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A NONSEPARABLE AMENABLE OPERATOR ALGEBRA WHICH IS NOT ISOMORPHIC TO A C*-ALGEBRA

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Abstract

It has been a long-standing question whether every amenable operator algebra is isomorphic to a (necessarily nuclear) C*-algebra. In this note, we give a nonseparable counterexample. Finding out whether a separable counterexample exists remains an open problem. We also initiate a general study of unitarizability of representations of amenable groups in C*-algebras and show that our method cannot produce a separable counterexample.

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1. Introduction

The notion of amenability for Banach algebras was introduced by Johnson [Jo72] in the 1970s and has been studied intensively since then (see a more recent monograph [Ru02]). For several natural classes of Banach algebras, the amenability property is known to single out the 'good' members of those classes. For example, Johnson's fundamental observation [Jo72] is that the Banach algebra $L^1(G)$ of a locally compact group G is amenable if and only if the group

G is amenable. Another example is the celebrated result of Connes [Co78] and Haagerup [Ha83] which states that a C*-algebra is amenable as a Banach algebra if and only if it is nuclear.

In this paper, we are interested in the class of *operator algebras*. By an operator algebra, we mean a (not necessarily self-adjoint) norm-closed subalgebra of $\mathbb{B}(H)$, the C*-algebra of the bounded linear operators on a Hilbert space H. It has been asked by several researchers whether every amenable operator algebra is isomorphic to a (necessarily nuclear) C*-algebra. The problem has been solved affirmatively in several special cases: for subalgebras of commutative C*-algebras [Še77], and subsequently for operator algebras generated by normal elements [CL95]; for subalgebras of compact operators [Gi06, Wi95]; for 1-amenable operator algebras [BL04, Theorem 7.4.18]; and for commutative subalgebras of finite von Neumann algebras [Ch13].

Here we give the first counterexample to the above problem. In fact, our counterexample is a subalgebra of the homogeneous C*-algebra $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$. Hence the result of [Še77] is actually quite sharp and the result of [Ch13] does not generalize to an arbitrary subalgebra of a finite von Neumann algebra.

THEOREM 1. *There is a unital amenable operator algebra $\mathcal{A}$ which is not isomorphic to a* C**-algebra. The algebra $\mathcal{A}$ is a subalgebra of $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ with density character $\aleph_1$, and is an inductive limit of unital separable subalgebras $\{\mathcal{A}_i\}_{i<\aleph_1}$, each of which is conjugated to a* C**-subalgebra of $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ by an invertible element $v_i \in \ell_\infty(\mathbb{N}, \mathbb{M}_2)$, such that $\sup_i \|v_i\|\|v_i^{-1}\| < \infty$. Moreover, for any $\varepsilon > 0$, one can choose $\mathcal{A}$ to be $(1+\varepsilon)$-amenable.*

Here, C-amenable means that the amenability constant is at most C (see [Ru02, Definition 2.3.15]). One drawback of our counterexample is that it is inevitably nonseparable, as explained by Theorem 8 below, and the existence of a separable counterexample remains an open problem. We note that if such an example exists, then there is one among the subalgebras of the finite von Neumann algebra $\prod_{n=1}^\infty \mathbb{M}_n$. Indeed, by Voiculescu's theorem [Vo91], the cone $C_0((0, 1], \mathcal{A})$ of a separable operator algebra $\mathcal{A}$ can be realized as a closed subalgebra of $\prod_{n=1}^\infty \mathbb{M}_n / \bigoplus_{n=1}^\infty \mathbb{M}_n$. The cone of $\mathcal{A}$ is amenable (see [Ru02, Exercise 2.3.6]), and its preimage $\tilde{\mathcal{A}}$ in $\prod_{n=1}^\infty \mathbb{M}_n$ is an extension of the cone by the amenable algebra $\bigoplus_{n=1}^\infty \mathbb{M}_n$; hence $\tilde{\mathcal{A}}$ is amenable (see [Ru02, Theorem 2.3.10]). $\tilde{\mathcal{A}}$ is not isomorphic to a C*-algebra, since it has $\mathcal{A}$ as a quotient and every closed two-sided ideal in a C*-algebra is automatically $*$-closed.

Note added in proof. In a recent preprint (http://arxiv.org/abs/1311.2982), L. W. Marcoux and A. Popov have proved that every abelian, amenable operator algebra is similar to an abelian C*-algebra. This subsumes the results of [Wi95, Ch13].

2. Proof of Theorem 1

Let $\mathcal{C}$ be a unital C*-algebra, Γ be a group, and $\pi : \Gamma \to \mathcal{C}$ be a representation, that is, $\pi(s)$ is invertible for every $s \in \Gamma$ and $\pi(st) = \pi(s)\pi(t)$ for every $s, t \in \Gamma$. The representation π is said to be *uniformly bounded* if $\|\pi\| := \sup_s \|\pi(s)\| < +\infty$. It is said to be *unitarizable* if there is an invertible element v in $\mathcal{C}$ such that $\mathrm{Ad}_v \circ \pi$ is a unitary representation. Here $\mathrm{Ad}_v(c) = vcv^{-1}$ for $c \in \mathcal{C}$. The element v is called a *similarity* element. A well-known theorem of Szökefalvi-Nagy, Day, Dixmier, and of Nakamura and Takeda, states that every uniformly bounded representation of an amenable group Γ into a von Neumann algebra is unitarizable. In fact the latter property characterizes amenability by Pisier's theorem [Pi07]. In particular, the operator algebra $\overline{\mathrm{span}}\, \pi(\Gamma)$ generated by a uniformly bounded representation π of an amenable group Γ is an amenable operator algebra which is isomorphic to a nuclear C*-algebra. See [Pi01, Ru02] for general information about uniformly bounded representations and amenable Banach algebras, respectively.

Let us fix the notation. Let $\mathbb{M}_2$ be the 2-by-2 full matrix algebra, $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ be the C*-algebra of the bounded sequences in $\mathbb{M}_2$, and $c_0(\mathbb{N}, \mathbb{M}_2)$ be the ideal of the sequences that converge to zero. We shall freely identify $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ with $\ell_\infty(\mathbb{N}) \otimes \mathbb{M}_2$, and $\ell_\infty(\mathbb{N}, \mathbb{M}_2)/c_0(\mathbb{N}, \mathbb{M}_2)$ with $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$, where $\mathcal{C}(\mathbb{N}) = \ell_\infty(\mathbb{N})/c_0(\mathbb{N})$. The quotient map from $\ell_\infty(\mathbb{N})$ (or $\ell_\infty(\mathbb{N}) \otimes \mathbb{M}_2$) onto $\mathcal{C}(\mathbb{N})$ (or $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$) is denoted by Q.

LEMMA 2. *Let Γ be an abelian group and $\pi : \Gamma \to \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ be a uniformly bounded representation. Then the amenable operator algebra*

$$\mathcal{A} := Q^{-1}(\overline{\mathrm{span}}\ \pi(\Gamma)) \subset \ell_\infty(\mathbb{N}, \mathbb{M}_2)$$

is isomorphic to a C**-algebra if and only if π is unitarizable.*

Proof. First of all, we observe that the operator algebra $\mathcal{A}$ is indeed amenable because it is an extension of an amenable Banach algebra $\overline{\mathrm{span}}\, \pi(\Gamma)$ by the amenable Banach algebra $c_0(\mathbb{N}, \mathbb{M}_2)$ (see [Ru02, Theorem 2.3.10]). Suppose now that π is unitarizable and $v \in \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ has the property that $\mathrm{Ad}_v \circ \pi$ is unitary. We may assume that v is positive, by taking the positive component from its polar decomposition. Since v is invertible, we can choose a representing sequence v_m, for $m \in \mathbb{N}$ of v such that each v_m is positive and moreover $1/\|v^{-1}\| \le v_m \le \|v\|$ for all m. In particular each v_m is invertible and $\|v_m\|\|v_m^{-1}\| \le \|v\|\|v^{-1}\|$ for all

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m. Now we have a representing sequence of an invertible lift $\tilde{v} \in \ell_\infty(\mathbb{N}, \mathbb{M}_2)$ of v such that $\|\tilde{v}\|\|\tilde{v}^{-1}\| = \|v\|\|v^{-1}\|$. Then $\tilde{v}\mathcal{A}\tilde{v}^{-1} = Q^{-1}(\overline{\operatorname{span}}\,(\mathrm{Ad}_v \circ \pi(\Gamma)))$ is a self-adjoint C*-subalgebra of $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$. Conversely, suppose that $\mathcal{A}$ is isomorphic to a C*-algebra, which is necessarily nuclear. Then thanks to the solution of Kadison's similarity problem for nuclear C*-algebras (see [Pi01, Theorem 7.16] or [Pi07, Theorem 1]), there is a $\tilde{v}$ in the von Neumann algebra $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ such that $\tilde{v}\mathcal{A}\tilde{v}^{-1}$ is a C*-subalgebra. Let $v = Q(\tilde{v}) \in \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$. Since $Q(\tilde{v}\mathcal{A}\tilde{v}^{-1})$ is a commutative C*-subalgebra of $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$, for every $s \in \Gamma$, the element $v\pi(s)v^{-1}$ is normal with its spectrum in the unit circle, which implies that $v\pi(s)v^{-1}$ is unitary. □

The above proof uses the fact that every (not necessarily separable) amenable C*-algebra is nuclear, as well as the solution to Kadison's similarity problem for nuclear C*-algebras. The reader may appreciate a more elementary and self-contained proof. Assume that θ is a bounded homomorphism of a unital C*-algebra $\mathcal{A}$ into $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$. We need to prove that θ is similar to a *-homomorphism. It suffices to show that every coordinate map is similar to a *-homomorphism and that the similarities are implemented by a uniformly bounded sequence v_n, for $n \in \mathbb{N}$, of operators. Consider the restriction of θ to the unitary group G of $\mathcal{A}$. At the nth coordinate we have a bounded homomorphism from G to $\mathrm{GL}(2, \mathbb{C})$. Since a bounded subgroup of $\mathrm{GL}(2, \mathbb{C})$ is included in a compact subgroup, by a standard averaging argument we find v_n such that $\mathrm{Ad}_{v_n} \circ \theta$ is a unitary representation of G. The operators v_n are easily seen to satisfy the required properties.

Proof of Theorem 1. We consider two 2-by-2 order-two invertible matrices which are not simultaneously unitarizable. For instance, let $s^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $s^1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$. Then by compactness, one has

$$\varepsilon(C) := \inf\{d(vs^0v^{-1}, \mathcal{U}) + d(vs^1v^{-1}, \mathcal{U}) : v \in \mathbb{M}_2^{-1},\ \|v\|\|v^{-1}\| \leq C\} > 0$$

for every $C > 0$. Here $\mathcal{U}$ denotes the unitary group of $\mathbb{M}_2$.

We shall need two families $\{E_i^0 : i \in \aleph_1\}$ and $\{E_i^1 : i \in \aleph_1\}$ of subsets of $\mathbb{N}$ such that: (i) $E_i^k \cap E_j^l$ is finite whenever $(i, k) \neq (j, l)$; and (ii) these two families are not *separated*, in the sense that there is no $F \subseteq \mathbb{N}$ such that both $E_i^0 \setminus F$ and $E_i^1 \cap F$ are finite for all i. The existence of such pair of families follows from [Lu47]. Luzin actually proved much more: he constructed a single family $\{E_i : i < \aleph_1\}$ of infinite subsets of $\mathbb{N}$ such that: (i) $E_i \cap E_j$ is finite whenever $i \neq j$; and (ii) whenever $X \subseteq \aleph_1$ is such that both X and $\aleph_1 \setminus X$ are uncountable, then the families $\{E_i : i \in X\}$ and $\{E_i : i \in \aleph_1 \setminus X\}$ cannot be separated (see appendix B below for Luzin's proof).

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The projections $p_i^k = Q(1_{E_i^k}) \in \mathcal{C}(\mathbb{N})$ are mutually orthogonal. For each pair (i,k), we define s_i^k in $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ by

$$s_i^k = p_i^k \otimes s^k + (1 - p_i^k) \otimes 1.$$

Let $\Gamma := \bigoplus_{i \in \aleph_1, k \in \{0,1\}} \mathbb{Z}/2\mathbb{Z}$ and $\{e_i^k\}$ be its standard basis. Then the map $e_i^k \mapsto s_i^k$ extends to a uniformly bounded representation $\pi : \Gamma \to \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ such that $\|\pi\| = \max\{\|s^0\|, \|s^1\|\}$. We claim that π is not unitarizable. Suppose for a contradiction that there is an invertible element $v \in \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ such that $\mathrm{Ad}_v \circ \pi$ is unitary. As in the proof of Lemma 2 we may assume that v is positive and find a representing sequence v_m, for $m \in \mathbb{N}$, of an invertible lift of v such that $\|v_m\|\|v_m^{-1}\| \leq \|v\|\|v^{-1}\|$ for all m. Let $\varepsilon = \varepsilon(\|v\|\|v^{-1}\|)$.

Now let $F^0 := \{m : d(v_m s^0 v_m^{-1}, \mathcal{U}) < \varepsilon/2\}$, and note that this set is disjoint from $F^1 := \{m : d(v_m s^1 v_m^{-1}, \mathcal{U}) < \varepsilon/2\}$. Therefore we have i such that $E_i^0 \setminus F^0$ is infinite or such that $E_i^1 \setminus F^1$ is infinite. If the former case applies, then

$$\limsup_{n \in E_i^0, n \to \infty} d(v_n s^0 v_n^{-1}, \mathcal{U}) \geq \varepsilon/2,$$

contradicting the assumption that v unitarizes π. The case where $E_i^1 \setminus F_1$ is infinite similarly leads to a contradiction. Thus, by Lemma 2, the preimage of $\overline{\mathrm{span}}\, \pi(\Gamma)$ in $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ is an amenable operator algebra which is not isomorphic to a C*-algebra. Its density character is equal to $\aleph_1 = |\Gamma|$.

Let Γ_i be a countable subgroup of Γ and denote the separable algebra $Q^{-1}(\overline{\mathrm{span}}\, \pi(\Gamma_i))$ by $\mathcal{A}_i$. Theorem 8 below shows that $\mathcal{A}_i$ is similar inside $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ to an amenable C*-algebra, with a similarity element v_i satisfying $\|v_i\|\|v_i^{-1}\| \leq \|\pi\|^2$. Furthermore, since every amenable C*-algebra is 1-amenable by results of Haagerup [Ha83], $\mathcal{A}_i$ is $\|\pi\|^4$-amenable. Now $\mathcal{A}$ is the inductive limit of the family $(\mathcal{A}_i)$ as Γ_i varies over all countable subgroups of Γ. Since each $\mathcal{A}_i$ is $\|\pi\|^4$-amenable, a routine argument with approximate diagonals shows that $\mathcal{A}$ is also $\|\pi\|^4$-amenable: for details see [Ru02, Proposition 2.3.17].

Finally, we explain how our example can be modified to have arbitrarily small amenability constant. For $0 < t < 1$, we keep $s^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ but replace s^1 with $s^1(t) = \begin{bmatrix} 1 & 0 \\ t & -1 \end{bmatrix}$ in our original construction. Denoting the resulting algebra by $\mathcal{A}(t)$, the previous arguments show that $\mathcal{A}(t)$ is $\|s^1(t)\|^4$-amenable, and $\|s^1(t)\|$ can be made arbitrarily close to 1. □

We note that a set-theoretical study of the cohomological nature of gaps similar to Luzin's was initiated in [Ta95].

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3. Unitarizability of uniformly bounded representations

In this section, we develop a general study of (non)unitarizability. First, we shall deal with separable C*-algebras. Let $\mathcal{A}$ be a unital C*-algebra and θ be a $*$-automorphism on $\mathcal{A}$. An element $a \in \mathcal{A}$ is called a *cocycle* if it satisfies

$$|||a||| := \sup_{n\geq 1} \left\| \sum_{k=0}^{n-1} \theta^k(a) \right\| < +\infty.$$

It is *inner* (or a coboundary) if there is $x \in \mathcal{A}$ such that $a = x - \theta(x)$. We recall that the *first bounded cohomology group* (see [Mo01]) of the $\mathbb{Z}$-module $(\mathcal{A}, \theta)$ is defined as

$$H_b^1(\mathcal{A}, \theta) = \{\text{cocycles}\}/\{\text{inner cocycles}\}.$$

When $\mathcal{A}$ is abelian and θ corresponds to a minimal homeomorphism of its spectrum then H_b^1 is trivial (see [Or00, Theorem 2.6]).

We note that every cocycle is approximately inner. Indeed, since $a_n := \sum_{k=0}^{n-1} \theta^k(a)$ satisfies $a_{n+1} = a + \theta(a_n)$, the element $x_n := n^{-1} \sum_{m=1}^{n} a_m$ satisfies $\|x_n\| \leq |||a|||$ and $\|a - (x_n - \theta(x_n))\| \leq 2n^{-1}|||a|||$. Suppose for a moment that θ is inner, $\theta = \mathrm{Ad}_u$ for a unitary element $u \in \mathcal{A}$, and $a \in \mathcal{A}$ is a cocycle. Then, $t = \begin{bmatrix} u & au \\ 0 & u \end{bmatrix}$ is an invertible element in $\mathbb{M}_2(\mathcal{A})$ such that $t^n = \begin{bmatrix} u^n & a_n u^n \\ 0 & u^n \end{bmatrix}$ for $n \geq 1$. Therefore $\sup_{n\in\mathbb{Z}} \|t^n\| \leq 1 + |||a|||$ and t gives rise to a uniformly bounded representation π_a of $\mathbb{Z}$ into $\mathcal{A}$.

LEMMA 3. *Let $\mathcal{A}$, u, a, and π_a be as above. Then the uniformly bounded representation π_a is unitarizable if and only if a is inner.*

See [Pi01, Lemma 4.5] or [MO10] for the proof of this lemma.

PROPOSITION 4. *Let $\mathcal{A}$ be a unital separable C*-algebra and θ be a $*$-automorphism of $\mathcal{A}$. Suppose that there are a (nonunital) θ-invariant C*-subalgebra $\mathcal{A}_0$, a state ϕ on $\mathcal{A}_0$, and a sequence of natural numbers $n(k)$ such that $(\phi \circ \theta^{n(k)})_{k=1}^{\infty}$ converges to 0 pointwise on $\mathcal{A}_0$. Then, $H_b^1(\mathcal{A}, \theta) \neq 0$.*

Proof. By a standard Hahn–Banach convexity argument, we construct an approximate unit $(h_n)_{n=0}^{\infty}$ of $\mathcal{A}_0$ such that $0 \leq h_n \leq 1$, $h_{n+1}h_n = h_n$, and $\|h_n - \theta(h_n)\| < 2^{-n}$ for all n. We note that $\phi'(h_n) \to 1$ for any state ϕ' on $\mathcal{A}_0$. Taking a state extension, we may assume that ϕ is defined on $\mathcal{A}$. Since $\mathcal{A}$ is separable, passing to a subsequence, we may assume that $\phi^k := \phi \circ \theta^{n(k)}$ converges pointwise to a state, say ψ, on $\mathcal{A}$.

Set $k(1) = 1$. By induction, one can find strictly increasing sequences $(m(j))_{j=1}^{\infty}$ and $(k(j))_{j=1}^{\infty}$ of natural numbers such that $\phi^{k(i)}(h_{m(j)}) > 1 - 2^{-j}$ for every $i \leq j$

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and $\phi^{k(j+1)}(h_{m(j)}) < 2^{-j}$ for every j. Let

$$x = \text{SOT-}\sum_{j=1}^{\infty}(h_{m(2j)} - h_{m(2j-1)}) \in \mathcal{A}^{**}.$$

We extend θ and ϕ on $\mathcal{A}^{**}$ by ultraweak continuity. One has $a := x - \theta(x) \in \mathcal{A}$, since it is a norm-convergent series in $\mathcal{A}_0$. By a telescoping argument, a is a cocycle.

Suppose for the sake of obtaining a contradiction that a is inner and $x - \theta(x) = y - \theta(y)$ for some $y \in \mathcal{A}$. Then, $y \in \mathcal{A}$ and $\theta(x - y) = x - y$. It follows that $\phi^{k(j)}(y) \to \psi(y)$ and $\phi^{k(j)}(x - y) = \phi(x - y)$. Hence the sequence $(\phi^{k(j)}(x))_{j=1}^{\infty}$ converges. However, for $j \geq 1$,

$$\phi^{k(2j)}(x) \geq \phi^{k(2j)}(h_{m(2j)} - h_{m(2j-1)}) \geq 1 - \frac{1}{2^{2j}} - \frac{1}{2^{2j-1}}$$

and

$$\phi^{k(2j+1)}(x) \leq \phi^{k(2j+1)}\left(\sum_{i=1}^{j} h_{m(2i)}\right) + \sum_{i=j+1}^{\infty}(1 - \phi^{k(2j+1)}(h_{m(2i-1)})) \leq \frac{1}{4}.$$

Hence, the sequence $(\phi^{k(j)}(x))_{j=1}^{\infty}$ does not converge, and we have a contradiction. □

Examples of $\mathcal{A}_0$, ϕ and θ as in the statement of Proposition 4 are the ideal $\mathbb{K}$ of compact operators on $\mathbb{B}(\ell_2(\mathbb{Z}))$, any one of its states, and the bilateral shift on $\ell_2(\mathbb{Z})$.

LEMMA 5. *For every unital separable* C^**-algebra* $\mathcal{A}$ *which is not of type* I, *there is a unitary element* $u \in \mathcal{A}$ *such that* $H_b^1(\mathcal{A}, \mathrm{Ad}_u) \neq 0$.

Proof. Let z be the bilateral shift on $\ell_2(\mathbb{Z})$ and take a self-adjoint element $h \in \mathbb{B}(\ell_2(\mathbb{Z}))$ such that $z = \exp(\sqrt{-1}h)$. Let $\mathcal{C} \subset \mathbb{B}(\ell_2(\mathbb{Z}))$ be the unital C^*-subalgebra generated by $\mathbb{K}$ and h, and let ϕ_0 be the vector state at δ_0. Since $\mathcal{C}$ is an extension of a commutative C^*-algebra by $\mathbb{K}$, it is nuclear. By Kirchberg's theorem and Glimm's theorem in tandem [Ki95, Corollary 1.4(vii)], there are a unital C^*-subalgebra $\mathcal{A}_1$ of $\mathcal{A}$ and a surjective $*$-homomorphism π from $\mathcal{A}_1$ onto $\mathcal{C}$. Let $g \in \mathcal{A}_1$ be a self-adjoint lift of h and let us have $u := \exp(\sqrt{-1}g) \in \mathcal{A}_1$, which is a unitary lift of z. Then $\mathcal{A}_0 = \pi^{-1}(\mathbb{K})$ is an Ad_u-invariant subalgebra and the state $\phi = \phi_0 \circ \pi$ satisfies $\phi \circ (\mathrm{Ad}_u)^n \to 0$ pointwise on $\mathcal{A}_0$. Hence the result follows from Proposition 4. □

Combining Lemmas 5 and 3, we arrive at the following theorem.

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THEOREM 6. *For every unital separable* C*-*algebra* $\mathcal{A}$ *which is not of type* I, *there is a uniformly bounded representation of* $\mathbb{Z}$ *into* $\mathbb{M}_2(\mathcal{A})$ *which is not unitarizable.*

Now, we shall deal with nonseparable C*-algebras. Our approach uses model theory of metric structures and the extension of Pedersen's techniques [Pe88] as presented in [FH13]. The following is [FH13, Definition 1.1], with a misleading typo corrected.

DEFINITION 7. Given a C*-algebra $\mathcal{M}$, a *degree-one *-polynomial* with coefficients in $\mathcal{M}$ is a linear combination of terms of the form axb, ax^*b and a with a, b in $\mathcal{M}$. A C*-algebra $\mathcal{M}$ is said to be *countably degree-one saturated* if for every countable family of degree-one $*$-polynomials $P_n(\bar{x})$ with coefficients in $\mathcal{M}$ and variables x_m, for $m \in \mathbb{N}$, and every family of compact sets $K_n \subset \mathbb{R}$, for $n \in \mathbb{N}$, the following are equivalent (writing $\bar{b}$ for $(b_1, b_2, \ldots)$ and $\mathcal{M}_{\leq 1}$ for the closed unit ball of $\mathcal{M}$).

(1) There are $b_m \in \mathcal{M}_{\leq 1}$, for $m \in \mathbb{N}$, such that $\|P_n(\bar{b})\| \in K_n$ for all n.

(2) For every $N \in \mathbb{N}$ there are $b_m \in \mathcal{M}_{\leq 1}$, for $m \in \mathbb{N}$, such that

$$\operatorname{dist}(\|P_n(\bar{b})\|, K_n) \leq \frac{1}{N}$$

for all $n \leq N$.

A type $\{P_n(\bar{x}) \in K_n : n \in \mathbb{N}\}$ satisfying (1) is said to be *realized* in $\mathcal{M}$ and a type satisfying (2) is said to be *consistent* with (or *approximately finitely realized in*) $\mathcal{M}$. Coronas of σ-unital C*-algebras, in particular the Calkin algebra $\mathcal{Q}(\ell_2)$ and $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$, as well as ultraproducts associated with nonprincipal ultrafilters on $\mathbb{N}$, are countably degree-one saturated [FH13, Theorem 1.4]. In each of these cases, given a consistent type, a realization $\bar{b}$ is assembled from the approximate realizations $\bar{b}^n$, for $n \in \mathbb{N}$, and a carefully chosen, appropriately quasicentral approximate unit e_n, for $n \in \mathbb{N}$, as $\bar{b} = \sum_n (e_n - e_{n+1})^{1/2} \bar{b}^n (e_n - e_{n+1})^{1/2}$. See [FH13] for details and more examples of countably degree-one saturated C*-algebras.

THEOREM 8. *Let* $\mathcal{M}$ *be a unital countably degree-one saturated* C*-*algebra. Then, every uniformly bounded representation* $\pi : \Gamma \to \mathcal{M}$ *of a countable amenable group* Γ *into* $\mathcal{M}$ *is unitarizable. Moreover a similarity element* v *can be chosen such that it satisfies* $\|v\|\|v^{-1}\| \leq \|\pi\|^2$.

Proof. The proof is analogous to the standard one (see [Pi01, Theorem 0.6]), modulo applying countable degree-one saturation. Consider the type in variable

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x over $\mathcal{M}$ consisting of conditions $\|x - x^*\| = 0$, $\|x\| \leq \|\pi\|^2$, $\| \|\pi\|^2 - x\| \leq \|\pi\|^2 - \|\pi\|^{-2}$, and $\|\pi(s)x\pi(s)^* - x\| = 0$ for all $s \in \Gamma$.

We now check that this type is consistent. Let $(F_n)_{n=1}^{\infty}$ be a Følner sequence of finite subsets of Γ. Then,

$$h_n = \frac{1}{|F_n|} \sum_{t \in F_n} \pi(t)\pi(t)^*,$$

are positive elements in $\mathcal{M}$ such that $\|\pi\|^{-2} \leq h_n \leq \|\pi\|^2$ and

$$\|\pi(s)h_n\pi(s)^* - h_n\| \leq \frac{|F_n \triangle sF_n|}{|F_n|}\|\pi\|^2 \to 0$$

for every $s \in \Gamma$. Hence this type is consistent and by countable degree-one saturation there is $h \in \mathcal{M}$ which realizes it. Therefore we have $h = h^*$, $\|h\| \leq \|\pi\|^2$, $\| \|\pi\|^2 - h\| \leq \|\pi\|^2 - \|\pi\|^{-2}$, and $\pi(s)h\pi(s)^* = h$ for every $s \in \Gamma$. It follows that h is a positive element such that $\|\pi\|^{-2} \leq h \leq \|\pi\|^2$ and the invertible elements $h^{-1/2}\pi(s)h^{1/2}$ satisfy

$$(h^{-1/2}\pi(s)h^{1/2})(h^{-1/2}\pi(s)h^{1/2})^* = h^{-1/2}\pi(s)h\pi(s)^*h^{-1/2} = 1,$$

that is , $h^{-1/2}\pi(s)h^{1/2}$ are unitary. □

Theorem 8 shows that the method used in the proof of Theorem 1 cannot be used to produce a separable counterexample.

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Appendix A. A correction for [Ch13]

We take the opportunity to fill a small gap in [Ch13]. The main result of that paper is only proved for commutative, amenable subalgebras of σ-finite, finite von Neumann algebras. It is then stated in [Ch13] that the general case follows from the σ-finite one because any finite von Neumann algebra $\mathcal{M}$ decomposes as a direct product $\prod_i \mathcal{M}_i$ where each $\mathcal{M}_i$ is σ-finite. However, the example of the present paper shows that similarity to a C*-algebra is not preserved by taking inductive limits, even with a uniform bound on the similarity elements, so more justification is needed. Instead, we may argue as follows. Let $\mathcal{A}$ be an amenable subalgebra of $\mathcal{M}$ and let $\mathcal{A}_i$ be its image under the projection $\mathcal{M} \to \mathcal{M}_i$. Applying the main result of [Ch13] to each $\mathcal{A}_i$, we obtain a uniformly bounded family $v_i \in \mathcal{M}_i$ such that $v_i \mathcal{A}_i v_i^{-1}$ is a commutative C*-subalgebra of $\mathcal{M}_i$. Take v to be the direct product of the v_i. Then $v\mathcal{A}v^{-1}$ is an amenable subalgebra of the commutative C*-algebra $\prod_i v_i \mathcal{A}_i v_i^{-1}$, and hence by [Še77] it is self-adjoint.

Appendix B. A construction of Luzin's gap

For the reader's convenience we prove Luzin's theorem. Following von Neumann, we identify $n \in \mathbb{N}$ with the set $\{0, 1, \ldots, n-1\}$. We construct a family E_i, for $i < \aleph_1$, of infinite subsets of $\mathbb{N}$ such that:

(1) $E_i \cap E_j$ is finite whenever $i \neq j$; and

(2) for every i and every $m \in \mathbb{N}$ the set $\{j < i : E_j \cap E_i \subseteq m\}$ is finite.

The construction is by recursion. For a finite i let $E_i = \{2^i(2k+1) : k \in \mathbb{N}\}$. Assume that $i < \aleph_1$ is infinite and the sets E_j, for $j < i$, were chosen to satisfy the requirements. Since i is countable, we can re-enumerate E_j, for $j < i$ as F_n, for $n \in \mathbb{N}$.

Now let $k(0) = 0$ and $k(n) = \min F_n \setminus (k(n-1) \cup \bigcup_{l<n} F_l)$ for $n \geq 1$. The sequence $\{k(n)\}$ is strictly increasing and $k(n) \in F_l$ implies $n \leq l$. Therefore $E_i = \{k(n) : n \in \mathbb{N}\}$ is infinite and $E_i \cap F_n \subseteq \{k(0), \ldots, k(n)\}$ is finite for all n. Finally, for any $m \in \mathbb{N}$ the set $\{n \in \mathbb{N} : F_n \cap E_i \subseteq m\} \subseteq \{n : k(n) < m\}$ is finite.

This describes the recursive construction of a family E_i, for $i < \aleph_1$, satisfying (1) and (2).

We claim that for any $X \subseteq \aleph_1$ such that X and $\aleph_1 \setminus X$ are uncountable the families $\{E_i : i \in X\}$ and $\{E_i : i \in \aleph_1 \setminus X\}$ cannot be separated. Assume otherwise, and fix $F \subseteq \mathbb{N}$ separating them. Since $E_i \setminus F$ is finite for all $i \in X$, there is an $m \in \mathbb{N}$ such that $X' = \{i \in X : E_i \setminus F \subseteq m\}$ is uncountable. By increasing m if necessary we can assure that $Y' = \{i \in \aleph_1 \setminus X : E_i \cap F \subseteq m\}$ is also uncountable.

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Pick $i \in Y'$ such that $X'' = \{j \in X' : j < i\}$ is infinite. Then for each $j \in X''$ we have $E_j \cap E_i \subseteq (E_j \setminus F) \cup (E_i \cap F) \subseteq m$. But this contradicts (2).

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