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Girsanov Transformations for Non-Symmetric Diffusions

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Abstract. Let X be a diffusion process, which is assumed to be associated with a (non-symmetric) strongly local Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E; m)$. For $u \in \mathcal{D}(\mathcal{E})_e$, the extended Dirichlet space, we investigate some properties of the Girsanov transformed process Y of X. First, let $\widehat{X}$ be the dual process of X and $\widehat{Y}$ the Girsanov transformed process of $\widehat{X}$. We give a necessary and sufficient condition for $(Y, \widehat{Y})$ to be in duality with respect to the measure $e^{2u}m$. We also construct a counterexample, which shows that this condition may not be satisfied and hence $(Y, \widehat{Y})$ may not be dual processes. Then we present a sufficient condition under which Y is associated with a semi-Dirichlet form. Moreover, we give an explicit representation of the semi-Dirichlet form.

1 Introduction

The Girsanov transformations for symmetric Markov processes have been extensively studied by many people (see [6–9, 11, 12, 14, 17] and the references therein). In these references, it is first shown that the Girsanov transformed processes are still symmetric. Then the symmetric Dirichlet forms associated with the Girsanov transformed processes are explicitly characterized. Finally, the Dirichlet forms are used to investigate some properties of the Girsanov transformed processes and the generalized Feynman–Kac semigroups. However, to our knowledge, the Girsanov transformations for non-symmetric Dirichlet processes have not been systematically studied. The aim of this paper is to investigate some properties of the Girsanov transformations for non-symmetric diffusions.

Let E be a metrizable Lusin space, *i.e.,* a space topologically isomorphic to a Borel subset of a complete separable metric space, $\mathcal{B}(E)$ the Borel σ-field of E, and m a σ-finite measure on $(E, \mathcal{B}(E))$. Let $X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ be a diffusion on E, which is assumed to be associated with a strongly local Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E; m)$ in the sense that $T_t f = P_t f\ m - a.e.$ for any $f \in B_b(E) \cap L^2(E; m)$ and $t \geq 0$, where $B_b(E)$ is the set of all bounded $\mathcal{B}(E)$-measurable functions on E, $(T_t)_{t\geq 0}$ is the L^2-semigroup corresponding to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, and $(P_t)_{t\geq 0}$ is the transition semigroup of X. It is well known that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ must be a quasi-regular Dirichlet form on $L^2(E; m)$ (cf. [15]). Let $\widehat{X} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\widehat{X}_t)_{t\geq 0}, (\widehat{P}_x)_{x\in E})$ be the dual process of X. Then $\widehat{X}$ is associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in the sense that $\widehat{T}_t f = \widehat{P}_t f\ m - a.e.$ for any $f \in B_b(E) \cap L^2(E; m)$ and $t \geq 0$, where $(\widehat{T}_t)_{t\geq 0}$ is the L^2-semigroup

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corresponding to $(\widehat{\mathcal{E}}, D(\widehat{\mathcal{E}}))$, the dual Dirichlet form of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, and $(\widehat{P}_t)_{t\geq 0}$ is the transition semigroup of $\widehat{X}$.

The notations and terminologies used in this paper follow [13,15]. We use $(\,\cdot\,,\,\cdot\,)_m$ to denote the inner product of $L^2(E;m)$. Let $U \subset E$ be an open set. Denote

$$\mathcal{D}(\mathcal{E})_{U^c} = \{f \in \mathcal{D}(\mathcal{E}) | f = 0 \quad m - a.e. \text{ on } U\}.$$

An increasing sequence $\{F_k\}_{k\geq 1}$ of closed subsets of E is called an $\mathcal{E}$-*nest* if $\bigcup_{k\geq 1} \mathcal{D}(\mathcal{E})_{F_k}$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to the $\mathcal{E}_1^{1/2}$-norm, where $\mathcal{E}_\alpha(f,f) := \mathcal{E}(f,f) + \alpha(f,f)_m$ for $\alpha \geq 0$. Let F be a subset of E. Then F is called $\mathcal{E}$-*exceptional* if there exists an $\mathcal{E}$-nest $\{F_k\}_{k\geq 1}$ such that $F \subset \bigcap_{k\geq 1} F_k^c$. A property of $(\mathcal{E}, D(E))$ or X is said to hold quasi-everywhere (q.e. for short) if it holds for any point in E except for an $\mathcal{E}$-exceptional set.

Denote by $(\mathcal{E}, \mathcal{D}(E)_e)$ the extended Dirichlet space of $(\mathcal{E}, D(E))$. Hereafter $\mathcal{D}(E)_e$ is the family of $\mathcal{B}(E)$-measurable functions u on E that is finite $m - a.e.$ and there is an $\mathcal{E}$-Cauchy sequence $\{u_n\} \subset \mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty} u_n = u \; m - a.e.$ on E. It is known that every element $u \in \mathcal{D}(E)_e$ admits an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$, where $\mathcal{E}$-quasi-continuous means that there exists an $\mathcal{E}$-nest $\{F_k\}_{k\geq 1}$ such that $\tilde{u}|_{F_k}$ is continuous on F_k for each k. For $u \in \mathcal{D}(\mathcal{E})_e$, we have the Fukushima's decompositions (cf. [13, 15, 16])

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u, \quad P_x - a.s. \text{ for } q.e. \quad x \in E,$$

and

$$\tilde{u}(X_t) - \tilde{u}(X_0) = \widehat{M}_t^u + \widehat{N}_t^u, \quad \widehat{P}_x - a.s. \text{ for } q.e. \quad x \in E,$$

where M_t^u and $\widehat{M}_t^u$ are square integrable martingales (MAFs) with respect to X and $\widehat{X}$, respectively; N_t^u and $\widehat{N}_t^u$ are continuous additive functional (CAF) of zero energy with respect to X and $\widehat{X}$, respectively.

We define a pair of local MAFs by

$$L_t := e^{M_t^u - \frac{1}{2}\langle M^u\rangle_t} \quad \text{and} \quad \widehat{L}_t := e^{\widehat{M}_t^u - \frac{1}{2}\langle \widehat{M}^u\rangle_t}.$$

Denote by ζ the lifetime of X. Then, by [18, §62] (cf. also [7,8],

$$dQ_x := L_t dP_x \quad \text{and} \quad d\widehat{Q}_x := \widehat{L}_t d\widehat{P}_x \text{ on } \mathcal{F}_t \bigcap \{t < \zeta\}, \quad x \in E$$

define unique families of probability measures $(Q_x)_{x\in E}$ and $(\widehat{Q}_x)_{x\in E}$ on $(\Omega, \mathcal{F}_\infty)$, respectively, where $\mathcal{F}_\infty := \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$. Note that X are $\widehat{X}$ are still Markov processes with the state space E under these measures. We denote them by

$$Y = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (Q_x)_{x\in E}) \quad \text{and} \quad \widehat{Y} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\widehat{Y}_t)_{t\geq 0}, (\widehat{Q}_x)_{x\in E})$$

and call them the Girsanov transformed processes of X and $\widehat{X}$, respectively. In fact, $Y_t(\omega) = X_t(\omega)$ and $\widehat{Y}_t(\omega) = \widehat{X}_t(\omega)$ for $\omega \in \Omega$. We use different notations to indicate that they are processes under the new measures $(Q_x)_{x\in E}$ and $(\widehat{Q}_x)_{x\in E}$.

If $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric Dirichlet form, then $X = \widehat{X}$ and one can show that $Y = \widehat{Y}$ is a symmetric Markov process with respect to $e^{2u}m$, whose associated symmetric Dirichlet form can be explicitly characterized via $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ (cf. [6–8]). When $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a general (non-symmetric) Dirichlet form, it is natural to consider the following two questions:

- Are the Girsanov transformed processes Y and $\widehat{Y}$ in duality with respect to $e^{2u}m$?
- Under what condition is Y associated with a (semi-)Dirichlet form? How can we characterize the associated (semi-)Dirichlet form?

This paper gives a complete answer to the first question and a partial answer to the second question.

The rest of this paper is organized as follows. In Section 2, we give a necessary and sufficient condition for Y and $\widehat{Y}$ to be in duality with respect to $e^{2u}m$. Moreover, we construct a counterexample which shows that this condition may not be satisfied. In Section 3, we first recall a recent result on perturbation of a non-symmetric Dirichlet form and its corresponding generalized Feynman–Kac semigroup, whose proof is sketched in the Appendix for the convenience of the reader. Then we give an explicit representation of the semi-Dirichlet form associated with Y under the additional assumption that N^u is of bounded variation. If, in addition, Y and $\widehat{Y}$ are in duality with respect to $e^{2u}m$, then the semi-Dirichlet form becomes a Dirichlet form.

2 Necessary and Sufficient Condition for Duality

For a fixed $\omega \in \Omega$, if $t < \zeta(\omega)$, we define the time-reversal operator r_t by: $r_t\omega(u) = \omega(t-u)$ if $0 \le u \le t$; $r_t\omega(u) = \omega(0)$ if $u > t$.

Lemma 2.1 *Let $t > 0$ and F be a non-negative $\mathcal{F}_t$-measurable function. Then*

$$E_m[F \circ r_t; t < \zeta] = \widehat{E}_m[F; t < \zeta]. \tag{2.1}$$

Proof Denote by $B_b^+(E)$ the set of all non-negative functions in $B_b(E)$. For $f_0, f_1 \in B_b^+(E)$ and $t_1 > 0$, we obtain by the duality of $\widehat{X}$ and X that

$$\widehat{E}_m[f_0(X_0)f_1(X_{t_1})] = E_m[(f_0(X_{t_1})f_1(X_0))] = E_m[(f_0(X_0)f_1(X_{t_1})) \circ r_{t_1}].$$

Assume that for $n \in \mathbf{N}$, $0 < t_1 < \cdots < t_n$ and $f_0, f_1, \dots, f_n \in B_b^+(E)$, we have

$$\widehat{E}_m[f_0(X_0)f_1(X_{t_1})\cdots f_n(X_{t_n})] = E_m[(f_0(X_0)f_1(X_{t_1})\cdots f_n(X_{t_n})) \circ r_{t_n}]. \tag{2.2}$$

Then by the Markov property, (2.2), and the duality of $\widehat{X}$ and X, we obtain that

$$\begin{aligned}
&\widehat{E}_m[f_0(X_0)\cdots f_n(X_{t_n})f_{n+1}(X_{t_{n+1}})]\\
&\quad= \widehat{E}_m[f_0(X_0)\cdots f_n(X_{t_n})\cdot \widehat{E}_{X_{t_n}}[f_{n+1}(X_{t_{n+1}-t_n})]]\\
&\quad= E_m[(f_0(X_0)\cdots f_n(X_{t_n})\cdot \widehat{E}_{X_{t_n}}[f_{n+1}(X_{t_{n+1}-t_n})]) \circ r_{t_n}]\\
&\quad= E_m[(f_0(X_{t_n})\cdots f_n(X_0)\cdot \widehat{E}_{X_0}[f_{n+1}(X_{t_{n+1}-t_n})]]\\
&\quad= \int_E f_{n+1}(x)E_x[E_{X_{t_{n+1}-t_n}}[f_n(X_0)f_{n-1}(X_{t_n-t_{n-1}})\cdots f_0(X_{t_n})]]m(dx)
\end{aligned}$$

$$
\begin{aligned}
&= \int_E f_{n+1}(x) E_x[f_n(X_{t_{n+1}-t_n}) \cdots f_0(X_{t_{n+1}})] m(dx) \\
&= E_m[f_{n+1}(X_0) f_n(X_{t_{n+1}-t_n}) \cdots f_0(X_{t_{n+1}})] \\
&= E_m[(f_0(X_0) \cdots f_n(X_{t_n}) f_{n+1}(X_{t_{n+1}})) \circ r_{t_{n+1}}].
\end{aligned}
$$

Thus (2.1) holds for any F with the form $\prod_k f_k(X_{t_k})$. By the monotone class theorem, (2.1) holds for any non-negative $\mathcal{F}_t$-measurable function F. ■

Following [10, (4.4)], we obtain the main result of this section.

Theorem 2.2 *Y and $\widehat{Y}$ are in duality with respect to $e^{2u}m$ if and only if*

$$
N_t^u + \frac{1}{2}\langle M^u \rangle_t = \widehat{N}_t^u + \frac{1}{2}\langle \widehat{M}^u \rangle_t \ \text{ for } t < \zeta, \ P_m - a.s. \tag{2.3}
$$

Proof Similar to [10, Theorem 2.1], one can show that a CAF A_t of the non-symmetric diffusion X with paths locally of bounded variation or merely of zero energy is necessarily even, *i.e.*, $A_t \circ r_t = A_t$ for all $t < \zeta$. Then (2.3) is equivalent to

$$
N_t^u + \frac{1}{2}\langle M^u \rangle_t = \widehat{N}_t^u \circ r_t + \frac{1}{2}\langle \widehat{M}^u \rangle_t \circ r_t \ \text{ for } t < \zeta, \ P_m - a.s. \tag{2.4}
$$

Set $\rho = e^{2\tilde{u}}$. Suppose that (2.3) holds. Then by (2.4),

$$
\begin{aligned}
&\frac{\rho(X_t)}{\rho(X_0)} \exp(\widehat{M}_t^u \circ r_t - \frac{1}{2}\langle \widehat{M}^u \rangle_t \circ r_t) \\
&\qquad = \exp[2\tilde{u}(X_t) - 2\tilde{u}(X_0) + \widehat{M}_t^u \circ r_t - \frac{1}{2}\langle \widehat{M}^u \rangle_t \circ r_t] \\
&\qquad = \exp[\tilde{u}(X_t) - \tilde{u}(X_0) - \widehat{N}_t^u \circ r_t - \frac{1}{2}\langle \widehat{M}^u \rangle_t \circ r_t] \\
&\qquad = \exp[\tilde{u}(X_t) - \tilde{u}(X_0) - N_t^u - \frac{1}{2}\langle M^u \rangle_t] \\
&\qquad = \exp[M_t^u - \frac{1}{2}\langle M^u \rangle_t]
\end{aligned}
$$

for $t < \zeta$, P_m-a.s. Denote by $(Q_t)_{t \geq 0}$ and $(\widehat{Q}_t)_{t \geq 0}$ the transition semigroups of Y and $\widehat{Y}$, respectively. Then by (2.1), for $f, g \in L^2(E; \rho m)$, we get

$$
\begin{aligned}
(\widehat{Q}_t f, g)_{\rho m} &= \widehat{E}_m[\rho(X_0) g(X_0) f(X_t) \widehat{L}_t] \\
&= E_m[\rho(X_t) g(X_t) f(X_0) \exp(\widehat{M}_t^u \circ r_t - \frac{1}{2}\langle \widehat{M}^u \rangle_t \circ r_t)] \\
&= E_m[\rho(X_0) f(X_0) g(X_t) \frac{\rho(X_t)}{\rho(X_0)} \exp(\widehat{M}_t^u \circ r_t - \frac{1}{2}\langle \widehat{M}^u \rangle_t \circ r_t)] \\
&= E_m[\rho(X_0) f(X_0) g(X_t) \exp(M_t^u - \frac{1}{2}\langle M^u \rangle_t)] \\
&= E_m[\rho(X_0) f(X_0) g(X_t) L_t] \\
&= (Q_t g, f)_{\rho m}.
\end{aligned}
$$

Hence Y and $\widehat{Y}$ are in duality with respect to $e^{2u}m$. This proves the "necessary part" of the theorem. The "sufficient part" of the theorem follows similarly by Lemma 2.1, the Markov property, and noting that both L_t and $\widehat{L}_t$ are multiplicative functionals. ■

Example 2.3. Let $E = \mathbf{R^d}$ and $m = dx$ be the Lebesgue measure on $\mathbf{R}^d$. Suppose that $a_{ij} \in C^1(\mathbf{R}^d)$, $1 \le i, j \le d$, satisfying the following conditions:

$$\sum_{i,j=1}^{d} \tilde{a}_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2_{\mathbf{R}^d}, \quad \forall x, (\xi_1, \xi_2, \dots, \xi_d) \in \mathbf{R^d} \text{ and } |\check{a}_{ij}(x)| \le C, \quad \forall x \in \mathbf{R^d},$$

where $\tilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$, $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji})$, $1 \le i, j \le d$, $\lambda, C \in (0, \infty)$. Define

$$\mathcal{E}(f, g) = \frac{1}{2}\sum_{i,j=1}^{d} \int_{\mathbf{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}dx, \quad \forall f, g \in C_0^\infty(\mathbf{R}^d).$$

Then $(\mathcal{E}, C_0^\infty(\mathbf{R}^d))$ is closable and its closure $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular (non-symmetric) Dirichlet form on $L^2(\mathbf{R^d}; dx)$. It is easy to see that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is strongly local.

Let $(X, \widehat{X})$ be the dual Markov processes associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and $(Y, \widehat{Y})$ the Girsanov transformed processes of $(X, \widehat{X})$ defined as in Section 1. Suppose that $u \in C_0^2(\mathbf{R}^d)$ satisfying

$$\sum_{i,j=1}^{d} \frac{\partial \check{a}_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} = 0. \tag{2.5}$$

Then Y and $\widehat{Y}$ are in duality with respect to $e^{2u}dx$.

Proof Note that

$$N_t^u = \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \frac{\partial}{\partial x_i}\Big[a_{ij}\frac{\partial u}{\partial x_j}\Big](X_s)ds, \tag{2.6}$$

$$\widehat{N}_t^u = \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \frac{\partial}{\partial x_i}\Big[a_{ji}\frac{\partial u}{\partial x_j}\Big](X_s)ds, \tag{2.7}$$

$$\langle M^u\rangle_t = \sum_{i,j=1}^{d}\int_0^t \Big(a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)(X_s)ds \text{ and} \tag{2.8}$$

$$\langle \widehat{M}^u\rangle_t = \sum_{i,j=1}^{d}\int_0^t \Big(a_{ji}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)(X_s)ds \tag{2.9}$$

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(cf. [16, Examples 5.1.6 and 5.2.13]). By (2.5) we get

$$
\begin{aligned}
(2.10)\qquad \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big[a_{ji}\frac{\partial u}{\partial x_j}\Big] &= \sum_{i,j=1}^{d} \frac{\partial a_{ji}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^{d} a_{ji}\frac{\partial^2 u}{\partial x_i \partial x_j} \\
&= \sum_{i,j=1}^{d} \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \\
&= \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big[a_{ij}\frac{\partial u}{\partial x_j}\Big].
\end{aligned}
$$

Hence we obtain by (2.10), (2.6), and (2.7) that $N_t^u = \widehat{N}_t^u$. By (2.8) and (2.9), we get $\langle M^u\rangle_t = \langle \widehat{M}^u\rangle_t$. Therefore (2.3) holds and Y and $\widehat{Y}$ are in duality with respect to $e^{2u}dx$ by Theorem 2.2. ■

Example 2.4 (Counterexample). Let $E = U$ be a bounded open subset of $\mathbf{R}^2$ and $m = dx$ the Lebesgue measure on U. Define $a_{11}(x) = 2 + x_1^2,\ a_{12}(x) = x_1,\ a_{21}(x) = 0,\ a_{22}(x) = 2 + x_2^2$,

$$
\mathcal{E}(f,g) = \frac{1}{2}\sum_{i,j=1}^{2}\int_U a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}dx, \quad \forall f,g \in H^{1,2}(U),
$$

$$
\mathcal{D}(\mathcal{E}) = H^{1,2}(U) = \Big\{ f \in L^2(U;dx) \;\Big|\; \frac{\partial f}{\partial x_i} \in L^2(U;dx), 1 \le i \le 2\Big\}.
$$

Then it is easy to see that the following conditions hold:

$$
\sum_{i,j=1}^{2} \tilde{a}_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2_{\mathbf{R}^2}, \quad \forall x, (\xi_1,\xi_2) \in \mathbf{R}^2,
$$

$$
|\check{a}_{ij}(x)| \le C, \quad \forall x \in \mathbf{R}^2,
$$

where $\tilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$, $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji})$, $1 \le i,j \le 2$, $\lambda, C \in (0,\infty)$. Hence $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a strongly local, regular (non-symmetric) Dirichlet form on $L^2(U;dx)$.

Let $(X,\widehat{X})$ be the dual Markov processes associated with $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ and $(Y,\widehat{Y})$ the Girsanov transformed processes of $(X,\widehat{X})$ defined as in Section 1. Suppose that $u \in C^2(\overline{U})$ satisfying

$$
(2.11)\qquad \frac{\partial u}{\partial x_2} \ge 0\, dx - a.e \text{ on } U \text{ and } \int_U \frac{\partial u}{\partial x_2}dx > 0.
$$

Then Y and $\widehat{Y}$ are not in duality with respect to $e^{2u}dx$.

Proof By Theorem 2.2, Y and $\widehat{Y}$ are in duality with respect to $e^{2u}dx$ if and only if (2.3) holds. Similar to Example 2.3, we have (2.6)–(2.9). Then $\langle M^u\rangle_t = \langle \widehat{M}^u\rangle_t$. Hence (2.3) is equivalent to

$$N_t^u = \widehat{N}_t^u \text{ for } t < \zeta, \quad P_{dx}\text{-a.s.} \tag{2.12}$$

Note that

$$\begin{aligned} N_t^u &= \frac{1}{2}\sum_{i,j=1}^{2}\int_0^t \frac{\partial}{\partial x_i}\Big[a_{ij}\frac{\partial u}{\partial x_j}\Big](X_s)ds \\ &= \frac{1}{2}\int_0^t\Big\{\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial x_j\partial x_i} + \Big[\sum_{i=1}^{2}\frac{\partial a_{ii}}{\partial x_i}\frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial x_2}\Big]\Big\}(X_s)ds \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \widehat{N}_t^u &= \frac{1}{2}\sum_{i,j=1}^{2}\int_0^t \frac{\partial}{\partial x_i}\Big[a_{ji}\frac{\partial u}{\partial x_j}\Big](\widehat{X}_s)ds \\ &= \frac{1}{2}\int_0^t\Big\{\sum_{i,j=1}^{2} a_{ji}\frac{\partial^2 u}{\partial x_j\partial x_i} + \sum_{i=1}^{2}\frac{\partial a_{ii}}{\partial x_i}\frac{\partial u}{\partial x_i}\Big\}(X_s)ds. \end{aligned} \tag{2.14}$$

By (2.13) and (2.14), (2.12) is equivalent to

$$\int_0^t \frac{\partial u}{\partial x_2}(X_s)\,ds = 0 \text{ for } t < \zeta, \quad P_{dx}\text{-a.s.} \tag{2.15}$$

Since $1 \in \mathcal{D}(E)$ and $\mathcal{E}(1,1) = 0$, X and $\widehat{X}$ are conservative. Then by (2.11), we get

$$\begin{aligned} E_{dx}\Big[\int_0^t \frac{\partial u}{\partial x_2}(X_s)ds\Big] &= \int_0^t\int_U p_s\Big(\frac{\partial u}{\partial x_2}\Big)dxds \\ &= \int_0^t\int_U \frac{\partial u}{\partial x_2}\widehat{p}_s 1dxds = t\int_U \frac{\partial u}{\partial x_2}dx > 0, \end{aligned}$$

where the third equality holds since $\widehat{X}$ is conservative so $\widehat{p}_s 1(x) = \widehat{E}_x[1(X_s)] = 1$. Therefore, (2.15) doesn't hold and Y and $\widehat{Y}$ are not in duality with respect to $e^{2u}dx$. ∎

3 Semi-Dirichlet Form Associated with Girsanov Transformed Process

If $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric Dirichlet form, then the Girsanov transformed process Y is symmetric with respect to $e^{2u}m$ for any $u \in \mathcal{D}(\mathcal{E})_e$. Moreover, the Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ associated with Y can be explicitly characterized via $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ (cf. [6–8]).

When $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a general (non-symmetric) Dirichlet form, the characterization of the (semi-)Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ associated with Y becomes much more difficult, since Y and $\widehat{Y}$ may not be in duality (cf. Theorem 2.2) and some powerful tools like Lyons-Zheng decomposition are mainly developed for symmetric Dirichlet forms. Our key observation in this section is that if N^u is of bounded variation, then an h-transformation of $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ can be characterized by the perturbation of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

We first recall a recent result on perturbation of non-symmetric Dirichlet forms. Let $\mu = \mu^+ - \mu^-$ be a smooth signed measure, where $\mu^+, \mu^- \in S$ and S denotes the set of all smooth measures on $(E, \mathcal{B}(E))$. Define the perturbation of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with respect to μ by

$$\begin{aligned} \mathcal{E}^\mu(f,g) &= \mathcal{E}(f,g) + \langle f,g\rangle_\mu, \quad f,g \in \mathcal{D}(\mathcal{E}^\mu),\\ \mathcal{D}(\mathcal{E}^\mu) &= \mathcal{D}(\mathcal{E}) \cap L^2(E; |\mu|), \end{aligned}$$

where $\langle f,g\rangle_\mu := \int_E fg\mu(dx)$. We use $A_t^{\mu^+}$ and $A_t^{\mu^-}$ to denote the positive CAFs (PCAFs) with the Revuz measures μ^+ and μ^-, respectively. Define $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$ and the corresponding generalized Feynmen–Kac semigroup by

$$P_t^\mu f(x) := E_x[e^{-A_t^\mu} f(X_t); t < \zeta], \quad t > 0,$$

provided the right-hand side makes sense.

Definition 3.1 A measure $\mu \in S$ is said to be of the Hardy class, denoted by $\mu \in S_H$, if there exist constants $\delta_\mu, \gamma_\mu \in (0,\infty)$ such that

$$\int_E \tilde{f}^2 d\mu \le \delta_\mu \mathcal{E}(f,f) + \gamma_\mu (f,f)_m, \quad \forall f \in \mathcal{D}(\mathcal{E}).$$

Lemma 3.2 (See [4].) *Let $\mu = \mu_1 - \mu_2 \in S - S$. Suppose that $\mu_2 \in S_H$ with $\delta_{\mu_2} < 1$. Then*

(i) *$(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is a lower semibounded bilinear form on $L^2(E; m)$ satisfying the weak sector condition.*
(ii) *$(P_t^\mu)_{t\ge 0}$ is the strongly continuous semigroup corresponding to $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$.*

For the convenience of the reader, a sketch of the proof of Lemma 3.2 is given in the Appendix.

Proposition 3.3 (See [16, Theorem 5.2.7].) *Let $u \in \mathcal{D}(\mathcal{E})_e$. Then N^u is of bounded variation if and only if there exist $\nu_1, \nu_2 \in S$ and an $\mathcal{E}$-nest $\{F_k\}_{k\ge 1}$ such that*

$$\mathcal{E}(u,v) = \int_E \tilde{v}\, d(\nu_1 - \nu_2), \forall v \in \bigcup_{k\ge 1} \mathcal{D}(\mathcal{E})_{F_k}.$$

Theorem 3.4 *Let $u \in \mathcal{D}(\mathcal{E})_e$. Suppose that N^u is of bounded variation with*

$$N_t^u = N_t^{(1)} - N_t^{(2)} \quad \textit{for } t < \zeta,$$

where $N^{(1)}, N^{(2)}$ are the PCAFs with the respective Revuz measures ν_1 and ν_2, and $\nu_2 \in S_H$ with $\delta_{\nu_2} < 1$. Define $\mu = (\frac{1}{2}\mu_{\langle u\rangle} + \nu_1) - \nu_2$, where $\mu_{\langle u\rangle}$ is the Revuz measure of $\langle M^u\rangle$. Then there exists a constant $\alpha_0 \geq 0$ such that for $\alpha \geq \alpha_0$, the α-subprocess of Y is associated with the semi-Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ on $L^2(E; e^{2u}m)$ defined by

$$\begin{aligned} \text{(3.1)} \qquad & \mathcal{Q}(f,g) = \mathcal{E}_\alpha(fe^u, ge^u) + \langle fe^u, ge^u\rangle_\mu, \quad \forall f,g \in \mathcal{D}(\mathcal{Q}), \\ & \mathcal{D}(\mathcal{Q}) = \{f \in L^2(E; e^{2u}m) \mid fe^u \in \mathcal{D}(\mathcal{E}^\mu)\}. \end{aligned}$$

Proof Denote by $(Q_t)_{t\geq 0}$ the transition semigroup of Y. That is,

$$Q_t f(x) = E_x[f(X_t)L_t; t < \zeta],$$

provided the right hand side makes sense. Set $h = e^{-u}$. Define the h-transformation for $(Q_t)_{t\geq 0}$ by

$$\text{(3.2)} \qquad Q_t^h f = h^{-1}Q_t(fh), \quad f \in L^2(E; m).$$

Note that for $f \in L^2(E; m)$,

$$\begin{aligned} Q_t^h f(x) &= h^{-1}Q_t(fh)(x) \\ &= h^{-1}(x)E_x[e^{M_t^u - \frac{1}{2}\langle M^u\rangle_t}(fh)(X_t)] \\ &= E_x[e^{-\tilde{u}(X_t)+\tilde{u}(X_0)+M_t^u-\frac{1}{2}\langle M^u\rangle_t} f(X_t)] \\ &= E_x[e^{-N_t^u - \frac{1}{2}\langle M^u\rangle_t} f(X_t)]. \end{aligned}$$

Define

$$\begin{aligned} \text{(3.3)} \qquad & \mathcal{Q}^h(f,g) = \mathcal{E}(f,g) + \langle f,g\rangle_\mu, \quad \forall f,g \in \mathcal{D}(\mathcal{Q}^h), \\ & \mathcal{D}(\mathcal{Q}^h) = \mathcal{D}(\mathcal{E}^\mu). \end{aligned}$$

By the assumption on N^u and Lemma 3.2, $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ is a lower semibounded bilinear form on $L^2(E; m)$ satisfying the weak sector condition and $(Q_t^h)_{t\geq 0}$ is its corresponding semigroup. Hence there exists a constant $\alpha_0 \geq 0$ such that for $\alpha \geq \alpha_0$ $(\mathcal{Q}_\alpha^h, \mathcal{D}(\mathcal{Q}_\alpha^h))$ is a coercive closed form on $L^2(E; m)$ with the corresponding semigroup $(e^{-\alpha t}Q_t^h)_{t\geq 0}$. By (3.2), (3.3) and (3.1), we find that $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is a coercive closed form on $L^2(E; e^{2u}m)$ with the corresponding semigroup $(e^{-\alpha t}Q_t)_{t\geq 0}$. Since $(e^{-\alpha t}Q_t)_{t\geq 0}$ is the transition semigroup of the α-subprocess of Y, $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ possesses the semi-Dirichlet property. Therefore, $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is the semi-Dirichlet form on $L^2(E; e^{2u}m)$ associated with the α-subprocess of Y. ■

Corollary 3.5 *Let $u \in \mathcal{D}(\mathcal{E})_e$. Suppose that the Girsanov transformed processes Y and $\widehat{Y}$ are in duality with respect to $e^{2u}m$ and N^u is of bounded variation with*

$$N_t^u = N_t^{(1)} - N_t^{(2)} \quad \textit{for } t < \zeta,$$

where $N^{(1)}, N^{(2)}$ are the PCAFs with the respective Revuz measures ν_1 and ν_2, and $\nu_2 \in S_H$ with $\delta_{\nu_2} < 1$. Define $\mu = (\frac{1}{2}\mu_{\langle u\rangle} + \nu_1) - \nu_2$, where $\mu_{\langle u\rangle}$ is the Revuz measure of $\langle M^u\rangle$. Then $(Y, \widehat{Y})$ are associated with the Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ on $L^2(E; e^{2u}m)$ defined by

$$(3.4) \qquad \begin{aligned} \mathcal{Q}(f,g) &= \mathcal{E}(fe^u, ge^u) + \langle fe^u, ge^u\rangle_\mu, \quad \forall f,g \in \mathcal{D}(\mathcal{Q}),\\ \mathcal{D}(\mathcal{Q}) &= \{f \in L^2(E; e^{2u}m) | fe^u \in \mathcal{D}(\mathcal{E}^\mu)\}. \end{aligned}$$

Proof Denote by $(\widehat{Q}_t)_{t\geq 0}$ the transition semigroup of $\widehat{Y}$. That is,

$$\widehat{Q}_t f(x) = \widehat{E}_x[f(X_t)\widehat{L}_t; t < \zeta],$$

provided the right hand side makes sense. Since Y and $\widehat{Y}$ are in duality with respect to $e^{2u}m$, $(Q_t)_{t\geq 0}$ and $(\widehat{Q}_t)_{t\geq 0}$ are in duality with respect to $e^{2u}m$. Then both of them are strongly continuous contraction Markovian semigroups on $L^2(E; e^{2u}m)$. Hence the $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ defined in (3.3) and the $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ defined in (3.4) are coercive closed forms on $L^2(E; m)$ and $L^2(E; e^{2u}m)$, respectively. Since $(Q_t)_{t\geq 0}$ and $(\widehat{Q}_t)_{t\geq 0}$ are (co-)associated with $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$, $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ possesses the Dirichlet property. Therefore, $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is the Dirichlet form on $L^2(E; e^{2u}m)$ associated with $(Y, \widehat{Y})$. ■

Example 3.6. Under the assumptions of Example 2.3, $(Y, \widehat{Y})$ are associated with the Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ on $L^2(\mathbf{R}^d; e^{2u}dx)$ satisfying

$$(3.5) \qquad \mathcal{Q}(f,g) = \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}e^{2u}dx + \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} \check{a}_{ij}\frac{\partial f}{\partial x_i}\frac{\partial u}{\partial x_j}ge^{2u}dx$$

for $f, g \in C_0^\infty(\mathbf{R}^d)$.

Proof From the proof of Example 2.3, we know that $(Y, \widehat{Y})$ are in duality with respect to $e^{2u}dx$ and

$$N_t^u = \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial}{\partial x_i}\Big[a_{ij}\frac{\partial u}{\partial x_j}\Big](X_s)ds, \ \langle M^u\rangle_t = \sum_{i,j=1}^d \int_0^t \Big(a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)(X_s)ds.$$

Obviously, N^u is of bounded variation. The Revuz measures of N^u and $\langle M^u\rangle$ are respectively given by

$$(3.6) \quad Lu\,dx = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)dx = \frac{1}{2}\Big[\sum_{i,j=1}^d \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^d a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\Big]dx$$

and

$$(3.7) \qquad \mu_{\langle u\rangle}(dx) = \sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx.$$

Since $a_{ij} \in C^1(\mathbf{R}^d)$, $1 \le i, j \le d$, and $u \in C_0^2(\mathbf{R}^d)$, $\left[\sum_{i,j=1}^d \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^d a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\right]$ is bounded on $\mathbf{R}^d$. Hence $\nu_2 \in S_H$ with $\delta_{\nu_2} < 1$. By Corollary 3.5, Y is associated with the Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$:

$$\mathcal{Q}(f,g) = \mathcal{E}(fe^u, ge^u) + \langle fe^u, ge^u\rangle_{\frac{1}{2}\mu_{\langle u\rangle}} + \langle fe^u, ge^u\rangle_{Ludx}, \quad f,g \in \mathcal{D}(Q). \tag{3.8}$$

Note that $\mathcal{D}(Q) = \{f \in L^2(\mathbf{R}^d; e^{2u}dx) | fe^u \in \mathcal{D}(\mathcal{E}^\mu)\}$ and $\left(\sum_{i,j=1}^d \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} a_{ij}\right)$ is bounded on $\mathbf{R}^d$ by the assumption. Then $C_0^\infty(\mathbf{R}^d) \subset L^2(\mathbf{R}^d; |\mu|)$ and thus $C_0^\infty(\mathbf{R}^d) \subset \mathcal{D}(Q)$. By the definition of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, (3.6), and (3.7), we get

$$\begin{aligned}\mathcal{E}(fe^u, ge^u) = {}& \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} e^{2u}dx + \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial u}{\partial x_j} ge^{2u}dx \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial g}{\partial x_j} fe^{2u}dx + \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} fge^{2u}dx,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\langle fe^u, ge^u\rangle_{Ludx} &= -\mathcal{E}(u, fge^{2u}) \\ &= -\sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} fge^{2u}dx - \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial f}{\partial x_j} ge^{2u}dx \\ &\quad - \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial g}{\partial x_j} fe^{2u}dx,\end{aligned} \tag{3.10}$$

$$\langle fe^u, ge^u\rangle_{\frac{1}{2}\mu_{\langle u\rangle}} = \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} fge^{2u}dx. \tag{3.11}$$

By (3.8)–(3.11), we get (3.5) for $f, g \in C_0^\infty(\mathbf{R}^d)$. ∎

Example 3.7. Under the assumptions of Example 2.4, define

$$\alpha_0 = \frac{1}{2}\max_{x\in\overline{U}}\Big\{\sum_{i=1}^2 2x_i\frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial x_2} + \sum_{i=1}^2 (2+x_i^2)\frac{\partial^2 u}{\partial x_i^2}\Big\}.$$

Then, for $\alpha \ge \alpha_0$, the α-subprocess of Y is associated with the semi-Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ on $L^2(U; e^{2u}dx)$ defined by

$$\begin{aligned}\mathcal{Q}(f,g) &= \frac{1}{2}\sum_{i,j=1}^2 \int_U a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} e^{2u}dx + \alpha\int_U fge^{2u}dx \\ &\quad + \frac{1}{2}\sum_{i,j=1}^2 \int_U \check{a}_{ij}\frac{\partial f}{\partial x_i}\frac{\partial u}{\partial x_j} ge^{2u}dx, \quad \forall f,g \in \mathcal{D}(\mathcal{Q}), \\ \mathcal{D}(\mathcal{Q}) &= H^{1,2}(U).\end{aligned}$$

Proof Similar to Example 3.6, we have

$$Lu = \frac{1}{2}\Big[\sum_{i,j=1}^{2}\frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\Big]$$
$$= \frac{1}{2}\Big[\sum_{i=1}^{2} 2x_i\frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial x_2} + \sum_{i=1}^{2}(2+x_i^2)\frac{\partial^2 u}{\partial x_i^2}\Big]$$

and (3.7). By the assumptions on u and a_{ij}, $1 \le i, j \le 2$, we find that $\mathcal{D}(\mathcal{E}^{\mu}) = \mathcal{D}(\mathcal{E}) = H^{1,2}(U)$. Since

$$\langle f, f\rangle_{\nu_2} \le \langle f, f\rangle_{|Lu|dx} \le \alpha_0 (f, f)_{dx}, \quad \forall f \in L^2(U; dx),$$

we get

$$\mathcal{E}_{\alpha}(f, f) + \langle f, f\rangle_{\mu} \ge \alpha(f, f) - \langle f, f\rangle_{\nu_2} \ge 0, \quad \forall f \in \mathcal{D}(\mathcal{E}^{\mu}), \alpha \ge \alpha_0.$$

The remainder of the proof is very similar to that of Example 3.6. The main difference is that here we need to use Theorem 3.4 instead of Corollary 3.5, since $(Y, \widehat{Y})$ are not in duality with respect to $e^{2u}dx$ (cf. Example 2.4). We omit the details. ■

4 Appendix: Sketch of the Proof of Lemma 3.2

Since $\mu_2 \in S_H$ with $\delta_{\mu_2} < 1$, there exist constants $\delta_{\mu_2}, \gamma_{\mu_2} \in (0, \infty)$ such that $\langle \tilde{f}, \tilde{f}\rangle_{\mu_2} \le \delta_{\mu_2}\mathcal{E}(f, f) + \gamma_{\mu_2}(f, f)_m$ for $f \in \mathcal{D}(\mathcal{E})$. Then, for $f \in \mathcal{D}(\mathcal{E}^{\mu})$,

$$\mathcal{E}^{\mu}(f, f) = \mathcal{E}(f, f) + \langle f, f\rangle_{\mu_1} - \langle f, f\rangle_{\mu_2}$$
$$\ge \mathcal{E}(f, f) + \langle f, f\rangle_{\mu_1} - \delta_{\mu_2}\mathcal{E}(f, f) - \gamma_{\mu_2}(f, f)_m.$$

Hence there exists a constant $\beta > 0$ such that

$$\mathcal{E}^{\mu}_{\beta}(f, f) \ge (1 - \delta_{\mu_2})\mathcal{E}(f, f). \tag{4.1}$$

Therefore, $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}^{\mu}))$ is lower semibounded.

For $f, g \in \mathcal{D}(\mathcal{E}^{\mu})$, set $\tilde{\mathcal{E}}(f, g) = \frac{1}{2}[\mathcal{E}(f, g) + \mathcal{E}(g, f)]$, $\check{\mathcal{E}}(f, g) = \frac{1}{2}[\mathcal{E}(f, g) - \mathcal{E}(g, f)]$, $\tilde{\mathcal{E}}^{\mu}(f, g) = \frac{1}{2}[\mathcal{E}^{\mu}(f, g) + \mathcal{E}^{\mu}(g, f)]$ and $\check{\mathcal{E}}^{\mu}(f, g) = \frac{1}{2}[\mathcal{E}^{\mu}(f, g) - \mathcal{E}^{\mu}(g, f)]$. Since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form, there exists a constant $K > 0$ such that

$$\check{\mathcal{E}}(f, g) \le K\mathcal{E}^{\frac{1}{2}}(f, f)\mathcal{E}^{\frac{1}{2}}(g, g) \tag{4.2}$$

(cf. [15, I.2.1]). By (4.1), (4.2), and noting that $\tilde{\mathcal{E}}^{\mu}$ is a symmetric form, we obtain that for $\alpha \geq \beta$,

$$\begin{aligned}
\mathcal{E}^{\mu}_{\alpha}(f,g) &= \tilde{\mathcal{E}}^{\mu}_{\alpha}(f,g) + \check{\mathcal{E}}^{\mu}(f,g) \\
&= \tilde{\mathcal{E}}^{\mu}_{\alpha}(f,g) + \check{\mathcal{E}}(f,g) \\
&\leq \tilde{\mathcal{E}}^{\mu}_{\alpha}(f,f)^{\frac{1}{2}} \tilde{\mathcal{E}}^{\mu}_{\alpha}(g,g)^{\frac{1}{2}} + K\mathcal{E}^{\frac{1}{2}}(f,f)\mathcal{E}^{\frac{1}{2}}(g,g) \\
&\leq \mathcal{E}^{\mu}_{\alpha}(f,f)^{\frac{1}{2}} \mathcal{E}^{\mu}_{\alpha}(g,g)^{\frac{1}{2}} + \frac{K}{1-\delta_{\mu_2}} \mathcal{E}^{\mu}_{\alpha}(f,f)^{\frac{1}{2}} \mathcal{E}^{\mu}_{\alpha}(g,g)^{\frac{1}{2}} \\
&\leq \Big(\frac{K}{1-\delta_{\mu_2}} + 1 \Big) \mathcal{E}^{\mu}_{\alpha}(f,f)^{\frac{1}{2}} \mathcal{E}^{\mu}_{\alpha}(g,g)^{\frac{1}{2}}.
\end{aligned}$$

The proof of (i) is complete.

Since $\mu_2 \in S$, similar to the proof of [2, Theorem 2.4], one finds that there exists an increasing sequence $\{F_n\}_{n\geq 1}$ of compact sets satisfying the following.

(1) $I_{F_n} \cdot \mu_2 \in S_{K_0}$, $n \geq 1$, where S_{K_0} is the set of all finite smooth measures of the Kato class.

(2) $\mu_2(E - \bigcup_{n\geq 1} F_n) = 0$.

(3) $\lim_{n\to\infty} \operatorname{Cap}(K - F_n) = 0$ for any compact set K.

Then $I_{F_n} \cdot \mu_2 \in S_H \bigcap S_{K_0}$ and $\delta_{I_{F_n}\cdot\mu_2} < 1$ since

$$\int_E \tilde{f}^2 I_{F_n} d\mu_2 \leq \int_E \tilde{f}^2 d\mu_2 \leq \delta_{\mu_2}\mathcal{E}^{\mu_1}(f,f) + \gamma_{\mu_2}(f,f)_m, \quad \forall f \in \mathcal{D}(\mathcal{E}). \tag{4.3}$$

Set $\mu_n = \mu_1 - I_{F_n} \cdot \mu_2$. By (4.3), there exists a constant $\alpha_0 \geq 0$ such that for $\alpha \geq \alpha_0$,

$$c_{\alpha}^{-1}\mathcal{E}^{\mu_1}_1(f,f) \leq \mathcal{E}^{\mu_n}_{\alpha}(f,f) \leq c_{\alpha}\mathcal{E}^{\mu_1}_1(f,f), \quad f \in \mathcal{D}(\mathcal{E}^{\mu}), \quad n \geq 1 \tag{4.4}$$

and

$$c_{\alpha}^{-1}\mathcal{E}^{\mu_1}_1(f,f) \leq \mathcal{E}^{\mu}_{\alpha}(f,f) \leq c_{\alpha}\mathcal{E}^{\mu_1}_1(f,f), \quad f \in \mathcal{D}(\mathcal{E}^{\mu}). \tag{4.5}$$

for some constant $c_\alpha > 1$ independent of n. Since $(\mathcal{E}^{\mu_1}, \mathcal{D}(\mathcal{E}^{\mu_1}))$ is a Dirichlet form by [19, Proposition 2.1.10] (cf. also [1, Proposition 3.1]), $(\mathcal{E}^{\mu_n}_{\alpha}, \mathcal{D}(\mathcal{E}^{\mu_n}))$ and $(\mathcal{E}^{\mu}_{\alpha}, \mathcal{D}(\mathcal{E}^{\mu}))$ are coercive closed forms on $L^2(E;m)$ by (4.4), (4.5), and (i).

Define

$$P^{\mu_n}_t f(x) := E_x[e^{-A^{\mu_n}_t} f(X_t); t < \zeta], \quad t > 0,$$

provided the right hand side makes sense. Then one can check that $(P^{\mu_n}_t)_{t\geq 0}$ is the strongly continuous semigroup on $L^2(E;m)$ corresponding to $(\mathcal{E}^{\mu_n}, \mathcal{D}(\mathcal{E}^{\mu_n}))$ (cf. [5, Lemma 2.8 and Theorem 2.9]). Moreover, let $(T^{\mu}_t)_{t\geq 0}$ be the strongly continuous semigroup on $L^2(E;m)$ corresponding to $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}^{\mu}))$. Then, for $f \in L^2(E;m)$, $P^{\mu_n}_t f$ converges weakly to $P^{\mu}_t f$ and converges strongly to $T^{\mu}_t f$ in $L^2(E;m)$ (cf. [4] for more details). Hence $P^{\mu}_t f = T^{\mu}_t f$ $m - a.e.$ The proof of (ii) is complete.

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