ON A TYPE OF MATRIX RING

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In this note we discuss a type of matrix ring that has nice properties concerning the injectivity and quasi-injectivity of one-sided ideals.

1. PRELIMINARIES

We consider associative rings with identity, and all modules are unitary. A module is said to be uniserial if its submodules are linearly ordered by inclusion. The uniserial modules encountered here will also be both Artinian and Noetherian and so have a (composition) length. A ring $R$ is called serial if both $R_R$ and $R_R$ are direct sums of finitely many uniserial modules.

Recall that a ring $R$ is called a right $V$-ring if all simple right $R$-modules are injective. If all right ideals of the ring $R$ are actually two-sided then $R$ is called a right duo ring.

For other undefined terminology, we refer the reader to the text [2] by Dung, Huynh, Smith and Wisbauer.

2. THE MATRIX RING

Here we define a type of matrix ring which was introduced first by Ivanov [4] in his study of the structure of non-local rings whose right ideals are quasi-injective. Since then these rings have proven to be very useful for finding examples or counter-examples of certain classes of rings (see, for example, Beidar, Fong, Ke and Jain[1], Huynh and Rizvi [3]).

Let $T$ be a ring having a two-sided ideal $M$ such that $D = T/M$ is a division ring. Let

$$V = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \subset \left\{ \begin{pmatrix} d & x \\ 0 & d \end{pmatrix} : d, x \in D \right\}.$$Then $V$ is a $D$-bimodule with $\dim_D(V) = \dim(V_D) = 1$ and $V \cdot V = 0$. Moreover, $VT = TV = V$. 

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Let \( n \in \mathbb{N}, \) with \( n \geq 3. \) We consider the \( n \times n \) matrix ring \( R \) of the form:

\[
R = \begin{pmatrix}
D & V & 0 & \cdots & 0 & 0 \\
0 & D & V & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & D & V \\
0 & 0 & 0 & \cdots & 0 & T
\end{pmatrix}
\]

We let \( R_i \) (respectively \( C_i \)) denote the right (respectively left) ideal of \( R \) which has the same \( i \)th row (resp. \( i \)th column) as \( R \), but all other rows (respectively columns) are zero. Then each \( R_i \) is a uniserial right \( R \)-module of length 2 for \( 1 \leq i \leq n-1 \), each \( C_i \) is a uniserial left \( R \)-module of length 2 for \( 1 < i \leq n-1 \), while \( C_1 \) is a simple left \( R \)-module.

Part (1) of Theorem 1 below was obtained also in [1], where this type of ring was used to describe rings in which all right ideals are quasi-injective. However the proofs in [1] involve complicated computation using elements. One might hope that the simple arguments we use here can streamline parts of [1] as well as provide a better understanding of this type of matrix ring.

**Theorem 1.** Let \( T \) and \( R \) be as above. Then

1. \( R \) is never left self-injective.
2. \( Q := R_1 \oplus \cdots \oplus R_{n-1} \) is a quasi-injective right \( R \)-module and \( P := C_2 \oplus \cdots \oplus C_{n-1} \) is a quasi-injective left \( R \)-module.
3. If \( T \) is a right nonsingular, right self-injective, right \( V \)-ring, and \( M \) is essential in \( TT \), then \( R \) is a right self-injective ring.
4. \( T \) is indecomposable as a ring if and only if \( R \) is indecomposable as a ring.
5. The ring \( R \) is never indecomposable if \( T \) is a von Neumann regular right \( V \)-ring which is not a division ring.

**Proof:**

1. Write \( RR = C_1 \oplus C_2 \oplus \cdots \oplus C_n \). If \( RR \) is injective, then \( R_1 \) is injective. However \( C_2 \) is a local left \( R \)-module of length 2 and \( \text{Soc}(C_2) \cong C_1 \). Hence \( \text{Soc}(C_2) \) splits in \( C_2 \), a contradiction. This proves (1).

2. Let \( M^* \) be the subset of \( C_n \) consisting of those matrices where the \((n,n)\)th entry is in \( M \). Since \( VM = 0 \), \( M^* \) is a two-sided ideal of \( R \). Using the decomposition \( RR = C_1 \oplus \cdots \oplus C_n \), we get \( R/M^* \cong C_1 \oplus \cdots \oplus C_{n-1} \oplus (C_n/M^*) \), a direct sum of uniserial left \( R/M^* \)-modules of length at most 2. (Clearly \( C_n/M^* \) is uniserial of length 2.) Similarly we have \( R/M^* \cong R_1 \oplus \cdots \oplus R_{n-1} \oplus (R_n/M^*) \), a direct sum of uniserial right \( R/M^* \)-modules of length at most 2. (Note that \( R_n/M^* \) is a simple right \( R \)-module.) Hence \( R/M^* \) is an Artinian serial ring with Jacobson radical square zero. This implies
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that every uniform right (or left) \((R/M^*)\)-module of length 2 is injective (see [2, 13.5]). Thus \(Q_R \) and \(R_P\) are quasi-injective, proving (2).

(3) Assume that \(T\) is a right nonsingular, right self-injective, and right V-ring, such that \(M\) is essential in \(T_T\). Then, by (2), we have \(R_R = Q \oplus R_n\), a direct sum of two quasi-injective right \(R\)-modules. Thus to prove the right self-injectivity of \(R\) we need only to show that \(Q\) is \(R_n\)-injective and \(R_n\) is \(Q\)-injective.

First note that the right \(R\)-module \(\text{Soc}(Q) \oplus M^*\) is essential in \(R_R\) and moreover \(\text{Soc}(Q) \cdot (\text{Soc}(Q) \oplus M^*) = 0\). This shows that \(\text{Soc}(Q)\) is a singular right \(R\)-module. Then, since \(R_n\) is a nonsingular right \(R\)-module, there are no nonzero homomorphisms \(Q_R \to R_n\) and so, trivially, \(R_n\) is \(Q\)-injective.

Next note that, as a right \(R\)-module, \(R_n\) is a \(V\)-module, that is, every \(R_n\)-subgenerated simple module is \(R_n\)-injective. Now let \(U\) be a submodule of \(R_n\) and let \(\varphi\) be a homomorphism of \(U\) to \(Q\). Since \(U/\text{Ker}\varphi\) is isomorphic to a submodule of \(Q\), \(\text{Soc}(U/\text{Ker}\varphi)\) is finitely generated, and hence \((R_n/\text{Ker}\varphi)\)-injective. Therefore \(\text{Soc}(U/\text{Ker}\varphi)\) splits in \(R_n/\text{Ker}\varphi\). Since \(\text{Soc}(U/\text{Ker}\varphi)\) is essential in \(U/\text{Ker}\varphi\), it follows that \(U/\text{Ker}\varphi = \text{Soc}(U/\text{Ker}\varphi)\). Thus there is a submodule \(W \subseteq R_n\) containing \(\text{Ker}\varphi\) such that \(R_n/\text{Ker}\varphi = (U/\text{Ker}\varphi) \oplus (W/\text{Ker}\varphi).\) This implies that we can extend \(\varphi\) to a homomorphism of \(R_n\) to \(Q\), proving that \(Q\) is \(R_n\)-injective. This establishes the right self-injectivity of \(R\).

(4) Assume that \(T\) is indecomposable as a ring. Since \(D = T/M\), we have

\[
R/M^* \cong \begin{pmatrix}
D & V & 0 & \ldots & 0 & 0 \\
0 & D & V & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ldots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & D & V \\
0 & 0 & 0 & \ldots & 0 & D
\end{pmatrix}
\]

It is easy to see that this matrix ring is right and left Artinian and indecomposable as a ring. Hence the ring \(R/M^*\) is also indecomposable.

Now, if \(R = A \oplus B\) is a ring decomposition with \(A\) and \(B\) both nonzero, then, since \(M^* = (A \cap M^*) \oplus (B \cap M^*)\), we have \(R/M^* \cong [A/(A \cap M^*)] \oplus [B/(B \cap M^*)]\). Hence one of these latter direct summands of \(R/M^*\) must be zero, say \(A/(A \cap M^*) = 0\). Then \(A = A \cap M^*\), and so \(A \subseteq M^* \subseteq R_n\). Thus \(R_n = A \oplus (B \cap R_n)\), a ring direct decomposition of \(R_n\) with \(B \cap R_n \neq 0\). This is a contradiction, because \(T \cong R_n\) and, by our assumption, \(T\) is an indecomposable ring.

Conversely, suppose that \(R\) is ring-indecomposable. Let’s assume that \(T = U \oplus W\), a ring-direct sum with \(U\) and \(W\) both nonzero. Then, adapting our definition of \(M^*\), we can define corresponding right ideals \(U^*\) and \(W^*\) in \(R\) to give a ring-direct sum
\[ R_n = U^* \oplus W^* \] for \( R_n \). It follows that \( R_n/M^* \cong \left[ U^*/(U^* \cap M^*) \right] \oplus \left[ W^*/(W^* \cap M^*) \right] \).

Since \( R_n/M^* \cong T/M \), a division ring, one of the two summands of \( R_n/M^* \) must be zero, say \( U^*/(U^* \cap M^*) = 0 \). Then \( U^* \subseteq M^* \), and so \( U \subseteq M \). It follows that \( VU = 0 \).

Hence \( V = VT = V(U \oplus W) = VU \oplus VW = VW \). Consequently, we get the following ring-direct decomposition of \( R \) in which the first summand is \( U^* \):

\[
R = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & U
\end{pmatrix} \oplus \begin{pmatrix}
D & V & 0 & \cdots & 0 & 0 \\
0 & D & V & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & D & V \\
0 & 0 & 0 & \cdots & 0 & W
\end{pmatrix}
\]

This contradiction establishes (3).

(4) This is clear from (3), because every von Neumann regular right duo ring is decomposable if and only if it is not a division ring.

We remark that in Theorem 1, statement (3) is not true if \( M \) is not essential in \( T_T \). Moreover, if \( M \) is not essential in \( T_T \), then the ring \( R \) is right nonsingular if \( T \) is right nonsingular.

From the proof of (3) we see that if \( M \) is essential in \( T_T \), then \( R \) contains an indecomposable ring-direct summand if and only if \( T \) has an indecomposable ring-direct summand.

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