NEW INEQUALITIES FOR PLANAR CONVEX SETS
WITH LATTICE POINT CONSTRAINTS

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We obtain new inequalities relating the inradius of a planar convex set with interior containing no point of the integral lattice, with the area, perimeter and diameter of the set. By considering a special sublattice of the integral lattice, we also obtain an inequality concerning the inradius and area of a planar convex set with interior containing exactly one point of the integral lattice.

1. INTRODUCTION

Let \( K \) be a compact, planar convex set with interior \( K^\circ \), and having area \( A = A(K) \), perimeter \( p = p(K) \), diameter \( d = d(K) \) and inradius \( r = r(K) \). Let \( \Gamma \) denote the integral lattice and let \( G(K^\circ, \Gamma) \) denote the number of points of \( \Gamma \) in \( K^\circ \). We prove new inequalities relating \( A, p, d \) and \( r \).

**Theorem 1.** Let \( K \) be a compact, planar, convex set with \( G(K^\circ, \Gamma) = 0 \). Then

\[
(2r - 1)A \leq 2\left(\sqrt{2} - 1\right) \approx 0.828,
\]

with equality when and only when \( K \) is congruent to the diagonal square shown in Figure 1.

**Figure 1.**
COROLLARY 1. Let $K$ be a compact, planar, convex set with $G(K^o, \Gamma) = 1$. Then

\[(2r - \sqrt{2})A \leq 4\left(2 - \sqrt{2}\right) \approx 2.343,\]

with equality when and only when $K$ is the square shown in Figure 2.

THEOREM 2. Let $K$ be a compact, planar, convex set with $G(K^o, \Gamma) = 0$. Then

\[(3) \quad (2r - 1)|A - 1| < \frac{1}{2},\]
\[(4) \quad (2r - 1)|p - 4| < 2,\]
\[(5) \quad (2r - 1)(d - 1) < 1.\]

The limiting infinite strip shows that the stated bounds are best possible.

2. PROOFS OF THEOREM 1 AND COROLLARY 1

We first prove two useful lemmas.

**Lemma 1.** Let $X_I$ be the Steiner symmetral of $X$ with respect to the line $l$. Then \(r(X_I) \geq r(X)\).

**Proof:** We first show that if $K \subseteq X$, then $K_I \subseteq X_I$. Let $PQ$ be any chord of $K$ perpendicular to $l$. Since $K \subseteq X$, the line $PQ$ intersects $X$ in a chord $AB$ with $|PQ| \leq |AB|$. Now Steiner symmetrisation maps chord $PQ$ to a chord $P'Q'$ on the line $PQ$, and having its midpoint on $l$ (see for example [1, p.90]). Similarly, the chord $AB$ is mapped to the chord $A'B'$ on the line $PQ$ and having midpoint on $l$. Since $|PQ| \leq |AB|$, the chord $P'Q'$ is a subset of the chord $A'B'$. Hence $K_I \subseteq X_I$.

Now let $C$ be an incircle of $X$. Then $C \subseteq X$ and $C_I \subseteq X_I$. But $C_I$ is congruent to $C$. It follows that $X_I$ contains a circle of radius $r(X)$. Therefore \(r(X_I) \geq r(X)\). \(\square\)

**Lemma 2.** Let $K$ be a compact, planar, convex set with $G(K^o, \Gamma) = 0$. Then there is a compact convex set $K_*$ with $G(K_*^o, \Gamma) = 0$ satisfying the following conditions:

(a) \(A(K_*) = A(K), \quad r(K_*) \geq r(K)\),
(b) $K_*$ is symmetric about the lines $x = 1/2, y = 1/2$. 


**Proof:** We use Steiner symmetrisation to obtain the set $K_*$. We first symmetrise $K$ with respect to the line $x = 1/2$ to obtain the set $K_1$. We recall that Steiner symmetrisation preserves convexity and areas so that $K_1$ is a convex set with $A(K_1) = A(K)$. Furthermore, by Lemma 1, $r(K_1) \geq r(K)$.

We now show that $G(K_1^\circ, \Gamma) = 0$. Since $G(K_1^\circ, \Gamma) = 0$, $K_1^\circ$ intersects the line $y = k$, where $k$ is an integer, either in the empty set or in a line segment of length at most 1. Hence the symmetric set $K_1^\circ$ intersects the line $y = k$ either in the empty set or between the points $(0, k)$ and $(1, k)$. Clearly, $G(K_1^\circ, \Gamma) = 0$.

We now symmetrise $K_1$ with respect to the line $y = 1/2$ to obtain $K_*$. Using the same arguments as above, we have $A(K_*) = A(K_1)$, $r(K_*) \geq r(K_1)$ and $G(K_1^\circ, \Gamma) = 0$. Hence $A(K_*) = A(K)$ and $r(K_*) \geq r(K)$. By construction, $K_*$ is symmetric about the lines $x = 1/2$ and $y = 1/2$ and the lemma is proved.

Let $f(K) = (2r(K) - 1)A(K)$. By Lemma 2 we have $f(K) \leq f(K_*)$. It therefore suffices to prove Theorem 1 for sets $K$ which are symmetric about the lines $x = 1/2$ and $y = 1/2$.

To fully utilise the symmetry of $K$ about the lines $x = 1/2$ and $y = 1/2$, we move the origin to the point $(1/2, 1/2)$. If $r \leq 1/2$, (1) is trivially true. Hence we may assume that $r > 1/2$. Since $K^\circ$ does not contain the points $P_1(1/2, 1/2)$, $P_2(-1/2, 1/2)$, $P_3(-1/2, -1/2)$ and $P_4(1/2, -1/2)$, it follows by the convexity of $K$ that for each $i = 1, \ldots, 4$, $K$ is bounded by a line $l_i$ through the point $P_i$, with $l_1$ and $l_3$ having negative slope and $l_2$ and $l_4$ having positive slope. Furthermore since $K$ is symmetric about the coordinate axes, $K$ is contained in a rhombus $Q$ determined by the lines $l_i$, $i = 1, \ldots, 4$. Since $K \subseteq Q$, $A(K) \leq A(Q)$ and $r(K) \leq r(Q)$ we have $f(K) \leq f(Q)$. It is therefore sufficient to maximise $f(K)$ over the set of all rhombi, $K = Q$, determined by the lines $l_i$, $i = 1, \ldots, 4$ (see Figure 3).

![Figure 3](https://www.cambridge.org/core/asset/1234567890abcdefg)

Let side $l_1$ make an acute angle of $\alpha$ with the $x$-axis and let it intercept the $x$ and $y$ axes in the points $X(x, 0)$ and $Y(0, y)$ respectively. Since $l_1$ passes through...
\[(1/2,1/2),\text{ similar triangles give}\]
\[
\frac{y}{x} = \frac{1/2}{x - 1/2},
\]
that is,
\[
\frac{1}{x} + \frac{1}{y} = 2.
\]
Multiplying both sides of the equation by \(r\), we get
\[
2r = \frac{r}{x} + \frac{r}{y} = \sin \alpha + \cos \alpha.
\]
Now
\[
A = 4A(\triangle OXY)
\]
\[
= 2xy
\]
\[
= \frac{2r^2}{\sin \alpha \cos \alpha}
\]
\[
= \frac{4r^2}{(\sin \alpha + \cos \alpha)^2 - 1}
\]
\[
= \frac{4r^2}{4r^2 - 1}
\]
\[
= 1 + \frac{1}{4r^2 - 1}.
\]
Hence
\[
(6) \quad f(K) = 2r - 1 + \frac{1}{2r + 1} = g(r).
\]
Now \((1/2)g'(r) = 1 - 1/(2r + 1)^2 > 0\). Hence \(g\) is an increasing function of \(r\). Noting that \(1/2 < r \leq \sqrt{2}/2\), the maximal value of \(g\) is therefore attained at \(r = \sqrt{2}/2\), that is, when and only when \(K\) is congruent to the diagonal square shown in Figure 1. In this case
\[
f(K) \leq 2(\sqrt{2} - 1) \approx 0.828.
\]
We next use Theorem 1 to prove Corollary 1. Let \(K\) now be a convex set with \(G(K^o, \Gamma) = 1\). Without loss of generality we may assume that the lattice point contained in \(K^o\) is the origin \(O\). Let \(\Gamma'\) be the sublattice of \(\Gamma\) with fundamental cell having vertices \((0, \pm 1), (\pm 1, 0)\). We first note that \(G(K^o, \Gamma') = 0\) (see Figure 2). Hence letting \(A'\) and \(r'\) be the area and the inradius respectively of \(K\) measured in the scale of \(\Gamma'\), and applying (1) to \(K\) with respect to \(\Gamma'\), we have
\[
(2r' - 1)A' \leq 2(\sqrt{2} - 1),
\]
with equality when and only when $K$ is congruent to the square of Figure 2. Since $\Gamma'$ is a rotation of $\Gamma$ scaled by a factor of $\sqrt{2}$, $A' = (1/2)A$ and $r' = (1/\sqrt{2})r$ where $A$ and $r$ are the area and the inradius respectively of $K$ measured in the scale of the integral lattice $\Gamma$. Hence

$$
\left(2, \frac{1}{\sqrt{2}}r - 1\right) \frac{A}{2} \leq 2(\sqrt{2} - 1).
$$

Simplifying, we get

$$
(2r - \sqrt{2})A \leq 4(2 - \sqrt{2}) \approx 2.343,
$$

with equality when and only when $K$ is congruent to the square of Figure 2.

3. PROOF OF THEOREM 2

We first note that if $r \leq 1/2$, inequalities (3) and (4) are trivially true. Hence we need only consider those cases for which $1/2 < r \leq \sqrt{2}/2$.

To prove (3), we first consider $A \leq 1$. Since $r > 1/2$, we have $A > \pi/4$ and so

$$
(2r - 1)A - 1 = (2r - 1)(1 - A) < \left(\sqrt{2} - 1\right)\left(1 - \frac{\pi}{4}\right) < \frac{1}{2}.
$$

Hence we may assume that $A > 1$. Using the same arguments as those given in Section 2, it suffices to consider a set $K$ where $K$ is a rhombus of the type described in Figure 3. Let $Q(r)$ denote such a rhombus with inradius $r$. From (6) we have

$$
(2r - 1)|A - 1| = (2r - 1)(A - 1) = \frac{1}{2r + 1} < \frac{1}{2}.
$$

Taking the infinite strip to be the limit of $Q(r)$ as $r$ tends to $1/2$, it is seen that the stated bound is best possible.

To prove (4), we first consider $p \leq 4$. Since $r > 1/2$, we have $p > \pi$ and so

$$
(2r - 1)|p - 4| = (2r - 1)(4 - p) < \left(\sqrt{2} - 1\right)(4 - \pi) < 2.
$$

Hence we may assume that $p > 4$. We note further that if $K$ is a convex polygon, $K$ may be partitioned into triangles by joining each vertex of $K$ to an in-centre of $K$. Summing the areas of these triangles gives

$$
A \geq \frac{1}{2}pr,
$$

with equality when and only when every edge of $K$ touches the unique incircle. Since any compact convex set may be approximated by a convex polygon, this inequality is
valid for all compact convex sets in the plane. By combining inequality (7) with (3) and noting that \( r > 1/2 \), we have

\[
(2r - 1) |p - 4| = (2r - 1)(p - 4) \leq (2r - 1) \left( \frac{2A}{r} - 4 \right) \leq 4(2r - 1)(A - 1) \leq 4 \cdot \frac{1}{2} = 2,
\]

obtaining (4). As before, taking the infinite strip to be the limit of \( Q(r) \) as \( r \) tends to 1/2, the stated bound is best possible.

Finally, to prove (5), we note that \((w - 1)(d - 1) \leq 1\) with equality when and only when \( K \) is a triangle of the type shown in Figure 4 (see [2]).

![Figure 4](image)

Since \( w \geq 2r \), we have

\[
(2r - 1)(d - 1) \leq (w - 1)(d - 1) \leq 1.
\]

Taking the infinite strip to be the limit of a sequence of triangles of the type shown in Figure 4 as \( w \) tends to \( 2r \), it can be seen that the stated bound is best possible.

REFERENCES