*Bull. Aust. Math. Soc.* **109** (2024), 101–109
doi:10.1017/S0004972723000175

# BOUNDS FOR FUNCTIONALS DEFINED ON A CERTAIN CLASS OF MEROMORPHIC FUNCTIONS

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(Received 14 January 2023; accepted 27 January 2023; first published online 3 March 2023)

**Abstract**

We obtain bounds for certain functionals defined on a class of meromorphic functions in the unit disc of the complex plane with a nonzero simple pole. These bounds are sharp in a certain sense. We also discuss possible applications of this result. Finally, we generalise the result to meromorphic functions with more than one simple pole.

2020 *Mathematics subject classification*: primary 30D30; secondary 30A10.

*Keywords and phrases*: analytic functions, meromorphic functions, Taylor coefficients.

## 1. Introduction and main result

We denote the set of all complex numbers by $\mathbb{C}$. Let $\mathcal{A}$ be the class of analytic functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D} \tag{1.1}$$

and $f(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$. We see that the functions in $\mathcal{A}$ must satisfy the normalisation $f(0) = 0 = f'(0) - 1$. For $f \in \mathcal{A}$, we define

$$d_f := \inf \left|\frac{f(z)}{z}\right|, \quad z \in \mathbb{D},$$

and for $f \in \mathcal{A}$ that are bounded in $\mathbb{D}$, let

$$D_f := \sup \left|\frac{f(z)}{z}\right|, \quad z \in \mathbb{D}.$$

Lewin obtained the following result.

THEOREM 1.1 (Lewin, [4]). *For $f \in \mathcal{A}$ with the expansion (1.1), $d_f \leq \exp(-|a_2|/2)$. If $f$ is bounded, then $D_f \geq \exp(|a_2|/2)$.*

The first author of this article would like to thank SERB, India, for its financial support through Core Research Grant (Ref. No. CRG/2022/001835).

In the same article, Lewin established that the estimates in Theorem 1.1 are best possible. He commented that although the bounds in Theorem 1.1 are not sharp in the case of univalent or bounded univalent functions, they nevertheless supply information which may be of help when dealing with conformal mappings (analytic and univalent mappings).

In this article, we allow the functions in $\mathcal{A}$ to possess a nonzero simple pole inside $\mathbb{D}$ and wish to see whether an analogue of Theorem 1.1 can be established after suitably defining the quantities $d_f$ and $D_f$ in this case. Therefore, we consider functions $f$ that are meromorphic having a simple pole at $z = p \in (0, 1)$ inside the unit disk $\mathbb{D}$, with the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}_p, \tag{1.2}$$

where $\mathbb{D}_p := \{z \in \mathbb{C} : |z| < p\}$ and such that $f$ does not vanish in $\mathbb{D}$ other than at the origin. Evidently, for such $f$, we have $f(0) = 0 = f'(0) - 1$. We denote the class of such functions by $\mathcal{F}(p)$. If $g$ is a meromorphic function having a simple pole at $pe^{i\beta}$, $\beta \in (0, 2\pi]$, $p \in (0, 1)$, and $g$ is nonvanishing in $\mathbb{D} \setminus \{0\}$ with $g(0) = 0$ and $g'(0) \neq 0$, then

$$f(z) = \frac{e^{-i\beta} g(ze^{i\beta})}{g'(0)} \in \mathcal{F}(p). \tag{1.3}$$

This shows that taking the pole $p$ in the interval $(0, 1)$ is sufficiently general. For $f \in \mathcal{F}(p)$, we define

$$d_p(f) := \inf_{z \in \mathbb{D}} \left| \frac{(z-p)f(z)}{z(1-pz)} \right|,$$

and if $(z - p)f$ is bounded in $\mathbb{D}$, we define

$$D_p(f) := \sup_{z \in \mathbb{D}} \left| \frac{(z-p)f(z)}{z(1-pz)} \right|.$$

These quantities can be thought of as analogous to $d_f$ and $D_f$ in [4]. The reason for multiplying $f/z$, $f \in \mathcal{F}(p)$, by the factor $(z - p)(1 - pz)$ is to make the resulting function holomorphic in $\mathbb{D}$. In addition, if $f$ has a holomorphic extension to the boundary $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ of $\mathbb{D}$, then

$$\left| \frac{(e^{i\theta} - p)f(e^{i\theta})}{e^{i\theta}(1 - pe^{i\theta})} \right| = |f(e^{i\theta})|, \quad \theta \in [0, 2\pi).$$

Thus, in such cases, finding the bounds of $d_p(f)$ and $D_p(f)$ will essentially give estimates for the distance between the origin and the image of the unit circle under $f$. In the second part of this paper, we will generalise these results to functions having more than one simple pole in $\mathbb{D}$.

We now state and prove our main result. We will adopt the main idea of the proof from [4], but as we approach the problem, we will realise that the proof itself and finding the extremal functions for which equalities hold in these estimates are not straightforward.

THEOREM 1.2. *Let $f \in \mathcal{F}(p)$ have the expansion (1.2) in $\mathbb{D}_p$. Then*

$$d_p(f) \le p\exp(-|pa_2 + p^2 - 1|/2),$$

*and if $(z-p)f$ is bounded in $\mathbb{D}$, then*

$$D_p(f) \ge p\exp(|pa_2 + p^2 - 1|/2).$$

*These bounds are best possible.*

PROOF. Let $s > 1$ be such that

$$\left|\frac{(z-p)f(z)}{pz(1-pz)}\right| \ge 1/s, \quad z \in \mathbb{D}.$$

Then we must have

$$\log\left|\frac{(z-p)f(z)}{pz(1-pz)}\right| \ge -\log s,$$

where we choose that branch of logarithm for which $\log f'(0) = 0$. A minor simplification of the above inequality yields

$$1 + \frac{\log|(z-p)/p(1-pz)|}{\log s} + \frac{\log|f(z)/z|}{\log s} \ge 0.$$

Now define

$$F(z) = 1 + \frac{\log\{(p-z)f(z)/pz(1-pz)\}}{\log s}, \quad z \in \mathbb{D},$$

which is analytic in $\mathbb{D}$ by choosing that branch of the logarithm for which $\log(f'(0)) = 0$. By virtue of the previous inequality, we have $\operatorname{Re} F(z) \ge 0$ with $F(0) = 1$. Now we can expand $F$ about the origin to get

$$F(z) = 1 + \left(\frac{a_2 + p - 1/p}{\log s}\right)z + \left(\frac{a_3 - a_2{}^2/2 + (p^2 - 1/p^2)/2}{\log s}\right)z^2 + \cdots, \quad z \in \mathbb{D}_p. \quad (1.4)$$

An application of Caratheodory's lemma (see [3]) for the function $F$ in $\mathbb{D}_p$ yields

$$\frac{|a_2 + p - 1/p|}{\log s} \le \frac{2}{p}.$$

Letting $d_p(f) = p/s$ gives the first estimate of the theorem. To obtain the second estimate of the theorem, we let

$$g(z) = \frac{(pz)^2(1-pz)^2}{(z-p)^2 f(z)}, \quad f \in \mathcal{F}(p),\ z \in \mathbb{D}.$$

Note that $g \in \mathcal{F}(p)$ as $(z-p)f$ is bounded in $\mathbb{D}$ and $g$ has the Taylor expansion

$$g(z) = z + (2/p - 2p - a_2)z^2 + \cdots, \quad z \in \mathbb{D}_p.$$

https://doi.org/10.1017/S0004972723000175 Published online by Cambridge University Press

We thus have $d_p(g)/p = p/D_p(f)$. Therefore,

$$\frac{1}{D_p(f)} = \frac{d_p(g)}{p^2} \le (1/p)\exp(-|2-2p^2-pa_2+p^2-1|/2)$$
$$= (1/p)\exp(-|pa_2+p^2-1|/2).$$

Consequently, the second inequality of the theorem follows.

The bounds obtained in the theorem are best possible in the following sense. We consider the functions $f_\alpha^\pm$ in $\mathcal{F}(p)$:

$$f_\alpha^\pm(z) = p\psi(z)\exp\Big(\pm\frac{\alpha\psi(z)}{1+\psi(z)}\Big), \quad \alpha \ge 0,\ z \in \mathbb{D},$$

where $\psi(z) = z(1-pz)/(p-z)$. A quick computation yields

$$f_\alpha^+(z) = z + \Big(\frac{\alpha}{p} + \frac{1}{p} - p\Big)z^2 + \cdots, \quad z \in \mathbb{D}_p.$$

Therefore, here $a_2 = \alpha/p + 1/p - p$, which gives $|a_2p + p^2 - 1| = \alpha$ and

$$|f_\alpha^+(e^{i\phi})| = p\exp(\alpha/2) = p\exp(|a_2p+p^2-1|/2),$$

where $\phi \in (0,\pi)\cup(\pi,2\pi)$. Again for the function $f_\alpha^-$, we have

$$f_\alpha^-(z) = z + \Big(-\frac{\alpha}{p} + \frac{1}{p} - p\Big)z^2 + \cdots, \ z \in \mathbb{D}_p,$$

which gives $a_2 = -\alpha/p + 1/p - p$ or equivalently $|a_2p + p^2 - 1| = \alpha$ and

$$|f_\alpha^-(e^{i\phi})| = p\exp(-\alpha/2) = p\exp(-|a_2p+p^2-1|/2),$$

where $\phi \in (0,\pi)\cup(\pi,2\pi)$. This shows that the estimates stated in the theorem are best possible. This completes the proof of the theorem. □

REMARK 1.3. (i) Note that the quantity $|pa_2 + p^2 - 1|$ in the bounds for $d_p(f)$ and $D_p(f)$ in Theorem 1.2, may be replaced by $|p^2(a_3 - \frac{1}{2}a_2^2) + (p^4-1)/2|$ as by Caratheodory's lemma, we also have

$$\frac{|a_3 - {a_2}^2/2 + (p^2 - 1/p^2)/2|}{\log s} \le \frac{2}{p^2},$$

for the function $F$ defined in $\mathbb{D}_p$. Furthermore, we comment here that if $pa_2 + p^2 - 1 = 0$, then we need to use the first nonvanishing coefficient in the expansion (1.4) to get the estimates for $d_p(f)$ and $D_p(f)$.

(ii) We observe that we recover Lewin's results (compare [4, Theorem A]) if we pass to the limit as $p \to 1-$ in the expression for the bounds obtained in Theorem 1.2.

We now illustrate the results obtained in Theorem 1.2 through some examples and indicate possible applications of the bounds.

EXAMPLE 1.4. Let

$$f(z) = \frac{-p\log(1+z)}{z-p}, \quad z \in \mathbb{D}.$$

We choose the branch of the logarithm such that $\log 1 = 0$. One can check that $f \in \mathcal{F}(p)$ and has the expansion

$$f(z) = z + (1/p - 1/2)z^2 + (1/p^2 - 1/2p + 1/3)z^3 + \cdots .$$

Here, $a_2 = 1/p - 1/2$ and as a result, an application of Theorem 1.2 yields

$$d_p(f) \le p\exp(-|p^2 - p/2|/2).$$

EXAMPLE 1.5. Let $f(z) = -pz\exp(z)/(z-p)$, $z \in \mathbb{D}$, with the expansion

$$f(z) = z + (1 + 1/p)z^2 + (1/2 + 1/p + 1/p^2)z^3 + \cdots , \quad z \in \mathbb{D}_p.$$

Thus, $pa_2 + p^2 - 1 = p^2 + p$ and $d_p(f) \le p\exp(-(p^2 + p)/2)$.

EXAMPLE 1.6 (Univalent case). Consider $f(z) = -zp/(z-p)(1-pz)$, $z \in \mathbb{D}$. It is a simple exercise to check that $f$ is one–one in $\mathbb{D}$ (see [1, 2]). The Taylor expansion of this function yields the second Taylor coefficient as $a_2 = p + 1/p$. Therefore, according to Theorem 1.2, we must have $d_p(f) \le p\exp(-p^2)$. Now for this function,

$$\left|\frac{(z-p)f(z)}{z(1-pz)}\right|_{|z|=1} = \frac{p}{|1-pe^{i\theta}|^2} \ge \frac{p}{(1+p)^2}.$$

Therefore, $d_p(f) \ge p/(1+p)^2$. Now, if $z = x$, $x \in (-1, 0)$, then

$$d_p(f) = \lim_{x\to -1} p/(1-px)^2 = p/(1+p)^2 < p\exp(-p^2)$$

for all $p \in (0, 1)$. Thus, the obtained bound in Theorem 1.2 is not sharp for this univalent function.

In the above three examples, it is difficult to give the exact estimate for the distance from the origin to the image of the unit circle under $f$, but nonetheless, we obtain some information about this distance.

EXAMPLE 1.7 (Existence of a zero). As an application of Theorem 1.2, we wish to investigate the existence of a zero for a meromorphic function $f$ with a nonzero pole other than at the origin. To this end, consider $p = 1/2$ and the function

$$f(z) = \frac{z + 15z^2 + iz^3 + 2z^4 - 4iz^5 + \frac{1}{5}z^6}{(1-2z)}, \quad z \in \mathbb{D}.$$

Suppose $f/z$ does not vanish in $\mathbb{D} \setminus \{0\}$. Then it is clear that $f \in \mathcal{F}(1/2)$. Expanding $f$ in a Taylor series about the origin for $|z| < 1/2$ gives

$$f(z) = z + 17z^2 + (34 + i)z^3 + \cdots .$$

Here, $a_2 = 17$. Therefore, an application of Theorem 1.2 yields

$$D_p(f) \ge p\, e^{|a_2p+p^2-1|/2} = \tfrac{1}{2}e^{31/8} = 24.08.$$

https://doi.org/10.1017/S0004972723000175 Published online by Cambridge University Press

However, then we see that

$$\left|\frac{(z-p)f(z)}{z(1-pz)}\right|_{|z|=1} = \left|\frac{(z-\frac{1}{2})(z+15z^2+iz^3+2z^4-4iz^5+\frac{1}{5}z^6)}{z(1-z/2)(1-2z)}\right|_{|z|=1} \leq 23.2.$$

This is a contradiction, and therefore $f/z$ must vanish at a nonzero point in $\mathbb{D}$.

## 2. Generalisation of the main result

In this section, we generalise Theorem 1.2 by allowing the functions in $\mathcal{F}(p)$ to have more than one nonzero simple pole in $\mathbb{D}$. This extension is possible if these poles in $\mathbb{D}$ lie on a line passing through the origin, that is, all the poles have the same argument. Thus, it will be sufficient to consider these nonzero poles in the interval $(0, 1)$ as we did for one nonzero pole in $\mathbb{D}$ (see (1.3)). More precisely, we consider functions $f$ that are meromorphic having simple poles at $z = p_1, p_2, \ldots, p_n \in (0, 1)$ inside the unit disk $\mathbb{D}$ with the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}_p, \tag{2.1}$$

where $p := \min\{p_1, p_2, \ldots, p_n\}$, $\mathbb{D}_p := \{z \in \mathbb{C} : |z| < p\}$ and $f$ does not vanish in $\mathbb{D}$ other than at the origin. For such $f$, we have $f(0) = 0 = f'(0) - 1$. We denote the class of such functions by $\mathcal{F}$. Let

$$B(z) = \prod_{i=1}^{n} \frac{z-p_i}{1-p_i z}, \quad z \in \mathbb{D}.$$

For $f \in \mathcal{F}$, we define

$$m_p(f) := \inf_{z\in\mathbb{D}} \left|\frac{B(z)f(z)}{z}\right|,$$

and if $(z-p_1)(z-p_2)\ldots(z-p_n)f$ is bounded in $\mathbb{D}$, we define

$$M_p(f) := \sup_{z\in\mathbb{D}} \left|\frac{B(z)f(z)}{z}\right|.$$

In the next theorem, we obtain estimates for $m_p(f)$ and $M_p(f)$.

THEOREM 2.1. *Let $f \in \mathcal{F}$ have the expansion (2.1) in $\mathbb{D}_p$. Then*

$$m_p(f) \leq \left(\prod_{i=1}^{n} p_i\right) \exp\left(-\frac{1}{2}p\left|a_2 + \sum_{i=1}^{n}\left(p_i - \frac{1}{p_i}\right)\right|\right),$$

*and if $(z-p_1)(z-p_2)\ldots(z-p_n)f$ is bounded in $\mathbb{D}$, then*

$$M_p(f) \geq \left(\prod_{i=1}^{n} p_i\right) \exp\left(\frac{1}{2}p\left|a_2 + \sum_{i=1}^{n}\left(p_i - \frac{1}{p_i}\right)\right|\right).$$

*These bounds are best possible.*

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PROOF. To prove this theorem, we use a similar technique to that in the proof of Theorem 1.2. Let $s > 1$ be such that

$$\left|\frac{B(z)f(z)}{z\prod_{i=1}^{n} p_i}\right| \geq 1/s, \quad z \in \mathbb{D}.$$

Therefore, we must have

$$\log\left|\frac{B(z)f(z)}{z\prod_{i=1}^{n} p_i}\right| \geq -\log s,$$

where we choose that branch of the logarithm for which $\log f'(0) = 0$. A minor simplification of this inequality yields

$$1 + \sum_{i=1}^{n} \frac{\log|(z - p_i)/p_i(1 - p_i z)|}{\log s} + \frac{\log|f(z)/z|}{\log s} \geq 0.$$

For $z \in \mathbb{D}$, we define

$$\begin{aligned} F(z) &= 1 + \frac{\log(-B(z)f(z)/z\prod_{i=1}^{n} p_i)}{\log s} \\ &= 1 + \sum_{i=1}^{n} \frac{\log((p_i - z)/p_i(1 - p_i z))}{\log s} + \frac{\log(f(z)/z)}{\log s}, \end{aligned}$$

which is analytic in $\mathbb{D}$ by choosing that branch of logarithm for which $\log(f'(0)) = 0$. By virtue of the previous inequality, we have $\operatorname{Re} F(z) \geq 0$ with $F(0) = 1$. Now we can expand $F$ about the origin to get

$$F(z) = 1 + \left(\frac{a_2 + \sum_{i=1}^{n}(p_i - 1/p_i)}{\log s}\right)z + \cdots, \quad z \in \mathbb{D}_p.$$

An application of Caratheodory's lemma for the function $F$ in $\mathbb{D}_p$ yields

$$\frac{|a_2 + \sum_{i=1}^{n}(p_i - 1/p_i)|}{\log s} \leq \frac{2}{p}.$$

Now, letting $m_p(f) = (\prod_{i=1}^{n} p_i)/s$, we obtain the first estimate of the theorem. To obtain the second estimate of the theorem, we let

$$g(z) = \frac{(z\prod_{i=1}^{n} p_i)^2}{(B(z))^2 f(z)}, \quad f \in \mathcal{F}, z \in \mathbb{D}.$$

Note that $g \in \mathcal{F}$ as $(z - p_1)(z - p_2)\ldots(z - p_n)f$ is bounded in $\mathbb{D}$ and $g$ has the Taylor expansion

$$g(z) = z + \left(-a_2 + 2\sum_{i=1}^{n}\left(\frac{1}{p_i} - p_i\right)\right)z^2 + \cdots, \quad z \in \mathbb{D}_p.$$

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We thus have $m_p(g)/(\prod_{i=1}^n p_i) = (\prod_{i=1}^n p_i)/M_p(f)$. Therefore, we deduce that

$$\begin{aligned}\frac{1}{M_p(f)} &= \frac{m_p(g)}{\left(\prod_{i=1}^n p_i\right)^2}\\ &\leq \frac{1}{\prod_{i=1}^n p_i}\exp\left(-\frac{1}{2}p\left|-a_2+2\sum_{i=1}^n\left(\frac{1}{p_i}-p_i\right)+\sum_{i=1}^n\left(p_i-\frac{1}{p_i}\right)\right|\right)\\ &= \frac{1}{\prod_{i=1}^n p_i}\exp\left(-\frac{1}{2}p\left|a_2+\sum_{i=1}^n\left(p_i-\frac{1}{p_i}\right)\right|\right).\end{aligned}$$

The above inequality follows by applying the first part of the theorem to the function $g$. Consequently, the second inequality of the theorem follows.

The bounds obtained in the theorem are best possible in the following sense. We consider the following functions in $\mathcal{F}$:

$$f_\alpha^\pm(z) = \frac{z\prod_{i=1}^n p_i}{B(z)}\exp\left(\pm\frac{\alpha\psi(z)}{1+\psi(z)}\right), \quad \alpha \geq 0, z \in \mathbb{D},$$

where $\psi(z) = z(1-pz)/(p-z)$. A little computation yields

$$f_\alpha^+(z) = z + \left(\frac{\alpha}{p}+\sum_{i=1}^n\left(\frac{1}{p_i}-p_i\right)\right)z^2 + \cdots, \quad z \in \mathbb{D}_p.$$

Here, $a_2 = \alpha/p + \sum_{i=1}^n(1/p_i - p_i)$, which in turn implies $p|a_2 + \sum_{i=1}^n(p_i - 1/p_i)| = \alpha$ and

$$|f_\alpha^+(e^{i\phi})| = \left(\prod_{i=1}^n p_i\right)\exp(\alpha/2) = \left(\prod_{i=1}^n p_i\right)\exp\left(\frac{1}{2}p\left|a_2+\sum_{i=1}^n\left(p_i-\frac{1}{p_i}\right)\right|\right),$$

where $\phi \in (0,\pi) \cup (\pi, 2\pi)$. Again for the function $f_\alpha^-$, we have

$$f_\alpha^-(z) = z + \left(-\frac{\alpha}{p}+\sum_{i=1}^n\left(\frac{1}{p_i}-p_i\right)\right)z^2 + \cdots, \quad z \in \mathbb{D}_p,$$

which gives $a_2 = -\alpha/p + \sum_{i=1}^n(1/p_i - p_i)$ or equivalently $p|a_2 + \sum_{i=1}^n(p_i - 1/p_i)| = \alpha$ and

$$|f_\alpha^-(e^{i\phi})| = \left(\prod_{i=1}^n p_i\right)\exp(-\alpha/2) = \left(\prod_{i=1}^n p_i\right)\exp\left(-\frac{1}{2}p\left|a_2+\sum_{i=1}^n\left(p_i-\frac{1}{p_i}\right)\right|\right),$$

where $\phi \in (0,\pi) \cup (\pi, 2\pi)$. This shows that the estimates stated in the theorem are best possible and completes the proof of the theorem. □

REMARK 2.2. We note that Theorem 2.1 reduces to Theorem 1.2 when $n = 1$.

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