4
Spin

4.1 Introduction

In our discussion of S-matrix theory in chapter 1, and in the development of Regge theory in chapter 2, we have for simplicity ignored the possibility that the external particles entering or leaving a given process may have intrinsic spin. Only the internal Reggeons have been permitted non-zero angular momentum. Since most hadronic scattering experiments use the spin $= \frac{1}{2}$ nucleon as the target, with beams of spin $= 0$ (π or K), spin $= \frac{1}{2}$ (p, n, $\bar{\text{p}}$, Λ etc.) or spin $= 1(\gamma)$, and since the particles produced in the final state may have any integer or half-integer spin, it is essential to rectify this deficiency before we can confront the predictions of Regge theory with the real world.

There are three important points to bear in mind while doing this. First, an experiment may include in the initial state particles whose spin orientations have been predetermined (polarization experiments), or may involve detection of the spin direction of some of the final-state particles, by secondary scattering or by observing their subsequent decay. So there are further experimental observables (in addition to σ^{tot} and $\text{d}\sigma/\text{d}t$) which show how the scattering probability depends on these spin directions. Secondly, the dependence of the scattering process on the spin vectors means that the Lorentz invariance and crossing properties of the scattering amplitudes will generally be more complicated than those for spinless particles. And finally, and most important for Regge theory, the total angular momentum of a given state, $\boldsymbol{J}$, will no longer be just the orbital angular momentum $\boldsymbol{l}$, as in chapter 2, but the vector sum of $\boldsymbol{l}$ and the spins of the particles, $\boldsymbol{\sigma}_j$, so that for the initial state for example

$$\boldsymbol{J} = \boldsymbol{l} + \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 \tag{4.1.1}$$

and care is needed in making an analytic continuation in J rather than l.

The two most commonly employed methods for discussing spin problems are invariant amplitudes, and centre-of-mass helicity amplitudes.

To obtain the invariant amplitudes each particle of spin σ_j is represented by a wave function $\psi(\sigma_z)$, the spin being quantized along

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a chosen z axis. For spin $= \frac{1}{2}$ particles these wave functions are just the usual four-component Dirac spinors $u(\sigma_z)$, $\sigma_z = \pm\frac{1}{2}$, while for spin $= 1$ we use the polarization vectors $\epsilon_\mu(\sigma_z)$, and higher-spin wave functions can be constructed by taking products of these with suitable Clebsh–Gordan coefficients. The transition amplitude for the scattering process $1+2 \to 3+4$ between these spin states is then written in the form (see for example Barut (1967), Pilkuhn (1967))

$$A_{fi} = \chi_f M_{fi} \chi_i \tag{4.1.2}$$

where the χ's are the spinor wave functions of the particles in the initial and final states ($\chi_i = \psi_1 \otimes \psi_2$, $\chi_f = \psi_3 \otimes \psi_4$), and the M-functions are matrices. Because of Lorentz invariance they may be decomposed in the form

$$M_{fi} = \Sigma_\alpha A_\alpha(s,t) Y_\alpha \tag{4.1.3}$$

where the A_α's are scalar functions of the invariants, and the Y_α's are all the different independent Lorentz invariant matrices which can be constructed from the spin operators (Dirac matrices, polarization vectors etc.) and the momentum vectors of the particles (see Scadron and Jones, 1968, and Cohen-Tannoudji *et al.*, 1968). For example in pseudo-scalar-meson–baryon scattering (spins $0+\frac{1}{2} \to 0+\frac{1}{2}$) it is found that there are only two independent terms in (4.1.3) (paying due regard to TCP invariance and the algebra of Dirac matrices), and in the now conventional notation of Chew *et al.* (1957) one writes

$$M = A(s,t) + B(s,t)\,\tfrac{1}{2}(p_1+p_3)_\mu \gamma^\mu \tag{4.1.4}$$

where p_1 and p_3 are the four-momenta of the pions in the initial and final states respectively, γ_μ is the Dirac matrix, and A, B are the required invariant amplitudes for the process.

This method has the advantage that, if the Y's are suitably chosen, the invariant amplitudes $A_\alpha(s,t)$ are free of kinematical singularities, and so have just the dynamical singularities generated by the unitarity equations. Also they can be crossed directly from one channel to another ($s \to t$ etc.) as the spin rotations etc. involved in going from one channel to another are taken care of by the Y's. So these invariant amplitudes are completely analogous to the spinless particle amplitudes of chapter 1. Their disadvantages are that the determination of a complete independent set of Y_α which satisfy TCP invariance and have no arbitrary zeros (which would introduce compensating kinematical poles in the A_α) is quite difficult for high spins, and their unitarity equations are complicated by the occurrence of spinors in the intermediate states, which necessitates the evaluation of the trace of a matrix product. Also the relation of invariant amplitudes to experimentally observable quantities is somewhat complicated, and, perhaps most serious for us, the angular-momentum decomposition of these amplitudes is non-trivial (see for example Durand (1967), Jones and Scadron (1967), Taylor (1967), for a discussion of covariant Reggeization).

For all these reasons the helicity representation of Jacob and Wick (1959) has become more popular. (A full discussion of helicity amplitudes may be found in Martin and Spearman (1970).)

As described in chapter 1 a helicity state for a particle of four-momentum p and spin σ is denoted by $|p, \sigma, \lambda\rangle$, where the helicity, λ, is the spin component along the direction of motion of the particle ((1.2.4): $\lambda \equiv \boldsymbol{\sigma}.\boldsymbol{p}/|\boldsymbol{p}|$), and has $2\sigma+1$ possible values, $\sigma, \sigma-1, \ldots, -\sigma$. These states are irreducible representations of the Lorentz group, and are invariant under rotations. A state containing two non-interacting particles is described by the direct product

$$|p_1, \sigma_1, \lambda_1\rangle \otimes |p_2, \sigma_2, \lambda_2\rangle \equiv |p_1, \sigma_1, \lambda_1, p_2; \lambda_2, \sigma_2\rangle \tag{4.1.5}$$

We work in the centre-of-mass system where $\boldsymbol{p}_1 = -\boldsymbol{p}_2$, and $s \equiv (p_1+p_2)^2$ is the square of the total energy (see (1.7.5)), and in this system, to avoid possible confusion, we shall denote the helicities by μ (λ will be used subsequently for helicities in the t-channel centre-of-mass system).

Thus for the scattering process $1+2 \to 3+4$, the s-channel centre-of-mass scattering amplitude may be written

$$\langle p_3, \sigma_3, \mu_3; p_4, \sigma_4, \mu_4| A |p_1, \sigma_1, \mu_1; p_2, \sigma_2, \mu_2\rangle$$
$$\equiv \langle \mu_3, \mu_4| A(s,t) |\mu_1 \mu_2\rangle \equiv A_{H_s}(s,t) \tag{4.1.6}$$

where the dependence on the p_i has been expressed in terms of the invariants s and t, as in chapter 1, and the spins σ_i, being internal quantum numbers (like Q, B, I, Y etc.), have been suppressed. For brevity we use

$$H_s \equiv \{\mu_1, \mu_2, \mu_3, \mu_4\} \tag{4.1.7}$$

for the helicities of the particles in the s-channel centre-of-mass system. These amplitudes are Lorentz invariant, except under reversal of the directions of the $\boldsymbol{p}_i$ (see below).

They have the advantage of being immediately applicable for particles of any spin, their unitarity equations are quite simple, requiring just a summation over intermediate-state helicity labels (see section 4.7), and, as we shall find below, they are directly related to experimental observables. Also their angular-momentum decomposition is comparatively easy. This is because for a two-particle state the orbital angular momentum is perpendicular to the direction of relative motion of the particles. So in the centre-of-mass frame the component of the total angular momentum in the direction of motion is just the difference of the helicities, which is fixed. Thus for the initial state $J_z = \mu_1 - \mu_2$ (the minus sign occurring because particle 2 is travelling in the $-z$ direction).

The disadvantage of these helicity amplitudes is that they are not free of kinematical singularities, so we must learn how to extract the necessary kinematical factors before we can write dispersion relations like (1.10.7), integrating just over the dynamical singularities. Also their crossing properties are non-trivial because the directions of motion of the particles are different in the s- and t-channel centre-of-mass systems, and so a given s-channel helicity amplitude crosses into a sum of t-channel amplitudes, and vice versa (see (4.3.7) below).

However, both of these problems have been solved for arbitrary spins, and so helicity amplitudes are now widely used for discussing spin problems and we shall employ them throughout this book. However, invariant amplitudes were invented first, and are still quite often invoked for pseudo-scalar-meson–baryon scattering and photo-production.

In the next section we shall briefly discuss the relation between helicity amplitudes and experimental observables, and then go on to consider their crossing properties. We then repeat the procedures of partial-wave decomposition and analytic continuation in angular momentum which we followed in chapter 2, showing the extra complications which spin introduces into Regge theory. We conclude the chapter with a review of the restrictions which unitarity places on the Regge singularities.

4.2 Helicity amplitudes and observables

For a given scattering process $1+2 \to 3+4$ there are $\prod_{i=1}^{4}(2\sigma_i+1)$ different helicity amplitudes, the different possible combinations of μ_i in (4.1.6). However, not all of these are independent because strong interactions are invariant under parity inversion and time reversal.

Under a parity inversion $((x,y,z) \overset{\mathrm{P}}{\to} (-x, -y, -z))$ the momentum

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vector $\boldsymbol{p} \overset{\mathrm{P}}{\rightarrow} -\boldsymbol{p}$, but since the spin, vector $\boldsymbol{\sigma}$ is an axial vector (i.e. transforms like a vector product $\boldsymbol{r} \times \boldsymbol{p} \overset{\mathrm{P}}{\rightarrow} (-\boldsymbol{r}) \times (-\boldsymbol{p}) = \boldsymbol{r} \times \boldsymbol{p}$) $\boldsymbol{\sigma} \overset{\mathrm{P}}{\rightarrow} \boldsymbol{\sigma}$. Hence the sign of the helicity (1.2.4) is reversed under a parity transformation, i.e. $\mu \overset{\mathrm{P}}{\rightarrow} -\mu$. Since the scattering process is invariant under P we have

$$\langle \mu_3, \mu_4 | A | \mu_1, \mu_2 \rangle = \eta \langle -\mu_3, -\mu_4 | A | -\mu_1, -\mu_2 \rangle \qquad (4.2.1)$$

where η is a phase factor $(= \pm 1)$. The phase convention usually adopted for helicity amplitudes, following Jacob and Wick (1959), is obtained by representing the parity inversion operator, P, as a reflection in the x–z plane, Y, followed by a rotation by π about the y axis. Also by convention the particle is travelling along the $\pm z$ axis, so for example

$$\mathrm{P}|p_1, \sigma_1, \mu_1\rangle = \mathrm{e}^{\mathrm{i}\pi J_y} Y |p_1, \sigma_1, \mu_1\rangle = P_1(-1)^{\sigma_1 - \mu_1} \mathrm{e}^{\mathrm{i}\pi J_y} |p_1, \sigma_1, -\mu_1\rangle \qquad (4.2.2)$$

where P_1 is the intrinsic parity of the particle, and the factor $(-1)^{\sigma_1 - \mu_1}$ appears because the reflection is achieved by the rotation matrix, $d^{\sigma}_{\mu', \mu}(\pi) = (-1)^{\sigma - \mu} \delta_{\mu' - \mu}$, from (B.7) and (B.8). Since the scattering plane is taken to be the x–z plane ($\phi = 0$) the phase factor in (4.2.1) is, remembering that 2 is travelling in the opposite direction to 1, etc.,

$$\eta = P_1 P_2 P_3 P_4 (-1)^{\sigma_1 - \mu_1 + \sigma_2 + \mu_2 - \sigma_3 + \mu_3 - \sigma_4 + \mu_4} \qquad (4.2.3)$$

(see Martin and Spearman (1970) p. 227).

Similarly time-reversal invariance implies that the amplitudes for $1 + 2 \rightarrow 3 + 4$ must equal those for $3 + 4 \rightarrow 1 + 2$, again apart from a phase factor, and with this convention

$$\langle \mu_3 \mu_4 | A | \mu_1 \mu_2 \rangle = (-1)^{\mu_3 - \mu_4 - \mu_1 + \mu_2} \langle \mu_1 \mu_2 | A | \mu_3 \mu_4 \rangle \qquad (4.2.4)$$

(Martin and Spearman (1970) p. 232).

These relations greatly reduce the number of amplitudes which we have to consider. Thus for a process with spins $0 + \frac{1}{2} \rightarrow 0 + \frac{1}{2}$, of the 4 possible helicity amplitudes only 2 are independent, while for $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{1}{2}$ only 6 of the 16 possible amplitudes are independent. Further restrictions may follow in some cases from the identity of the particles (depending on whether they obey Fermi or Bose statistics).

In general in a scattering experiment it is impossible to determine completely the spin orientations of all the particles. This means that one is not able to deal with pure helicity states in which each particle has a well defined spin projection, but must consider mixed states (statistical ensembles) which are incoherent sums of the different helicity states, occurring with various probabilities (see for example Schiff (1968) p. 378).

The simplest experiment is one in which no attempt is made to determine any of the spin directions, so that all the $2\sigma_i+1$ helicity states for each particle are equally probable. In this case we simply have to average over all the possible helicity states which could occur in the initial state, and sum over all those which may occur in the final state, so instead of (1.8.16) the unpolarized differential cross-section in terms of the amplitudes (4.1.6) is

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s q_{s12}^2} \frac{1}{(2\sigma_1+1)(2\sigma_2+1)} \sum_{H_s} |A_{H_s}(s,t)|^2 \qquad (4.2.5)$$

where the sum over H_s is over all the $2\sigma_i+1$ values of each μ_i ($i = 1, \ldots, 4$). Similarly the total cross-section, $1+2 \rightarrow$ all, for scattering from an initially unpolarized state is related via the optical theorem (1.9.6) to the forward elastic scattering amplitudes $1+2 \rightarrow 1+2$ by

$$\sigma_{12}^{\text{tot}} = \frac{1}{2q_{s12}\sqrt{s}} \frac{1}{(2\sigma_1+1)(2\sigma_2+1)} \sum_{\mu_1\mu_2} \operatorname{Im}\{\langle\mu_1\mu_2| A^{\text{el}}(s,0)|\mu_1\mu_2\rangle\} \qquad (4.2.6)$$

It is possible to obtain information about the spin dependence of the scattering process by doing experiments with polarized particles, that is to say particles for which the average spin projection in some chosen direction is different from zero. This can be achieved for example by a polarization experiment in which the target proton is placed in a strong magnetic field along a chosen y axis at very low temperatures giving, say, a more than 50% probability that $\sigma_y = +\frac{1}{2}$ rather than $-\frac{1}{2}$. Or, if one of the final-state particles is unstable we can determine the average spin orientation of that particle from the angular distribution of its decay products.

We describe such a mixed-spin state for a given particle, i, by a spin density matrix, $\rho_{mm'}$, a $(2\sigma_i+1)$ by $(2\sigma_i+1)$ Hermitian matrix of unit trace, such that the expectation value (or average value) of some spin-dependent observable, O, in this state is given by

$$\langle O\rangle = \operatorname{tr}(O\boldsymbol{\rho}) \qquad (4.2.7)$$

(tr = trace). Thus suppose we observe the angular distribution (θ, ϕ) of the two-body decay of one of the final-state particles (4 say), so that the full process is $1+2 \to 3+4$, $4 \to a+b$. Then the scattering amplitude will take the form (Jackson 1965)

$$\sum_m A_{\mu_1\mu_2\mu_3 m} A(m \to ab;\, \theta, \phi) \tag{4.2.8}$$

where $A_{\mu_1\mu_2\mu_3 m}$ is the probability amplitude for producing particle 4 with helicity $\mu_4 = m$, and $A(m \to ab;\, \theta, \phi)$ is the probability amplitude for the decay of 4 from this helicity state into $a+b$, with particle a travelling in the direction specified by the polar angles θ, ϕ relative to the direction of motion of particle 4. (These angles are measured in the rest frame of particle 4.) So the production angular distribution for this process will be

$$W(\theta, \phi) \propto \sum_{\mu_1\mu_2\mu_3} \left|\sum_m A_{\mu_1\mu_2\mu_3 m} A(m \to ab;\, \theta, \phi)\right|^2 \tag{4.2.9}$$

Hence if we define the production spin density matrix for particle 4 by

$$\rho_{mm'} \equiv \frac{\sum\limits_{\mu_1\mu_2\mu_3} A_{\mu_1\mu_2\mu_3 m} A^*_{\mu_1\mu_2\mu_3 m'}}{\sum\limits_{\mu_1\mu_2\mu_3\mu_4} |A_{\mu_1\mu_2\mu_3\mu_4}|^2} \tag{4.2.10}$$

which is normalized so that $\mathrm{tr}(\boldsymbol{\rho}) = 1$, and define the decay density matrix by

$$R_{mm'} = A(m \to ab)\, A^*(m' \to ab) \tag{4.2.11}$$

then the angular distribution (4.2.9) will be given by

$$W(\theta, \phi) = \mathrm{tr}(\boldsymbol{\rho}\mathbf{R}^*) \tag{4.2.12}$$

Thus if we know $\mathbf{R}$, $\boldsymbol{\rho}$ can be determined directly from $W(\theta, \phi)$ and this gives further information about the A_{H_s} in addition to (4.2.5).

To obtain R we let $\boldsymbol{q}$ and $-\boldsymbol{q}$ be the momenta of a and b, respectively, in the rest frame of particle 4, and $\hat{\boldsymbol{q}}$ a unit vector in the direction of $\boldsymbol{q}$. The final state after the decay is then $|\hat{\boldsymbol{q}}, \mu_a, \mu_b\rangle$. For a parity conserving decay the decay amplitude takes the form (when suitably normalized)

$$A(m \to \mu_a \mu_b) = \left(\frac{2\sigma_4 + 1}{4\pi}\right)^{\frac{1}{2}} \mathscr{D}^{\sigma_4 *}_{m\mu}(\phi, \theta, 0) \tag{4.2.13}$$

where $\mathscr{D}$ is the rotation matrix (B.3) corresponding to the rotation of a system having angular momentum σ_4 from the direction of motion of particle 4 (in which m is its spin projection) to the direction $\hat{\boldsymbol{q}}$ (in which $\mu \equiv \mu_a - \mu_b$ is its spin projection) θ is the angle between $\hat{\boldsymbol{q}}$ and $\boldsymbol{p}_4$, and ϕ the azimuthal angle about $\hat{\boldsymbol{q}}$. Using the representation (B.4)

$$\mathscr{D}^{\sigma_4}_{m\mu}(\phi, \theta, 0) = e^{im\phi} d^{\sigma_4}_{m\mu}(\theta) \tag{4.2.14}$$

and summing over the helicities, μ_a, μ_b, we find that the normalized angular distribution is

$$W(\theta,\phi) = \frac{2\sigma_4+1}{4\pi} \sum_{mm'} \sum_{\mu_a\mu_b} \rho_{mm'}\, e^{i(m-m')\phi}\, d^{\sigma_4}_{m\mu}(\theta)\, d^{\sigma_4}_{m'\mu}(\theta) \quad (4.2.15)$$

Thus for the decay of a spin = 1 particle into two spin = 0 particles (e.g. $\rho \to \pi\pi$) we have

$$W_1(\theta,\phi) = \frac{3}{4\pi}[\cos^2\theta\,\rho_{00} + \tfrac{1}{2}\sin^2\theta\,(\rho_{11}-\rho_{-1-1}) - \sin^2\theta\,\mathrm{Re}\{\rho_{1-1}e^{2i\phi}\}$$
$$-\frac{1}{\sqrt{2}}\sin 2\theta\,\mathrm{Re}\{\rho_{10}\,e^{i\phi} - \rho_{-1,0}\,e^{-i\phi}\}] \quad (4.2.16)$$

It is then quite easy to take suitable moments of the observed experimental distributions to invert (4.2.16) to give the ρ's directly, e.g.

$$\rho_{00} = \tfrac{1}{2}\int d\Omega\,(5\cos^2\theta - 1)\,W_1(\theta,\phi)$$
$$\rho_{11}+\rho_{-1-1} = \tfrac{1}{2}\int d\Omega\,(3 - 5\cos^2\theta)\,W_1(\theta,\phi) \quad (4.2.17)$$

Similar, but slightly more complicated expressions are obtained for parity-violating weak decays such as $\Lambda \to p\pi^-$ since the decay amplitude corresponding to (4.1.23) will then involve two terms, one even under parity reflections and the other odd (see Jackson 1965).

Because of the parity relation (4.2.1) not all the production density matrix elements are independent, but

$$\rho_{-m-m'} = (-1)^{m-m'}\rho_{mm'} \quad (4.2.18)$$

Also the Hermitian nature of the density matrix implies that ρ_{mm} is real, which, together with the normalization condition that $\mathrm{tr}(\rho) = \sum_m \rho_{mm} = 1$, leaves only the following independent real observables

$$\left.\begin{array}{ll} \rho_{mm} & 0 \leqslant m \leqslant \sigma_4 \\ \mathrm{Re}\{\rho_{mm'}\} & |m'| < m \leqslant \sigma_4 \\ \rho_{m-m} & \text{for (integral } \sigma_4) \end{array}\right\} \quad (4.2.19)$$

If both the final-state particles decay there are similar joint production density matrices

$$\rho^{mm'}_{nn'} \equiv \frac{\sum_{\mu_1\mu_2} A_{\mu_1\mu_2 mn}\, A_{\mu_1\mu_2 m'n'}}{\sum_{\mu_1\mu_2\mu_3\mu_4} |A_{\mu_1\mu_2\mu_3\mu_4}|^2} \quad (4.2.20)$$

which can be obtained from the joint decay distribution

$$W(\theta_3\phi_3;\, \theta_4\phi_4).$$

For spin $= \frac{1}{2}$ particles it is more usual to re-express the density matrix in terms of the polarization vector $\boldsymbol{P}$ defined by

$$\rho_{mm'}(\tfrac{1}{2}) = \tfrac{1}{2}(1+\boldsymbol{P}.\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix}1+P_z & P_x - \mathrm{i}P_y \\ P_x+\mathrm{i}P_y & 1-P_z\end{pmatrix} \qquad (4.2.21)$$

where $\boldsymbol{\sigma}$ is the Pauli matrix, and where as usual the z axis is along the direction of motion, and **y** is perpendicular to the production plane. Parity conservation (4.2.18) requires $P_x = P_z = 0$. Thus for example for $\pi + \mathrm{p} \rightarrow \pi + \mathrm{p}$, with a polarized proton target,

$$P_y = \langle \sigma_y \rangle = \mathrm{tr}\,(\rho\sigma_y) = -2\,\mathrm{Im}\{\rho_{\frac{1}{2}-\frac{1}{2}}\} = -2\frac{A_{++}A^*_{+-}}{|A_{++}|^2+|A_{+-}|^2} \qquad (4.2.22)$$

where $\pm \equiv \pm\frac{1}{2}$ for the nucleon helicities, and the pion helicity label $(=0)$ is omitted. This can be determined directly from the left–right asymmetry of the scattering cross-section about the y–z plane.

4.3 Crossing of helicity amplitudes

To discuss the Regge pole exchange contributions to a scattering process it is necessary to be able to cross from the t-channel centre-of-mass scattering amplitude $A^t(s,t)$, for the process $1+\overline{3}\rightarrow\overline{2}+4$ in which the Reggeon appears as a physical particle, to the s-channel centre-of-mass amplitude $A^s(s,t)$, which describes the process $1+2\rightarrow 3+4$. For spinless-particle scattering the crossing relation is simply

$$A^s(s,t) = A^t(s,t) \qquad (4.3.1)$$

from the crossing postulate (section 1.6).

However, for helicity amplitudes things are not quite so simple because the helicities are defined in terms of the spin projections in the directions of motion of the various particles, so if we change the directions of motion the helicities will change too. Moreover, we have to make not just a physical Lorentz transformation, but a complex Lorentz transformation in which we pass from the values of the momenta appropriate for a physical process in the t channel, to those appropriate for the s channel, where the four-momenta of particles 2 and 3 are reversed. Thus great care is needed in following the path of continuation of the kinematical factors involved in the Lorentz transformation. However, it can be shown (Trueman and Wick 1964) that with a suitable choice of path the helicities are unchanged by

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crossing so that (apart from a possible phase factor)

$$\langle\lambda_3\lambda_4|\,A^s(s,t)\,|\lambda_1\lambda_2\rangle = \langle\lambda_2\lambda_4|\,A^t(s,t)\,|\lambda_1\lambda_3\rangle \qquad (4.3.2)$$

where the λ's are t-channel centre-of-mass-frame helicities (i.e. the spin projections of the particles in their directions of motion in that frame). It is then necessary to re-express (4.3.2) in terms of s-channel helicities, and to achieve this we use the fact that under a general Lorentz transformation a helicity state is transformed as

$$|\mathrm{p},\sigma,\lambda\rangle \to \sum_{\lambda'} \mathscr{D}^{\sigma}_{\lambda'\lambda}(R)\,|p',\sigma,\lambda'\rangle \qquad (4.3.3)$$

where $\mathscr{D}$ is the rotation matrix (B.3) and p' is the Lorentz transformed four-momentum. But the momenta appear only in the Lorentz scalars s and t, and so

$$\langle\mu_3\mu_4|\,A^s(s,t)\,|\mu_1\mu_2\rangle = \sum_{\lambda_i} d^{\sigma_1}_{\lambda_1\mu_1}(\chi_1)\,d^{\sigma_2}_{\lambda_2\mu_2}(\chi_2) \times d^{\sigma_3}_{\lambda_3\mu_3}(\chi_3)\,d^{\sigma_4}_{\lambda_4\mu_4}(\chi_4)\langle\lambda_2\lambda_4|\,A^t(s,t)|\,\lambda_1\lambda_3\rangle \qquad (4.3.4)$$

where we have used (B.4) to express the rotation matrices in terms of the rotation functions $d^{\sigma}_{\lambda\mu}$, and χ_i is the angle of rotation for particle i between its direction of motion in the s- and t-channel centre-of-mass frames. In terms of s and t these angles are given by (see for example Martin and Spearman (1970) p. 337)

$$\left.\begin{aligned}
\cos\chi_1 &= \frac{-(s+m_1^2-m_2^2)(t+m_1^2-m_3^2)-2m_1^2\Delta}{(\lambda(s,m_1,m_2)\,\lambda(t,m_1,m_3))^{\frac{1}{2}}}\\
\cos\chi_2 &= \frac{(s+m_2^2-m_1^2)(t+m_2^2-m_4^2)-2m_2^2\Delta}{(\lambda(s,m_1,m_2)\,\lambda(t,m_2,m_4))^{\frac{1}{2}}}\\
\cos\chi_3 &= \frac{(s+m_3^2-m_4^2)(t-m_3^2-m_1^2)-2m_3^2\Delta}{(\lambda(s,m_3,m_4)\,\lambda(t,m_1,m_3))^{\frac{1}{2}}}\\
\cos\chi_4 &= \frac{-(s+m_4^2-m_3^2)(t+m_4^2-m_2^2)-2m_4^2\Delta}{(\lambda(s,m_3,m_4)\,\lambda(t,m_2,m_4))^{\frac{1}{2}}}\\
\sin\chi_i &= \frac{2m_i\phi^{\frac{1}{2}}}{(\lambda(s,m_i,m_j)\,\lambda(t,m_i,m_k))^{\frac{1}{2}}} \quad \text{(j,k chosen as for $\cos\chi_i$ above)}
\end{aligned}\right\} \qquad (4.3.5)$$

where $$\Delta \equiv m_2^2-m_4^2-m_1^2+m_3^2 \qquad (4.3.6)$$

and ϕ and λ are defined in (1.7.23) and (1.7.11).

It is often convenient to rewrite (4.3.4) as

$$A_{H_s}(s,t) = \sum_{H_t} M(H_s,H_t)\,A_{H_t}(s,t) \qquad (4.3.7)$$

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where $H_s \equiv \{\mu_1, \mu_2, \mu_3, \mu_4\}$, $H_t \equiv \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, and $\boldsymbol{M}$ is the helicity crossing matrix given in (4.3.4). It is of course a square matrix with $\prod_{i=1}^{4}(2\sigma_i+1)$ rows and columns, but the number of elements can often be reduced because of the parity and time-reversal relations (4.2.1) and (4.2.4).

As an example we consider $\pi + \mathrm{p} \to \pi + \mathrm{p}$ elastic scattering in the s channel, for which the t channel is $\pi\pi \to \mathrm{p}\bar{\mathrm{p}}$. The crossing relation reads

$$A_{H_s}(s,t) = \sum_{\lambda_2\lambda_4} \mathrm{d}^{\frac{1}{2}}_{\lambda_2\mu_2}(\chi_2)\, d^{\frac{1}{2}}_{\lambda_4\mu_4}(\chi_4)\, A_{H_t}(s,t) \tag{4.3.8}$$

with $\chi_2 = \pi - \chi_4$ given by substituting the appropriate masses in (4.3.5). So using (B.19) and the relations $A_{++} = A_{--}$, $A_{+-} = -A_{-+}$ from (4.2.1) (where $\pm \equiv \pm\frac{1}{2}$ as in (4.2.22)) we find the crossing relation becomes

$$\left.\begin{aligned} A^s_{++}(s,t) &= \sin\chi_4 A^t_{++}(s,t) - \cos\chi_4 A^t_{+-}(s,t) \\ A^s_{+-}(s,t) &= \cos\chi_4 A^t_{++}(s,t) + \sin\chi_4 A^t_{+-}(s,t) \end{aligned}\right\} \tag{4.3.9}$$

These amplitudes are related to the invariant amplitudes $A(s,t)$ and $B(s,t)$ of (4.1.4) by (Cohen-Tannoudji, Salin and Morel 1968)

$$\left.\begin{aligned} A^s_{++} &= \left(\frac{1+z_s}{2}\right)^{\frac{1}{2}} [2m_{\mathrm{N}} A(s,t) + (s - m_{\mathrm{N}}^2 - m_\pi^2) B(s,t)] \\ A^s_{+-} &= -\left(\frac{1-z_s}{2}\right)^{\frac{1}{2}} s^{-\frac{1}{2}}[(s + m_{\mathrm{N}}^2 - m_\pi^2) A(s,t) + (s - m_{\mathrm{N}}^2 + m_\pi^2)\, m_{\mathrm{N}} B(s,t)] \end{aligned}\right\} \tag{4.3.10}$$

and

$$\left.\begin{aligned} A^t_{++} &= -(t - 4m_{\mathrm{N}}^2)^{\frac{1}{2}} A(s,t) + m_{\mathrm{N}}(t - 4m_\pi^2)^{\frac{1}{2}} z_t B(s,t) \\ &\equiv -(t - 4m_{\mathrm{N}}^2)^{\frac{1}{2}} A'(s,t) \\ A^t_{+-} &= \tfrac{1}{2}(t - 4m_\pi^2)^{\frac{1}{2}} t^{\frac{1}{2}} (1 - z_t^2)^{\frac{1}{2}} B(s,t) \end{aligned}\right\} \tag{4.3.11}$$

Since the invariant amplitudes are free of kinematical singularities these equations directly exhibit the kinematical singularities of the helicity amplitudes. ($A'(s,t)$ defined in (4.3.11) will be used below.)

The rotation matrices $d^\sigma_{\lambda\mu}$ are orthogonal, and so the crossing matrix is too. Hence, we can also write the differential cross-section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s q_{s12}^2} \frac{1}{(2\sigma_1+1)(2\sigma_2+1)} \sum_{H_t} |A_{H_t}(s,t)|^2 \tag{4.3.12}$$

Equations (4.2.5) and (4.3.12) are equivalent in both the s- and t-channel physical regions so it does not matter whether one uses s- or t-channel helicity amplitudes. However, outside the physical regions the crossing matrix has singularities so care is needed in

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interpreting the equivalence of these two equations. The density matrices (4.2.10) are obviously not the same with the two sets of amplitudes, though both frames are quite commonly used. Equation (4.2.10) gives what are called the s-channel or 'helicity frame' density matrices, while the similar expressions with λ's substituted for the μ's gives the t-channel or 'Gottfried–Jackson' density matrices (named after their originators Gottfried and Jackson (1964)). The crossing matrix of (4.3.7) enables one to transform from one set of density matrices to the other.

4.4 Partial-wave amplitudes with spin

Our main motive for introducing helicity amplitudes has been to provide a basis for defining partial-wave amplitudes, so that we can make an analytic continuation in the total angular momentum, J, similar to that made in chapter 2.

The initial state, $|p_1, \sigma_1, \mu_1; p_2, \sigma_2, \mu_2\rangle$, has the two particles travelling in opposite directions along the z axis in the s-channel centre-of-mass system. It can be decomposed into partial waves of angular momentum J by

$$|p_1, \sigma_1, \mu_1; p_2, \sigma_2, \mu_2\rangle = (16\pi)^{\frac{1}{2}} \sum_{J=|\mu|}^{\infty} (2J+1)^{\frac{1}{2}} |s, J, \mu, \mu_1, \mu_2\rangle \quad (4.4.1)$$

where
$$\mu \equiv \mu_1 - \mu_2 \quad (4.4.2)$$

is the z component of J, $s = (p_1+p_2)^2$ as usual, and the factor $[16\pi(2J+1)]^{\frac{1}{2}}$ gives a convenient normalization. We have absorbed the spin labels, $\sigma_{1,2}$, into the implicit particle-type label on the right-hand side of (4.4.1) (see section 1.2).

Similarly, in the final state the particles are travelling in opposite directions at polar angles, θ, ϕ, relative to the z axis (see fig. 2.1 (c)), and the corresponding decomposition is

$$|p_3, \sigma_3, \mu_3; p_4, \sigma_4, \mu_4\rangle = (16\pi)^{\frac{1}{2}} \sum_{J=|\mu'|}^{\infty} \sum_{\mu''=-J}^{J} (2J+1)^{\frac{1}{2}}$$
$$\times \mathscr{D}^J_{\mu''\mu'}(\phi, \theta, -\phi) |s, J. \mu'', \mu_3, \mu_4\rangle \quad (4.4.3)$$

using (4.4.1), (B.1) and (B.3), where

$$\mu' \equiv \mu_3 - \mu_4 \quad (4.4.4)$$

is the component of J along the direction of motion, and μ'' is the component of J along the z axis. $\mathscr{D}^J_{\mu''\mu'}(\phi, \theta, -\phi)$ is the rotation matrix defined in (B.3) corresponding to the rotation from the θ, ϕ direction to the z axis.

Because of angular-momentum conservation we can define a partial-wave scattering amplitude for scattering in each J, i.e.

$$A_{HJ}(s) \equiv \langle s, J, \mu'', \mu_3, \mu_4 | A | s, J, \mu, \mu_1, \mu_2 \rangle \tag{4.4.5}$$

$$H \equiv \{\mu_1, \mu_2, \mu_3, \mu_4\} \tag{4.4.6}$$

where $\mu'' = \mu$ to conserve the z component of $\boldsymbol{J}$, and so the full scattering amplitude (4.1.6) may be written (using (4.4.1), (4.4.3) and (4.4.5)) as

$$A_{H_s}(s,t) = 16\pi \sum_{J=M}^{\infty} (2J+1) A_{HJ}(s) \mathscr{D}^{J*}_{\mu\mu'})\phi, \theta, -\phi) \tag{4.4.7}$$

where
$$M \equiv \max\{|\mu|, |\mu'|\} \tag{4.4.8}$$

If we take the scattering plane to be the x–z plane $\phi = 0$, so, from (B.4), (4.4.7) simplifies to

$$A_{H_s}(s,t) = 16\pi \sum_{J=M}^{\infty} (2J+1) A_{HJ}(s) d^J_{\mu\mu'}(z_s) \tag{4.4.9}$$

which may be compared to (2.2.2) for spinless scattering.

The partial-wave amplitudes can be obtained from (4.4.9) using the orthogonality relation (B.14), viz.

$$A_{HJ}(s) = \frac{1}{32\pi} \int_{-1}^{1} A_{H_s}(s,t) d^J_{\mu\mu'}(z_s) \, dz_s \tag{4.4.10}$$

It is evident that for spinless scattering where $\mu_i = 0$, $i = 1, \ldots, 4$, (4.4.10) reduces to (2.2.1) because of (B.18).

The values of J in the series (4.4.9) are either integer or half-odd-integer depending on whether the number of fermions in the s channel is even or odd (i.e. J is integer for boson–boson and fermion–fermion scattering, but half-odd-integer for boson–fermion scattering). The sum starts at $J = M$ (defined in (4.4.8)) not 0 or $\frac{1}{2}$, because, as we noted in section 4.1, there is no component of $\boldsymbol{l}$ in the direction of motion of the particles, so for the initial state

$$J_z = \sigma_{1z} + \sigma_{2z} = \mu_1 - \mu_2 \equiv \mu$$

(with a similar expression for the final-state particles in their direction of motion) and obviously one must have $J \geqslant |J_z|$.

Following similar arguments to those in section 2.2 we find that the unitarity relation for these partial-wave amplitudes is

$$A^{if}_{HJ}(s_+) - A^{if}_{HJ}(s_-) = \frac{4iq_{sn}}{\sqrt{s}} \sum_{H_n} A^{in}_{HJ}(s_+) A^{nf}_{HJ}(s_-) \tag{4.4.11}$$

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like (2.2.7), but where the sum runs over all the possible helicities of the intermediate state $|n\rangle$.

Like (2.2.2), the series (4.4.9) is valid only until we reach the nearest dynamical t-singularity (i.e. only inside the small Lehmann ellipse) and to continue outside the neighbourhood of the s-channel physical region it is necessary to make an analytic continuation. However, unlike the $P_l(z_s)$, the $d^J_{\mu\mu'}(z_s)$ are not in general entire functions of z_s, and so there are additional 'kinematical' singularities which we must also take into account. They can be read off directly from (B.9), for since the Jacobi polynomials are entire functions of z, the singularities of the $d^J_{\mu\mu'}(z_s)$ stem just from the half-angle factor

$$\xi_{\mu\mu'}(z_s) \equiv \left(\frac{1-z_s}{2}\right)^{\frac{1}{2}|\mu-\mu'|}\left(\frac{1+z_s}{2}\right)^{\frac{1}{2}|\mu+\mu'|} = \left(\sin\frac{\theta_s}{2}\right)^{|\mu-\mu'|}\left(\cos\frac{\theta_s}{2}\right)^{|\mu+\mu'|} \tag{4.4.12}$$

and so occur at $z_s = \pm 1$. They have a rather simple physical interpretation in that for forward scattering, $z_s = 1$, μ and μ' are the projections of $\boldsymbol{J}$ along the z axis in the initial and final states, respectively. Since angular momentum is to be conserved the scattering amplitude must obviously vanish as $z_s \to 1$ unless $\mu = \mu'$. The same applies for backward scattering ($z_s = -1$) where μ and $-\mu'$ are the corresponding z-components of $\boldsymbol{J}$.

It is thus convenient to define s-channel helicity amplitudes free of these kinematical singularities in t by

$$\hat{A}_{H_s}(s,t) \equiv A_{H_s}(s,t)\,[\xi_{\mu\mu'}(z_s)]^{-1} \tag{4.4.13}$$

These amplitudes will satisfy the same sort of fixed-s dispersion relations, involving integrals over the dynamical singularities in t, as do spinless-particle scattering amplitudes. Note, however, that (4.4.13) still has kinematical s-singularities, which we shall discuss later (see section 6.2).

We could of course repeat the discussion of this section for t-channel helicity amplitudes to obtain the partial-wave series

$$A_{H_t}(s,t) = 16\pi \sum_{J=M}^{\infty} (2J+1)\, A_{HJ}(t)\, d^J_{\lambda\lambda'}(z_t) \tag{4.4.14}$$

where $\quad \lambda \equiv \lambda_1 - \lambda_3, \quad \lambda' \equiv \lambda_2 - \lambda_4, \quad M \equiv \max\{|\lambda|, |\lambda'|\} \qquad (4.4.15)$

and $\quad \hat{A}_{H_t}(s,t) \equiv A_{H_t}(s,t)\,[\xi_{\lambda\lambda'}(z_t)]^{-1} \qquad (4.4.16)$

will be free of kinematical singularities in s. The inverse of (4.4.14) is (like (4.4.10))

$$A_{HJ}(t) = \frac{1}{32\pi}\int_{-1}^{1} A_{H_t}(s,t)\, d^J_{\lambda\lambda'}(z_t)\, \mathrm{d}z_t \tag{4.4.17}$$

(We have for simplicity dropped the channel label for the helicities of the partial-wave amplitudes in (4.4.10) and (4.4.17) as they are always implied by the channel invariants.)

4.5 The Froissart–Gribov projection

Since $\hat{A}_{H_t}(s,t)$ defined in (4.4.16) has no kinematical s-singularities it satisfies a dispersion relation in s at fixed t like (1.10.7), i.e.

$$\hat{A}_{H_t}(s,t) = \frac{1}{\pi}\int_{s_T}^{\infty}\frac{D_{sH}(s',t)}{s'-s}\,ds' + \frac{1}{\pi}\int_{u_T}^{\infty}\frac{D_{uH}(u',t)}{u'-u}\,du' \qquad (4.5.1)$$

where D_{sH} is the discontinuity of $\hat{A}_H$ across the dynamical s-cuts above the threshold s_T (and correspondingly for D_{uH}). Bound-state poles, if they occur, can be added as in (1.10.7).

This expression can be employed, following the method of section 2.3, to define partial-wave amplitudes even outside the region of convergence of the partial-wave series. Substituting (4.5.1) into (4.4.17), remembering (4.4.16) and (2.3.2), we obtain (Calogero, Charap and Squires 1963*b*, Drechsler 1968)

$$A_{HJ}(t) = \frac{1}{32\pi}\int_{-1}^{1} dz_t\, d^J_{\lambda\lambda'}(z_t)\,\xi_{\lambda\lambda'}(z_t)\left\{\frac{1}{\pi}\int_{z_T}^{\infty}\frac{D_{sH}(s',t)}{z'-z_t}\,dz' + \frac{1}{\pi}\int_{-z_T}^{-\infty}\frac{D_{uH}(u',t)}{z'-z_t}\,dz'\right\} \qquad (4.5.2)$$

($z_T \equiv z_s(s_T,t)$), which, with the generalized Neumann relation (B.21), gives the Froissart–Gribov projection (cf. (2.3.4))

$$A_{HJ}(t) = \frac{1}{16\pi^2}\int_{z_T}^{\infty} dz_t\{D_{sH}(s,t)\,e^J_{\lambda\lambda'}(z_t)\,\xi_{\lambda\lambda'}(z_t) + (-1)^{J-\lambda}D_{uH}(s,t)\,e^J_{\lambda-\lambda'}(z_t)\,\xi_{\lambda-\lambda'}(z_t)\} \qquad (4.5.3)$$

where (B.23) has been used for the second term.

If the asymptotic behaviour is $A_{H_t} \underset{s\to\infty}{\sim} s^{\alpha}$, then $\hat{A}_{H_t} \sim s^{\alpha-M}$ from (4.4.16) since $\xi_{\lambda\lambda'}(z_t) \sim s^M$, and since from (B.25) $e^J_{\lambda\lambda'}(z) \sim s^{-J-1}$, the criterion for the convergence of (4.5.3) is the same as for (2.3.4), i.e. $J > \alpha$.

As $J \to \infty$ we find from (B.26) that the first term in (4.5.3) tends to zero like

$$\underset{J\to\infty}{\sim} J^{-\frac{1}{2}}\,e^{-(J+\frac{1}{2})\zeta(z_T)} \qquad (4.5.4)$$

but the second term behaves like

$$\underset{J\to\infty}{\sim} J^{-\frac{1}{2}}\,e^{-(J+\frac{1}{2})\zeta(z_T)}\,e^{-i\pi(J-\lambda)} \qquad (4.5.5)$$

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and so diverges as $J \to \mathrm{i}\infty$. So (4.5.3) does not satisfy the conditions for Carlson's theorem, and (as in section 2.5) before we can make an analytic continuation in J we have to introduce amplitudes of definite signature. These are defined by replacing $(-1)^{J-v}$ by the signature $\mathcal{S} = \pm 1$, where

$$\left.\begin{aligned} v &\equiv 0 \quad \text{for physical } J = \text{integer} \\ v &\equiv \tfrac{1}{2} \quad \text{for physical } J = \text{half-odd-integer} \end{aligned}\right\} \tag{4.5.6}$$

(Note that whether λ is integral or half-integral depends on the physical J values.) Hence

$$A^{\mathcal{S}}_{HJ}(t) \equiv \frac{1}{16\pi^2}\int_{z_{\mathrm{T}}}^{\infty} \mathrm{d}z_t \{D_{sH}(s,t)\, e^J_{\lambda\lambda'}(z_t)\, \xi_{\lambda\lambda'}(z_t) + \mathcal{S}(-1)^{\lambda-v} D_{uH}(s,t)\, e^J_{\lambda-\lambda'}(z_t)\, \xi_{\lambda-\lambda'}(z_t)\} \tag{4.5.7}$$

For $\mathcal{S} = \pm 1$ these amplitudes coincide with the physical $A_{HJ}(t)$ for $J-v = $ even/odd, so instead of (4.4.14) we can write

$$A_{H_t}(s,t) = 16\pi \sum_{J=M}^{\infty} (2J+1)\,(A^+_{HJ}(t)\, d^+_{\lambda\lambda'}(J,z) + A^-_{HJ}(t)\, d^-_{\lambda\lambda'}(J,z)) \tag{4.5.8}$$

if we define

$$d^{\mathcal{S}}_{\lambda\lambda'}(J,z) \equiv \tfrac{1}{2}[d^J_{\lambda\lambda'}(z) + \mathcal{S}(-1)^{\lambda-v} d^J_{\lambda-\lambda'}(-z)] \tag{4.5.9}$$

Note that $d^{\pm}_{\lambda\lambda'}(J,z)$ vanishes for $J-v = $ odd/even because of the symmetry relation (B.7).

Scattering amplitudes of definite signature are defined by

$$A^{\mathcal{S}}_{H_t}(s,t) = 16\pi \sum_{J=M}^{\infty} (2J+1)\, A^{\mathcal{S}}_{HJ}(t)\, d^{\mathcal{S}}_{\lambda\lambda'}(J,z) \tag{4.5.10}$$

Equation (4.5.7) may be used to define definite-signature partial-wave amplitudes for all J. The physical J values are of course those having integer $J-v$, with $J \geqslant |\lambda|$ for the initial state ($1+\bar{3}$ in the t channel) and $J \geqslant |\lambda'|$ for the final state. So $J \geqslant M$ defined in (4.4.15). Because these are the values of J which make physical sense, they are known as 'sense–sense' or ss values, and the amplitudes for these values of J are called ss amplitudes. When we continue in J we may arrive at integer values of $J-v$ with $J < M$, but $J \geqslant N$ where

$$N \equiv \min\{|\lambda|, |\lambda'|\} \tag{4.5.11}$$

If say $|\lambda| > |\lambda'|$ then this J value makes physical sense for the final state, but not for the initial state (and vice versa if $|\lambda| < |\lambda'|$). These are called 'sense–nonsense' or sn values of J. And of course for integer $J-v$, $J < N$, we have nonsense–nonsense or nn amplitudes which do not make physical sense for either the incoming or outgoing states (Gell–Mann 1962). It is sometimes convenient to refer to all integer $J-v$ with $J < M$ as 'nonsense' values of J.

4.6 The Sommerfeld–Watson representation

The partial-wave series (4.5.10) can be rewritten as a contour integral in J, like (2.7.5), viz

$$A^{\mathscr{S}}_{H_t}(s,t) = -\frac{16\pi}{2\mathrm{i}}\int_{C_1}\frac{2J+1}{\sin\pi(J+\lambda')}A^{\mathscr{S}}_{HJ}(t)\,d^{\mathscr{S}}_{-\lambda\lambda'}(J,-z_t)\,\mathrm{d}J \quad (4.6.1)$$

where the contour C_1 encloses the physical values $J \geqslant M$, but avoids any singularities of the $A_{HJ}(t)$ as in fig. 4.1. The $(-1)^{J+\lambda'}$ from the residues of the poles of $(\sin\pi(J+\lambda'))^{-1}$ is cancelled by the use of $d^{\mathscr{S}}_{-\lambda\lambda'}(J,-z)$ instead of $d^{\mathscr{S}}_{\lambda\lambda'}(J,z)$ because of the symmetry relation (B.7).

Then when we open up the contour to C_2 of fig. 4.1 we reveal any Regge poles and cuts of $A_{HJ}(t)$, and also obtain contributions from integer values of $J-v$ in the region $-\frac{1}{2} < J < M$, i.e. from the sn and nn values defined above, so we have (substituting the integrand of (4.6.1) where indicated)

$$\begin{aligned}A^{\mathscr{S}}_{H_t}(s,t) = &-\frac{16\pi}{2\mathrm{i}}\int_{C_2}[(4.6.1)] - 16\pi^2\frac{2\alpha_i(t)+1}{\sin\pi(\alpha(t)+\lambda')}\beta_H(t)d^{\mathscr{S}}_{-\lambda\lambda'}(\alpha(t),-z_t)\\ &-\frac{16\pi}{2\mathrm{i}}\int^{\alpha_c(t)}\frac{2J+1}{\sin\pi(J+\lambda')}\Delta^{\mathscr{S}}(J,t)\,d^{\mathscr{S}}_{-\lambda\lambda'}(J,-z_t)\,\mathrm{d}J\\ &-\sum_{J=N}^{M-1}-\sum_{J=v}^{N-1}16\pi(2J+1)\,A^{\mathscr{S}}_{HJ}(t)\,d^{\mathscr{S}}_{-\lambda\lambda'}(J,-z_t) \end{aligned} \quad (4.6.2)$$

The first term is the usual background integral, $\sim s^{-\frac{1}{2}}$. For simplicity we have assumed that there is just one pole at $J = \alpha(t)$, and one branch point at $J = \alpha_c(t)$, in $\mathrm{Re}\{J\} > -\frac{1}{2}$, and evidently these terms have the usual asymptotic behaviour $\sim s^{\alpha(t)}$, and $\sim s^{\alpha_c(t)}$ respectively, from (B.14). The final terms contain the sn and nn contributions.

At a sn point $J = J_0$ say, where $J_0 = v$ is an integer with $N \leqslant J_0 < M$, we can see from (B.12) that $d^J_{\lambda\lambda'}(z)$ (and hence $d^{\mathscr{S}}_{-\lambda\lambda'}(J,-z)$) vanishes like $(J-J_0)^{\frac{1}{2}}$, and so there will be no contribution from these terms unless $A_{HJ} \sim (J-J_0)^{-\frac{1}{2}}$. We shall discuss this possibility further in section 4.8, but if for the moment we assume that this does not happen

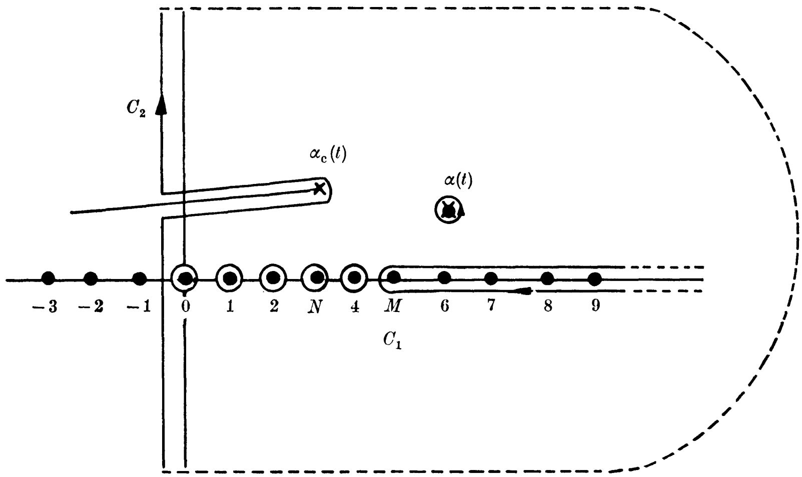

FIG. 4.1 The Sommerfeld–Watson transform for a helicity amplitude with $M = 5$ and $N = 3$. The contour C_1 encloses the integers $J \geqslant M$. When it is opened up to C_2 we get contributions from the Regge pole at $\alpha(t)$, from the branch cut starting at $\alpha_c(t)$, and from the integer values $-\frac{1}{2} < J < M$.

the first summation can be neglected. Similarly from (B.12) we find that at the nn points $J = J_0$ with $J_0 - v$ integer, $v \leqslant J_0 < N$, $d^J_{\lambda\lambda'} \sim (J - J_0)$ and so these terms also vanish unless there are fixed poles,

$$A_{HJ} \sim (J - J_0)^{-1},$$

a possibility which we shall also reconsider in section 4.8.

If we wish to explore the region $\mathrm{Re}\{J\} < -\frac{1}{2}$ we can again employ the Mandelstam method described in section 2.9, using the relation (B.28) instead of (A.18). The symmetry of the rotation functions (B.27) ensures that, from (4.5.7),

$$A_{HJ}(t) = (-1)^{\lambda-\lambda'} A^{\mathscr{S}'}_{H\,-J-1}(t), \quad J - v = \text{half-odd-integer} \tag{4.6.3}$$

(where $\mathscr{S}' \equiv \mathscr{S}$ for $v = 0$ and $\mathscr{S}' \equiv -\mathscr{S}$ for $v = \frac{1}{2}$) as long as (4.5.7) converges, and so the contribution of the poles of $[\cos\pi(J+\lambda')]^{-1}$ in the two terms of (B.28) cancel pairwise for $J < M$. So we get

$$\begin{aligned} A^{\mathscr{S}}_{H_t}(s,t) = {} & 16\pi(2\alpha(t)+1)\,\beta_H(t)\,\frac{e^{\mathscr{S}'}_{\lambda-\lambda'}(-\alpha-1,-z_t)}{\cos\pi(\alpha+\lambda')} \\ & + \frac{16\pi}{2\mathrm{i}}\int^{\alpha_c(t)} \frac{2J+1}{\cos\pi(J+\lambda')}\,\Delta^{\mathscr{S}}(J,\lambda')\,e^{\mathscr{S}'}_{\lambda-\lambda'}(-J-1,-z_t)\,\mathrm{d}J \\ & + \text{possible fixed poles or cuts} \\ & + \text{background integral} \end{aligned} \tag{4.6.4}$$

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where, following (4.5.9), we have defined

$$e^{\mathscr{S}}_{\lambda\lambda'}(J,z) \equiv \tfrac{1}{2}[e^J_{\lambda\lambda'}(z) + \mathscr{S}(-1)^{\lambda-v} e^J_{\lambda\lambda'}(-z)] \tag{4.6.5}$$

Equation (B.25) ensures that the pole and cut terms will have the asymptotic behaviour $\sim s^{\alpha(t)}$ and $\sim s^{\alpha_c(t)}$, respectively, but now the background-integral contour can be pulled as far to the left as we like.

For Regge poles it is rather unfortunate that the helicity states which we use are not eigenstates of the parity operator, because of course the Reggeons do have a definite parity. (Cuts do not have definite parity so the above formalism is quite satisfactory for them – see chapter 8.) It is therefore sometimes more convenient to analytically continue in J amplitudes of definite parity, which are defined as follows (Gell–Mann *et al.* 1964).

A given t-channel partial-wave helicity state $|J,\lambda,\lambda_1,\lambda_2\rangle$ transforms under the parity operator as

$$P|J,\lambda,\lambda_1,\lambda_3\rangle = P_1 P_3 (-1)^{J-\sigma_1-\sigma_3} |J,\lambda,-\lambda_1,-\lambda_3\rangle \tag{4.6.6}$$

where P_1, P_3 are the intrinsic parities of the particles, and, as discussed in section 4.2, the helicities change sign. The phase factor $(-1)^{J-\sigma_1-\sigma_3}$ corresponds to the Condon and Shortley phase conventions for the relative phases of the helicity states as used in (4.2.2) and in the reflection properties of the rotation matrices (B.7) (see Jacob and Wick 1959). Thus we may define definite parity states by

$$|J,\lambda,\lambda_1,\lambda_3,\eta\rangle \equiv \frac{1}{\sqrt{2}}\{|J,\lambda,\lambda_1,\lambda_3\rangle + \eta P_1 P_3 (-1)^{\sigma_1+\sigma_3-v}|J,\lambda,-\lambda_1,-\lambda_3\rangle\} \tag{4.6.7}$$

where $\eta = \pm 1$ for natural/unnatural parity. A state is said to have natural parity if $P = (-1)^{J-v}$ and unnatural parity if $P = (-1)^{J-v-1}$, results which are readily obtained from (4.6.7) using (4.6.6). These states are physical for $J-v$ even/odd depending on the signature, and so we have the relation

$$P = \eta\mathscr{S} \tag{4.6.8}$$

Since parity is conserved in strong interactions, scattering amplitudes occur only between states of the same parity, and a definite-parity partial-wave amplitude is given by

$$\begin{aligned}\langle J,\lambda',\lambda_2,\lambda_4,\eta|\, A^{\mathscr{S}}(t)\, |J,\lambda,\lambda_1,\lambda_3,\eta\rangle &\equiv A^{\mathscr{S}\eta}_{HJ}(t) \\ &\equiv \langle\lambda_2,\lambda_4|\, A^{\mathscr{S}}_J(t)\, |\lambda_1\lambda_3\rangle + \eta P_1 P_3 (-1)^{\sigma_1+\sigma_3-v} \langle\lambda_2,\lambda_4|\, A_J(t)\, |-\lambda_1,-\lambda_3\rangle\end{aligned} \tag{4.6.9}$$

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Hence we can define the so-called 'parity-conserving helicity amplitudes' free of kinematical singularities in s by

$$\hat{A}^{\mathscr{S}\eta}_{H_t}(s,t) \equiv \langle\lambda_2\lambda_4\eta|\,\hat{A}^{\mathscr{S}}(s,t)\,|\lambda_1\lambda_3\eta\rangle = \langle\lambda_2\lambda_4|\,A^{\mathscr{S}}(s,t)\,|\lambda_1\lambda_3\rangle\xi^{-1}_{\lambda\lambda'}(z_t)$$
$$+\eta P_1 P_3(-1)^{\lambda'+M+\sigma_1+\sigma_3-v}\langle\lambda_2\lambda_4|\,\hat{A}^{\mathscr{S}}(s,t)\,|-\lambda_1-\lambda_3\rangle\,\xi^{-1}_{-\lambda\lambda'}(z_t) \quad (4.6.10)$$

The partial-wave series for this amplitude is

$$\hat{A}^{\mathscr{S}\eta}_{H_t}(s,t) = 16\pi\sum_{J=M}^{\infty}(2J+1)\,A^{\mathscr{S}}_{HJ}(t)\,\frac{d^{\mathscr{S}}_{\lambda\lambda'}(J,z_t)}{\xi_{\lambda\lambda'}(z_t)}$$
$$+\eta P_1 P_3(-1)^{\lambda'+M+\sigma_1+\sigma_3-v}A^{\mathscr{S}}_{\bar{H}J}(t)\,\frac{d^{\mathscr{S}}_{-\lambda\lambda'}(z_t)}{\xi_{-\lambda\lambda'}(z_t)} \quad (4.6.11)$$

where we have introduced $\bar{H} \equiv \{-\lambda_1, -\lambda_3, \lambda_2, \lambda_4\}$. Or, using (4.6.9),

$$\hat{A}^{\mathscr{S}\eta}_{H_t}(s,t) = 16\pi\sum_{J=N}^{\infty}(2J+1)\,(A^{\mathscr{S}\eta}_{HJ}(t)\,d^{\mathscr{S}+}_{\lambda\lambda'}(J,z_t)+A^{\mathscr{S}\bar{\eta}}_{HJ}(t)\,\hat{d}^{\mathscr{S}-}_{\lambda\lambda'}(J,z_t)) \quad (4.6.12)$$

with $\bar{\eta} \equiv -\eta$ and

$$\hat{d}^{\mathscr{S}\eta}_{\lambda\lambda'}(J,z) \equiv \frac{1}{2}\left[\frac{d^{\mathscr{S}}_{\lambda\lambda'}(J,z)}{\xi_{\lambda\lambda'}(z)}+\eta(-1)^{\lambda'+M}\frac{d^{\mathscr{S}}_{-\lambda\lambda'}(J,z)}{\xi_{-\lambda\lambda'}(z)}\right] \quad (4.6.13)$$

Thus we see that the total amplitude contains contributions from partial-wave amplitudes of both parities, but asymptotically, from (4.6.12), (4.5.9), (B.17) and (B.13),

$$d^{\mathscr{S}\eta}_{\lambda\lambda'}(J,z) \sim \left(\frac{z}{2}\right)^{J-M}\left(\frac{1+\eta}{2}\right)+O(z^{J-M-1}), \quad \mathrm{Re}\{J\} > -\tfrac{1}{2} \quad (4.6.14)$$

so to leading order $\hat{d}^{\mathscr{S}+}_{\lambda\lambda'}$ dominates over $\hat{d}^{\mathscr{S}-}_{\lambda\lambda'}$. It is only in this asymptotic sense that (4.6.12) can be regarded as a definite-parity amplitude.

If we now make a Sommerfeld–Watson transform of (4.6.12), and use the Mandelstam method like (4.6.4), we find that a Regge contribution is given by

$$\hat{A}^{\mathscr{S}\eta}_{H_t}(s,t) = 16\pi(2\alpha(t)+1)\,\beta_H(t)\,\frac{\hat{e}^{\mathscr{S}+}_{\lambda-\lambda'}(-\alpha-1,z_t)}{\cos\pi(\alpha+\lambda')} \quad (4.6.15)$$

where, in analogy with (4.6.13), we have introduced

$$\hat{e}^{\mathscr{S}\eta}_{\lambda\lambda'}(J,z) \equiv \tfrac{1}{4}(1+\mathscr{S}\mathrm{e}^{\pm i\pi(J-v)})\left[\frac{e^{J}_{\lambda\lambda'}(z)}{\xi_{\lambda\lambda'}(z)}+\eta(-1)^{\lambda'+M}\frac{e^{J}_{-\lambda\lambda'}(J,z)}{\xi_{\lambda\lambda'}(z)}\right] \quad (4.6.16)$$

But to leading order there is no difference between (4.6.15) and the Regge pole contribution in (4.6.4).

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4.7 Restrictions on Regge singularities from unitarity

We have already noted in section 2.4 how the application of s-channel unitarity leads to the Froissart bound, and hence to the restriction that the t-channel Regge singularities cannot be above 1 for $t \leqslant 0$. This applies also in the presence of spin, since the Regge power behaviours are unchanged.

There are also some important restrictions which stem from t-channel unitarity. For spinless-particle elastic scattering in the t channel, $1+\bar{3} \rightarrow 1+\bar{3}$, the unitarity condition reads (from (2.2.7) and (2.6.8), with $s \rightarrow t$)

$$B_l^{\mathscr{S}}(t_+) - B_l^{\mathscr{S}}(t_-) = 2\mathrm{i}\rho_l(t) B_l^{\mathscr{S}}(t_+) B_l^{\mathscr{S}}(t_-) \tag{4.7.1}$$

$$\rho_l(t) \equiv (q_{t13})^{2l+1} \frac{2}{\sqrt{t}} \tag{4.7.2}$$

for $t_{\mathrm{T}} < t < t_{\mathrm{I}}$, where t_{T} is the elastic threshold, and t_{I} the inelastic threshold. Since $B_l(t)$ is a real analytic function we have

$$(B_{l^*}^{\mathscr{S}}(t+\mathrm{i}\epsilon))^* = B_l^{\mathscr{S}}(t-\mathrm{i}\epsilon) \tag{4.7.3}$$

for real t (where $* \equiv$ complex conjugate), and so we can rewrite (4.7.1) as

$$B_l^{\mathscr{S}}(t) - (B_{l^*}^{\mathscr{S}}(t))^* = 2\mathrm{i}\rho_l(t) B_l^{\mathscr{S}}(t) (B_{l^*}^{\mathscr{S}}(t))^* \tag{4.7.4}$$

To start with we only know that this equation is valid for right-signature integer values of l, but both sides of (4.7.4) satisfy the boundedness condition for Carlson's theorem (section 2.7) and hence the equation remains true if we continue in l. Note that, from the discussion in section 2.6, (4.7.4) is true for non-integer l only because we removed the kinematical threshold singularities in defining $B_l^{\mathscr{S}}(t)$ in (2.6.8).

It is evident that (4.7.1) cannot be satisfied by a fixed l-plane pole of the form

$$B_l^{\mathscr{S}}(t) \approx \frac{\beta(l,t)}{l-l_0}, \quad l \rightarrow l_0 \tag{4.7.5}$$

for if we inserted (4.7.5) into (4.7.1) we would have a single pole at l_0 on the left-hand side equated to a double pole on the right-hand side. A pole whose position changes with t, say at $l = \alpha(t)$, can satisfy (4.7.1) as long as $\alpha(t_+) \neq \alpha(t_-)$, i.e. as long as $\mathrm{Im}\{\alpha(t)\} \neq 0$ (for $t > t_{\mathrm{T}}$). We have seen examples of this in section 3.4 where unitarity has converted the fixed pole of the Born term into a moving pole with a right-hand cut.

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The only way in which (4.7.4) can be satisfied with a fixed pole is if there is also an l-plane cut passing through l_0 for all $t_T < t < t_I$. Then one approaches $l = l_0$ on different sides of this cut in B_l and $B^*_{l^*}$, and the pole can be present on one side of the cut but not the other, in which case there is no problem (see section 8.3). But in the absence of cuts all poles must be moving poles, i.e. their positions must be functions of t.

For particles with spin we define corresponding partial-wave helicity amplitudes

$$B^{\mathscr{S}}_{HJ}(t) = A^{\mathscr{S}}_{HJ}(t)\,(q_{t13})^{-2L} \tag{4.7.6}$$

where L is the lowest possible orbital angular momentum at threshold for the given J ($L = J - Y^+_{13}$ where $Y^+_{13} = \sigma_1 + \sigma_3$ or $\sigma_1 + \sigma_3 - 1$ depending on the parity – this will be discussed in section 6.2.) Then the unitarity condition can be written in the form

$$\mathbf{B}^{\mathscr{S}}_J(t) - (\mathbf{B}^{\mathscr{S}}_{J^*}(t))^\dagger = 2\mathrm{i}(\mathbf{B}^{\mathscr{S}}_{J^*}(t))^\dagger\,\boldsymbol{\rho}_J(t)\,\mathbf{B}^{\mathscr{S}}_J(t) \tag{4.7.7}$$

where the B's have been expressed as matrices, the various initial- and final-state helicities labelling the rows and columns ($\dagger \equiv$ Hermitian conjugate = complex conjugate transposed matrix, i.e. $B^\dagger_{ij} = B^*_{ji}$). Here $\boldsymbol{\rho}_J(t)$ is a diagonal matrix of kinematical factors

$$(\rho_{HJ}(t))_{nn} = (q_{tn})^{2L_n+1}\frac{2}{\sqrt{t}} \tag{4.7.8}$$

So in (4.7.7) the sum over intermediate-state helicities is represented as a matrix product. Above the inelastic threshold, two-body inelastic processes can similarly be incorporated by increasing the numbers of rows and columns to represent the unitarity equation (2.2.11).

A fixed pole at $J = J_0$ in (4.7.7) implies that

$$\boldsymbol{\beta}(J_0,t)\,\boldsymbol{\beta}^\dagger(J_0,t) = 0, \quad \text{i.e. } \boldsymbol{\beta} = 0 \tag{4.7.9}$$

so again fixed poles on the real J axis are forbidden, but if J_0 has an imaginary part (4.7.7) simply gives

$$\boldsymbol{\beta}(J_0,t_+)\,\boldsymbol{\beta}(J_0,t_-) = 0 \tag{4.7.10}$$

which does not require $\boldsymbol{\beta} = 0$. So in principle there could be fixed poles even in the absence of cuts, but not on the real axis. However, there does not seem to be any reason why such fixed poles at complex values of J should occur. We shall find in the next section that fixed poles do occur on the real axis at wrong-signature nonsense points, and these clearly must have shielding Regge cuts.

If we define the partial-wave S-matrix by

$$\mathbf{S}(J,t) = \mathbf{1} + 2\mathrm{i}\boldsymbol{\rho}_J(t)\,\mathbf{B}(J,t) \tag{4.7.11}$$

where $\mathbf{1}$ is the unit matrix, the unitarity relation (4.7.7) reads

$$\mathbf{S}(J,t)\,\mathbf{S}^{\dagger}(J,t) = \mathbf{1} \quad \text{or} \quad \mathbf{S}(J,t)\,\frac{\operatorname{cof}(\mathbf{S}^{\dagger})}{\det(\mathbf{S}^{\dagger})} \tag{4.7.12}$$

(where cof $\equiv$ cofactor matrix and det $\equiv$ determinant). Thus for a two-channel process this becomes

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \frac{\begin{pmatrix} S_{22}^{*} & -S_{21}^{*} \\ -S_{12}^{*} & S_{11}^{*} \end{pmatrix}}{S_{11}^{*}S_{22}^{*} - S_{12}^{*}S_{21}^{*}} \tag{4.7.13}$$

so if $\mathbf{S}$ has a simple pole of the form $\boldsymbol{\beta}(J-\alpha)^{-1}$, the vanishing of the denominator on the right-hand side requires that

$$\beta_{22}\beta_{11} = \beta_{12}\beta_{21} \tag{4.7.14}$$

so that one can write

$$\beta_{ij} = \beta_i\beta_j \tag{4.7.15}$$

i.e. the Regge pole residue must factorize, as could have been anticipated from our discussion in section 1.5. This result has been proved for an arbitrary number of channels by Charap and Squires (1962).

4.8 Fixed singularities and SCR

The rotation functions, $e^{J}_{\lambda\lambda'}$, used in (4.5.7) to define partial-wave amplitudes of any J, have fixed J singularities stemming from the square bracket in (B.24) at unphysical values of J. ($F(a,b,c,d)$ is an entire function of its arguments.) Since $x!$ has poles at $x = -1, -2, -3, \ldots$, we see that for $J = J_0$ (where $J_0 - v$) is an integer

$$\left.\begin{aligned} e^{J}_{\lambda\lambda'}(z) &\sim (J-J_0)^{-\frac{1}{2}}, \quad N \leqslant J_0 < M \quad \text{and} \quad -M \leqslant J_0 < -N \\ &\sim (J-J_0)^{-1}, \quad -N \leqslant J_0 < N \quad \text{and} \quad J_0 < -M \end{aligned}\right\} \tag{4.8.1}$$

Thus for $J < -M$ the pole residue is just $d^{J_0}_{\lambda\lambda'}(z)$ (see (B.29)) and so for $J \to J_0 < -M$

$$\begin{aligned} A^{\mathscr{S}}_{HJ}(t) \to \frac{1}{J-J_0}\,\frac{1}{16\pi^2}\int_{z_{\mathrm{T}}}^{\infty} \mathrm{d}z_t \{ & D_{sH}(s,t)\,\xi_{\lambda\lambda'}(z_t)\,d^{J_0}_{\lambda\lambda'}(z_t) \\ & + \mathscr{S}(-1)^{\lambda-v} D_{uH}(s,t)\,\xi_{\lambda-\lambda'}(z_t)\,d^{J_0}_{\lambda-\lambda'}(z_t)\} \end{aligned} \tag{4.8.2}$$

But such a real-axis fixed pole is incompatible with unitarity, as we found in the previous section, and so the integral in (4.8.2) must

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vanish, i.e.

$$\int_{z_T}^{\infty} dz_t \{D_{sH}(s,t)\,\xi_{\lambda\lambda'}(z_t)\,d^{J_0}_{\lambda\lambda'}(z_t) + \mathscr{S}(-1)^{\lambda-v} D_{uH}(s,t)\,\xi_{\lambda-\lambda'}(z_t)\,d^{J_0}_{\lambda-\lambda'}(z_t)\} = 0 \qquad (4.8.3)$$

or taking the asymptotic limit of the rotation functions (for $J < -\frac{1}{2}$ from (B.13a))

$$\int_{z_T}^{\infty} ds\,\{D_{sH}(s,t) + \mathscr{S}(-1)^{M-v} D_{uH}(s,t)\}\, s^{-J_0-1+M} = 0 \qquad (4.8.4)$$

which is needed for all $J_0 < -M$.

Such integrals are known as 'superconvergence relations', or SCR for short. For example, with spinless particle scattering (N, $M = 0$) $Q_l(z)$ in (2.5.3) has poles for all negative integers, $J_0 = -1, -2, \ldots$, from (A.32), and the SCR becomes

$$\int_{s_T}^{\infty} D^{\mathscr{S}}_s(s,t)\, s^n\, ds = 0, \quad n = 0, 1, 2, \ldots \qquad (4.8.5)$$

Similar SCR must hold in potential scattering if a trajectory is to pass below $l = -(1+n)$ (see section 3.3b).

Of course the integral (4.5.7) will diverge for $J > J_0$ if there are Regge poles and cuts in $\mathrm{Re}\{J\} > J_0$, and it is only after all such pole and cut contributions have been removed that the SCR obtain. Since the Froissart bound requires that poles and cuts must not be above 1 for $t \leqslant 0$, we find from (4.8.3) and (B.14) that it is essential for

$$\int_{s_T}^{\infty} ds\,\{D_{sH}(s,t) + \mathscr{S}(-1)^{M+v}(D_{uH}(s,t)\}\, s^n = 0, \quad n = M, M-1, \ldots, 1 \qquad (4.8.6)$$

whatever Regge singularities occur, otherwise the fixed singularities (4.8.1) would give contributions to the asymptotic behaviour which violate this bound.

But there will still be $(J-J_0)^{\frac{1}{2}}$ branch points in the partial-wave amplitudes for $N \leqslant J_0 < M$ and $-M \leqslant J_0 < -N$ from the cancellation of the SCR zero with (4.8.1). These can conveniently be joined pairwise by kinematical cuts running from $J = M-1-k$ to $-M+k$, $k = 0, 1, \ldots, M-1$. They do not contribute to the asymptotic behaviour because the $d^J_{\lambda\lambda'}$ also vanish like $(J-J_0)^{\frac{1}{2}}$ at these points, as we noted when discussing (4.6.2).

However, Gribov and Pomeranchuk (1962) demonstrated that in fact these SCR cannot hold at wrong-signature nonsense values of J, and that $A^{\mathscr{S}}_{HJ}(t)$ will therefore have fixed poles (or infinite square-root branch points) at these points. This is because, from (2.6.19), the imaginary part of the partial-wave helicity amplitude contains a contribution from the 'third' double spectral function of the form

$$\operatorname{Im}\{A^{\mathscr{S}}_{HJ}(t)\} = \frac{1}{16\pi^2}\int_{a(t)}^{b(t)} \mathrm{d}z' \rho^{su}_{H}(s', u')\, e^{J}_{\lambda\lambda'}(z'_t)\,(1-\mathscr{S}\,\mathrm{e}^{-\mathrm{i}\pi(J-v)}) \tag{4.8.7}$$

This vanishes for physical J-values, i.e. at right-signature points, and is obviously absent from situations like potential scattering (without Majorana exchange forces) which have no third double spectral function. But at the wrong-signature nonsense points of hadronic scattering amplitudes the fixed singularities of (4.8.1) will occur, and this time their residues will certainly not vanish due to SCRs because, at least for some regions of t where the integral in s runs over the elastic part of the double spectral function (see fig. 2.6), we can be sure from (3.5.34) that the integrand is always positive. So the SCRs (4.8.3), (4.8.4), (4.8.5) hold only for J_0 such that $(-1)^{J_0-v} = \mathscr{S}$. (We shall return to this point in section 7.2.)

Because of the unitarity equation (4.7.7), each helicity amplitude will acquire the singularities of the others, so fixed singularities will in fact occur at all wrong-signature $J_0 = \sigma_{\mathrm{T}} - k$, $k = 2, 4, 6, \ldots$ or $1, 3, 5, \ldots$ since σ_{T} ($\equiv \max\{\sigma_1+\sigma_3, \sigma_2+\sigma_4\}$) gives the largest possible value of M. Of course the occurrence of wrong-signature fixed poles for $J_0 > 1$ does not violate the Froissart bound since the vanishing of the signature factor ensures that they will not contribute to the asymptotic behaviour. But these real-axis fixed poles are incompatible with the unitarity equation, and so the occurrence of Gribov–Pomeranchuk poles proves that Regge cuts must exist, as we shall find in chapter 8.

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