Thesis (Heriot-Watt University, 2020).
- [16] B. Farb and L. Mosher, Problems on the geometry of finitely generated solvable groups, in *Crystallographic groups and their generalizations (Kortrijk, 1999)*, Contemporary Mathematics, vol. 262 (American Mathematical Society, Providence, RI, 2000), 121–134.
- [17] J. L. Fernández and P. Fernández, Asymptotic normality and greatest common divisors. *Int. J. Number Theory* **11**(1) (2015), 89–126.
- [18] P. Flajolet, S. Gerhold and B. Salvy, On the non-holonomic character of logarithms, powers, and the n th prime function, *Electron. J. Combin.* **11**(2) (2004/2006), Article 2, 16.
- [19] I. Gekhtman and W. Yang, Counting conjugacy classes in groups with contracting elements, ArXiv e-prints (2018). arXiv:1810.02969.
- [20] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete mathematics*, 2nd edition (Addison-Wesley Publishing Company, Reading, MA, 1994). A foundation for computer science.
- [21] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.* (53) (1981), 53–73.
- [22] V. Guba and M. Sapir, On the conjugacy growth functions of groups, *Illinois J. Math.* **54**(1) (2010), 301–313.
- [23] M. Hall, Jr., *The theory of groups* (The Macmillan Co., New York, NY, 1959).
- [24] M. Hull and D. Osin, Conjugacy growth of finitely generated groups, *Adv. Math.* **235** (2013), 361–389.
- [25] A. Mann, *How groups grow*, London Mathematical Society Lecture Note Series, vol. 395 (Cambridge University Press, Cambridge, 2012).
- [26] G. A. Margulis, Certain applications of ergodic theory to the investigation of manifolds of negative curvature, *Funkcional. Anal. i Priložen.* **3**(4) (1969), 89–90.
- [27] V. Mercier, Conjugacy growth series of some wreath products, ArXiv e-prints (2016). arXiv:1610.07868.
- [28] D. Osin, Small cancellations over relatively hyperbolic groups and embedding theorems, *Ann. Math. (2)* **172**(1) (2010), 1–39.
- [29] D. V. Osipov, The discrete Heisenberg group and its automorphism group, *Mat. Zametki* **98**(1) (2015), 152–155.
- [30] P. Pansu, Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un, *Ann. Math. (2)* **129**(1) (1989), 1–60.
- [31] I. Rivin, Some properties of the conjugacy class growth function, in *Group theory, statistics, and cryptography*, Contemporary Mathematics, vol. 360 (American Mathematical Society, Providence, RI, 2004), 113–117.

- [32] R. P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49 (Cambridge University Press, Cambridge, 1997). With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.
- [33] M. Stoll, Rational and transcendental growth series for the higher Heisenberg groups, *Invent. Math.* **126**(1) (1996), 85–109.