ON GROUPS WITH DECOMPOSABLE COMMUTATOR SUBGROUPS

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(Received 16 April, 1977)

1. Introduction. Let G be a group. We define $\lambda(G)$ to be the smallest integer n such that every element of the commutator subgroup G' is a product of n commutators. Ito [**4**] has shown that $\lambda(A_n)=1$ for all n. Thompson [**7**] has shown that $\lambda(\mathrm{SL}_n(q))=1$ for all n and q. In fact, there is no known simple group G such that $\lambda(G)>1$. However, there do exist such perfect groups (cf. [**7**]).

G is said to be a *C-group* if G is the commutator subgroup of some group. First we show that if $G_1,\ldots,G_s$ satisfy certain conditions, then there exists a group K with $K'=\prod_{i=1}^{s} G_i$ and $\lambda(K)>k$. As a consequence of this, we show that if G is a finite, non-perfect C-group and k is a positive integer, then there exist a group K and a positive integer s so that $K'=\prod_{i=1}^{s} G$ and $\lambda(K)>k$.

In the next section, we derive some applications to the case where G' is abelian, and in particular where G' is cyclic. Finally, we construct nilpotent groups G of class 2 satisfying $\lambda(G)=n$ and either G' is finite of rank n^2 or G' is finitely generated of rank n^2-n+1 (where the rank of a finitely generated abelian group is the minimal number of elements in a generating set). The former improves a result of Gallagher [**1**], while the latter answers a question of Liebeck [**5**].

2. C_p-groups.

Definition. Let p be a prime. A group G is said to be a *C_p-group* if there exists a group K with a subgroup H satisfying:

(i) $K'=G\subset H$,
(ii) $H'\neq G$, and
(iii) $[K:H]=p$.

Theorem 1. *Suppose G_i is a C_p-group for $i=1,2,\ldots,s=1+p+\cdots+p^{2k}$. Then there exists a group K so that $K'=\prod_{i=1}^{s} G_i$ and $\lambda(K)>k$.*

Proof. Choose K_i, H_i as in the definition above, $1\le i\le s$. Then $K_i=\langle H_i, u_i\rangle$, where $u_i^p\in H_i$. Pick elements $x_1,\ldots,x_{2k+1}\in\prod_{i=1}^{s} K_i$, where $x_i=(u_1^{\lambda_{i1}},\ldots,u_s^{\lambda_{is}})$ and $(\lambda_{1j},\lambda_{2j},\ldots,\lambda_{2k+1,j})$, $1\le j\le s$, range over all s one dimensional subspaces of $\prod_{i=1}^{2k+1}\mathbb{Z}_p$. Set $K=\langle H_1,\ldots,H_s,x_1,\ldots,x_{2k+1}\rangle\subset\prod_{i=1}^{s}K_i$. Clearly $K'=\prod_{i=1}^{s}G_i$.

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Let w be a typical product of k commutators in K. Then we have

$$w=\prod_{j=1}^{k}\left[y_j\prod_{r=1}^{2k+1}x_r^{e_{jr}},\ z_j\prod_{r=1}^{2k+1}x_r^{f_{jr}}\right], \tag{1}$$

where $y_j, z_j \in K$. Then $y_j=(a_{j1},\ldots,a_{js})$ and $z_j=(b_{j1},\ldots,b_{js})$ for $j=1,\ldots,k$, where a_{jm}, $b_{jm}\in H_m$ for $m=1,\ldots,s$.

Since $H_i \supset K_i'$, $H_i \lhd K_i$, and hence $H_i' \lhd K_i$. Thus $H=\prod_{i=1}^{s} H_i \lhd K$. Consider (1) in K/H. A straightforward calculation gives

$$w=\prod_{j=1}^{k}\left([a_{j1},u_1^{\alpha_{1j}}][u_1^{\beta_{1j}},b_{j1}],\ldots,[a_{js},u_s^{\alpha_{sj}}][u_s^{\beta_{sj}},b_{js}]\right)$$

in K/H, where $\alpha_{ij}=\sum_{r=1}^{2k+1} e_{jr}\lambda_{ri}$ and $\beta_{ij}=\sum_{r=1}^{2k+1} f_{jr}\lambda_{ri}$.

The system of $2k$ equations

$$\sum_{r=1}^{2k+1} e_{jr}w_r=0,$$
$$\sum_{r=1}^{2k+1} f_{jr}w_r=0, \qquad j=1,\ldots,k,$$

in the unknowns $w_1,\ldots,w_{2k+1}$ has a non-trivial solution in $\mathbb{Z}_p$. Hence there exists t, $1\le t\le s$, such that $(w_1,\ldots,w_{2k+1})=(\lambda_{1t},\ldots,\lambda_{2k+1,t})$ is a solution. So $\alpha_{tj}\equiv\beta_{tj}\equiv 0 \pmod p$ for $j=1,\ldots,k$, and therefore the tth component of w in K/H is 1. Thus we have shown that if $c_i\in G_i-H_i'$ for $i=1,\ldots,s$, then the element $(c_1,\ldots,c_s)$ is not a product of k commutators in K.

As a corollary to this we get our desired result.

THEOREM 2. *If G is a non-perfect, finite C-group, then, given a positive integer k, there exist a positive integer s and a group K so that $K'=\prod_{i=1}^{s} G$ and $\lambda(K)>k$.*

Proof. Macdonald [**6**] has shown that if G is a finite C-group, then there exists K finite with $K'=G$. Choose K of minimal order such that $K'=G$. Since G is not perfect, $K\neq G$, and hence there exists a prime p so that $p\mid[K:G]$. Since K/G is a finite abelian group whose order is a multiple of p, there exists a subgroup H of K satisfying $H\subset G$ and $[K:H]=p$. By the minimality of K, $H'\neq G$, and thus G is a C_p-group. The result now follows from Theorem 1.

3. Abelian commutator subgroups.

LEMMA 1. *An abelian group is a C_2-group.*

Proof. Let A be an abelian group. Consider $G=(A\times A)\times_s\langle x\rangle$, where $x^2=1$ and

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$x(a, b)x = (b, a)$. Then $G' = \{(a, a^{-1}) : a \in A\} \cong A$. Let $H = A \times A$. Then $H' = 1$ and $[G : H] = 2$.

We remark that in the group above $\lambda(G) = 1$ as $[x, (1, a)] = (a, a^{-1})$. From Lemma 1 and Theorem 1 we have the following theorem.

THEOREM 3. *Suppose G is a direct product of $2^{2k+1} - 1$ abelian groups. Then there exists a group K with $K' = G$ and $\lambda(K) > k$.*

COROLLARY. *If G is a cyclic group of order n, where $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, $s \geq 2^{2k+1} - 1$, then there exists K with $K' = G$ and $\lambda(K) > k$.*

If $2 \mid n$ the result can be improved slightly. However, if n is odd, this is the best possible example (cf. [**3**]). Macdonald [**6**] has proved the corollary in the case n is odd and $s \geq 2^{2k+2} - 1$.

4. Nilpotent groups. Let $R = M_{2n}(\mathbb{Z})$ be the ring of $2n \times 2n$ matrices with integer entries. Set $E_{kl} = (a_{ij}) \in R$ where $a_{ij} = 0$ if $(i, j) \neq (k, l)$ and $a_{kl} = 1$. Let $U = \{(b_{ij}) \in M_3(R) : b_{ij} = 0$ if $i > j$, and $b_{ii} = 1\}$. Then U is a group under matrix multiplication. Let G be the subgroup of U generated by $\{(b_{ij}) \in U : b_{12} = b_{23}^t = \sum_{k=1}^{2n} \lambda_k E_{k1}\}$. If we denote the set of skew-symmetric matrices of R by S, then $G' = \{(b_{ij}) \in U : b_{12} = b_{23} = 0,\ b_{13} \in S\}$. However, $x = (b_{ij}) \in G'$ is a commutator if and only if rank $b_{13} \leq 2$. Hence $x = (b_{ij}) \in G'$ is a product of m commutators if and only if rank $b_{13} \leq 2m$. Clearly, then $\lambda(G) \doteq n$. Note also that $G' \subset Z(G)$, and hence G is nilpotent of class 2.

Let $A = \{(i, j) : 1 \leq i < j \leq 2n,\ i + j \leq 2n + 1\}$. Notice that $|A| = n^2$. Choose integers k_{ij}, $(i, j) \in A$, so that there exists a prime p with $p \mid k_{ij}$. Let $H = \{(a_{ij}) \in S : a_{ij}$ is a multiple of k_{ij} for $(i, j) \in A\}$. Note that H is an additive subgroup of S. Set $x_0 = (a_{ij}) \in S$, where $a_{ij} = 0$ if $i + j \neq 2n + 1$, and $a_{ij} = 1$ if $i + j = 2n + 1$ and $1 \leq i \leq n$. Then the coset $x_0 + H$ consists only of matrices of rank $2n$, since if $y \in x_0 + H$, det $y \equiv 1 \pmod p$.

Now consider $K = \{(b_{ij}) \in G' : b_{13} \in H\}$. K is a central subgroup of G, and hence is normal. By the above remarks, the element $y = \overline{(b_{ij})} \in G'/K$ and $b_{13} = x_0$ is not a product of fewer than n commutators. Hence $\lambda(G/K) = n$. Also $(G/K)' = G'/K \cong \prod_{(i,j) \in A} \mathbb{Z}/k_{ij}\mathbb{Z}$. Thus we have proved the following theorem.

THEOREM 4. *If G is a finitely generated abelian group with rank $G \geq n^2$, then there exists a group K such that $G = K' \subset Z(K)$ and $\lambda(K) = n$.*

If we take $k_{ij} = p$, a fixed prime for all $(i, j) \in A$, then $G' = \prod_{i=1}^{n^2} \mathbb{Z}_p$. If T is a maximal torsion-free central subgroup of G, then $K = G/T$ is a finite nilpotent group such that $K' = G'$ and $\lambda(K) = \lambda(G)$. Let L be the sylow p-subgroup of K. Then $L' = K'$ and $\lambda(L) = n$. Gallagher [**1**] has shown that if G is a p-group and $|G'| < p^{n(n+1)}$, then $\lambda(G) \leq n$. Our example shows that $n(n+1)$ can not be replaced by $(n+1)^2 + 1$. For $n = 1$, p^4 is the best bound (cf. [**2**]).

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We now will construct another example. For each $(i, j) \in A$, set $k_{ij} = 0$ if $i + j < 2n + 1$ and $k_{ij} = m_i > 1$ if $i + j = 2n + 1$. Let H be as above. If $y \in x_0 + H$, $\det y = \prod_{i=1}^{n} (1 + \lambda_i m_i)^2 \neq 0$. Again set $K = \{(b_{ij}) \in G' : b_{13} \in H\}$. Arguing as above, we see that $\lambda(G/K) = n$, and $(G/K)' = G'/K \cong \prod_{i=1}^{n^2-n} \mathbb{Z} \times \prod_{i=1}^{n} \mathbb{Z}/m_i\mathbb{Z}$. In particular, if we assume that the m_i are pairwise relatively prime, then rank $G'/K = n^2 - n + 1$. Hence we have constructed a group N of nilpotency class 2 with $\lambda(N) = n$, and N' generated by $n^2 - n + 1$ elements. Liebeck [5] showed that if $N' \subset Z(N)$ can be generated by 2 elements, then $\lambda(N) = 1$. The above example shows that 2 cannot be replaced by 3. This answers a question posed in [5].

Acknowledgement. The author would like to express his gratitude to Robert Steinberg for his help in the construction of the examples in Section 4.

REFERENCES

1. P. X. Gallagher, The generation of the lower central series, *Canad. J. Math.* **17** (1965), 405–410.
2. R. Guralnick, *Expressing group elements as products of commutators,* Ph.D. Thesis, UCLA (1977).
3. R. Guralnick, On cyclic commutator subgroups, to appear.
4. N. Ito, A theorem on the alternating group A_n $(n \geq 5)$, *Math. Japon.* **2** (1951), 59–60.
5. H. Liebeck, A test for commutators, *Glasgow Math. J.* **17** (1976), 31–36.
6. I. D. Macdonald, On cyclic commutator subgroups, *J. London Math. Soc.* **38** (1963), 419–422.
7. R. C. Thompson, Commutators in the special and general linear groups, *Trans. Amer. Math. Soc.* **101** (1961), 16–33.

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https://doi.org/10.1017/S0017089500003578 Published online by Cambridge University Press