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EXOTIC LEFT-ORDERINGS OF THE FREE GROUPS FROM THE DEHORNOY ORDERING

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Abstract

We show that the restriction of the Dehornoy ordering to an appropriate free subgroup of the three-strand braid group defines a left-ordering of the free group on k generators, $k > 1$, that has no convex subgroups.

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1. Introduction

A group G is said to be left-orderable if there exists a strict total ordering of its elements such that $g < h$ implies $fg < fh$ for all f, g, h in G. To each left-ordering $<$ of a group G, we can associate the set $P = \{g \in G \mid g > 1\}$, which is called the positive cone associated to the left-ordering $<$. The positive cone P satisfies $P \cdot P \subset P$, and $P \sqcup P^{-1} \sqcup \{1\} = G$. Conversely, any subset P satisfying these two properties defines a strict total ordering of the elements of G via $g < h$ if and only if $g^{-1}h \in P$. Any ordering defined in this way is easily seen to be invariant under left multiplication.

We may strengthen our conditions on a left-ordering $<$ of G by requiring that, for all $g, h > 1$ in G, there must exist a positive integer n such that $g < hg^n$. In this case, the ordering is called Conradian (after the work of Conrad in [2]). It has since been observed that, equivalently, we may ask that this condition hold for $n = 2$ [9].

Finally, the strongest condition we may require of an ordering $<$ of G is that the ordering be invariant under multiplication from both sides, that is, $g < h$ implies $fg < fh$ and $gf < hf$ for all f, g, h in G. Equivalently, we may require that the positive cone associated to the ordering $<$ of G be preserved by conjugation. If either of these equivalent conditions is satisfied by the ordering $<$ of G, then the ordering is said to be a bi-ordering.

An important structure associated to a given left-ordering $<$ of G is the set of convex subgroups of G. A subgroup $H \subset G$ is said to be convex in G (with respect to the ordering $<$) if whenever f, h are in H and g is in G, the implication $f < g < h \Rightarrow g \in H$ holds.

Owing to work of Conrad and Hölder, the convex subgroups of bi-orderings and Conradian orderings are very well understood [2]. This leaves us with understanding the set of convex subgroups for the case of left-orderings that are neither bi-orderings nor Conradian orderings. This problem seems to be quite difficult, as constructing Conradian orderings and bi-orderings of a group G is in general somewhat easier than constructing left-orderings of a group that are not Conradian orderings.

Two of the primary methods for constructing non-Conradian orderings of a group G are given by the following proposition and theorem.

PROPOSITION 1.1. *Let G be a group, K a subgroup of G left-ordered by the ordering $\prec$, and G/K the set of left cosets of K in G. Suppose that G/K is ordered by the ordering $\prec'$, satisfying $gK \prec' hK \Rightarrow fgK \prec' fhK$ for all f, g, h in G. Then a left-ordering $<$ can be defined on G, according to the rule that, for every g in G, $1 < g$ if $g \in K$ and $1 \prec g$, or if $g \notin K$ and $K \prec' gK$.*

THEOREM 1.2 (Conrad [2]). *A group G is left-orderable if and only if G acts effectively by order-preserving automorphisms on a linearly ordered set.*

In both of these cases, at least some of the convex subgroups of the constructed ordering are obvious. In Proposition 1.1, the subgroup K is a convex subgroup in the left-ordering $<$ of G. In Theorem 1.2, the stabilizers under the G-action of points in the given linearly ordered set correspond to convex subgroups (see [2] or [9] for details of the construction). In light of the fact that both of these known methods for producing left-orderings of a group result in a left-ordering that (often) admits convex subgroups, it is quite surprising to find that some non-Conradian left-orderings may contain no proper, nontrivial convex subgroups whatsoever. In this paper, we will left-order the free groups of finite rank in such a way that the free group contains no proper, nontrivial convex subgroups with respect to our constructed ordering. The construction relies heavily on the Dehornoy ordering of the braid group B_3.

The existence and a construction of such orderings of the free groups seems to have appeared only in [7]. The construction there, unlike our present setting, deals with creating a very unusual effective action on the rationals. Our present approach is in simpler algebraic terms.

It is also worth noting that admitting a Conradian or bi-ordering that has no proper, nontrivial convex subgroups is a very restrictive condition on the group G, as the following theorem shows.

THEOREM 1.3 [2]. *Suppose that G admits a Conradian or bi-ordering which has no proper, nontrivial convex subgroups. Then G is a subgroup of* $(\mathbb{R}, +)$.

In the case where G admits a non-Conradian left-ordering having no proper, nontrivial convex subgroups, it is not likely that the structure of G must be so restricted. While we will see that free groups admit such left-orderings, there are also nonfree, non-abelian groups that admit such left-orderings as well [1, Example 7.2.3]. It has also recently been shown that the braid groups themselves admit many left-orderings with no convex subgroups; see [10].

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2. A left-ordering of F_2 having no convex subgroups

As a warm-up for the general case, which will be slightly more involved, we deal first with the free group on two generators.

We begin by defining the Dehornoy left-ordering of the braid groups (also known as the 'standard' ordering), whose positive cone we shall denote P_D [3, 4]. Recall that for each integer $n \geq 2$, the Artin braid group B_n is the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1.$$

DEFINITION 2.1. Let w be a word in the generators $\sigma_i, \ldots, \sigma_{n-1}$. Then w is said to be i-positive if the generator σ_i occurs in W with only positive exponents, i-negative if σ_i occurs with only negative exponents, and i-neutral if σ_i does not occur in w.

We then define the positive cone of the Dehornoy ordering as follows.

DEFINITION 2.2. The positive cone $P_D \subset B_n$ of the Dehornoy ordering is the set

$$P_D = \{\beta \in B_n : \beta \text{ is } i\text{-positive for some } i \leq n-1\}.$$

Let $\beta \in B_n$ be any braid. An extremely important property of this ordering is that the conjugate $\beta\sigma_k\beta^{-1}$ is always i-positive for some i, for every generator σ_k in B_n. This property is referred to as the subword property [4].

Recall that the commutator subgroup $[B_3, B_3]$ is isomorphic to the free group F_2 on two generators. The commutator subgroup is generated by the braids $\beta_1 = \sigma_2\sigma_1^{-1}$ and $\beta_2 = \sigma_1\sigma_2\sigma_1^{-2}$ [8]. Of course we can instead take as free generators the 1-positive braids $\beta_1^{-1} = \sigma_1\sigma_2^{-1}$ and $\beta_2^{-1}\beta_1^{-1} = \sigma_1^2\sigma_2^{-2}$, since any generating set of F_2 with only two elements will freely generate F_2.

Define a positive cone $P \subset F_2$ by $P = [B_3, B_3] \cap P_D$, with associated ordering $<$ of F_2. Thus, the left-ordering $<$ of F_2 is the restriction of the Dehornoy ordering $<_D$ of B_3 to the (free) commutator subgroup $[B_3, B_3]$.

THEOREM 2.3. *The ordering $<$ of F_2 has no proper, nontrivial convex subgroups.*

PROOF. Let $C \subset F_2 = [B_3, B_3]$ be a nontrivial convex subgroup. Then we may choose $1 < \beta \in F_2$ that is 1-positive (no nontrivial 1-neutral braids lie in $[B_3, B_3]$, because they do not have zero total exponent).

There are now two cases to consider.

Case 1. Suppose that β commutes with σ_2. Then $\beta = \Delta_3^{2p}\sigma_2^q$ for some integers p, q ([5], here $\Delta = \sigma_1\sigma_2\sigma_1$). Since $\beta \in [B_3, B_3]$, we know that $q = -6p$, since β must have zero total exponent, and $p > 0$ because we have chosen β to be 1-positive. Then we have that $\Delta_3^2 < \Delta_3^{4p}\sigma_2^{-12p} = \beta^2$, so that $\langle\beta\rangle$ is cofinal in the Dehornoy ordering [4]. Therefore, there exist integers k, l such that in F_2 we have

$$1 < \sigma_1\sigma_2^{-1} < \beta^k \quad \text{and} \quad 1 < \sigma_1^2\sigma_2^{-2} < \beta^l,$$

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and thus $\sigma_1\sigma_2^{-1}, \sigma_1^2\sigma_2^{-2} \in C$ by convexity. Therefore we must have $C = F_2$, as C contains both generators of F_2.

Case 2. Suppose that β and σ_2 do not commute. Let $k > 0$, and observe that $\beta\sigma_2^k\beta^{-1}$ is a 1-positive braid by the subword property, so that the commutator $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k}$ is also 1-positive. Next, because β is 1-positive, the braid $\sigma_2^k\beta^{-1}\sigma_2^{-k}$ is 1-negative, so that $\sigma_2^k\beta^{-1}\sigma_2^{-k} < 1$, and thus $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k} < \beta$. Thus, we have shown that $1 < \beta\sigma_2^k\beta^{-1}\sigma_2^{-k} < \beta$, so that $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k}$ must lie in the subgroup C, by convexity.

Now both the braids β and $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k}$ lie in the convex subgroup C, so the braid $\sigma_2^k\beta^{-1}\sigma_2^{-k}$ (and hence its inverse $\sigma_2^k\beta\sigma_2^{-k}$) must also lie in C, for any choice of positive integer k.

We now refine our choice of braid $\beta \in C$. Suppose that β is represented by the 1-positive braid word $\sigma_2^u\sigma_1 w$, where u is any integer, and w is a 1-positive, 1-neutral or empty word. Choose $k > 0$ so that $u' = k + u > 0$, and set $\beta' = \sigma_2^k\beta\sigma_2^{-k}$, so that β' is represented by the 1-positive braid word $\sigma_2^{u'}\sigma_1 w\sigma_2^{-k}$. Note that $\beta' \in C$, from our work above.

We will now show that C must contain both generators of F_2. Observe that the braid represented by the word $\sigma_2\sigma_1^{-1}\sigma_2^{u'}\sigma_1 w\sigma_2^{-k}$ is 1-positive, as $\sigma_2(\sigma_1^{-1}\sigma_2^{u'}\sigma_1)w\sigma_2^{-k} = \sigma_2(\sigma_2\sigma_1^{u'}\sigma_2^{-1})w\sigma_2^{-k}$, and $u' > 0$. Therefore

$$1 < \sigma_2\sigma_1^{-1}\sigma_2^{u'}\sigma_1 w\sigma_2^{-k} \Rightarrow \sigma_1\sigma_2^{-1} < \sigma_2^{u'}\sigma_1 w\sigma_2^{-k} = \beta' \in C,$$

and since $1 < \sigma_1\sigma_2^{-1}$, this implies that $\sigma_1\sigma_2^{-1} \in C$ by convexity.

Considering the second generator $\sigma_1^2\sigma_2^{-2}$, observe that the braid represented by the word $\sigma_2^2\sigma_1^{-2}\sigma_2^{u'}\sigma_1 w\sigma_2^{-k}$ is 1-positive, as we compute

$$\sigma_2^2\sigma_1^{-1}(\sigma_1^{-1}\sigma_2^{u'}\sigma_1)w\sigma_2^{-k} = \sigma_2^2\sigma_1^{-1}(\sigma_2\sigma_1^{u'}\sigma_2^{-1})w\sigma_2^{-k}$$

and

$$\sigma_2^2(\sigma_1^{-1}\sigma_2\sigma_1)\sigma_1^{u'-1}\sigma_2^{-1}w\sigma_2^{-k} = \sigma_2^2(\sigma_2\sigma_1\sigma_2^{-1})\sigma_1^{u'-1}\sigma_2^{-1}w\sigma_2^{-k},$$

where $u' > 0$. Therefore

$$1 < \sigma_2^2\sigma_1^{-2}\sigma_2^{u'}\sigma_1 w\sigma_2^{-k} \Rightarrow \sigma_1^2\sigma_2^{-2} < \sigma_2^{u'}\sigma_1 w\sigma_2^{-k} = \beta' \in C,$$

and since $1 < \sigma_1^2\sigma_2^{-2}$, we conclude from convexity of C that $\sigma_1^2\sigma_2^{-2} \in C$.

Thus, C contains both generators of F_2, so that $C = F_2$. □

3. Left-ordering the free groups of rank greater than two

We now extend our results to cover those free groups F_k with $k > 2$. Let $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1^2\sigma_2^{-2}$ denote the generators of F_2, and we let K_n denote the kernel of the map $F_2 \to \mathbb{Z}_{n-1}$ defined by $y \mapsto 0$, $x \mapsto 1$. Here $\mathbb{Z}_{n-1}$ is the cyclic group

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of order $n-1$. We will employ a proof very similar to that of Theorem 2.3, by considering $K_n \subset F_2 = [B_3, B_3]$, and showing that the restriction of the Dehornoy ordering to K_n has no convex subgroups. First we need to find a generating set for K_n.

LEMMA 3.1. *The subgroup K_n is free of rank n, with basis*

$$y,\, x^{n-1},\, xyx^{n-2},\, x^2yx^{n-3},\, \ldots,\, x^{n-2}yx.$$

PROOF. From [11, Lemma 7.56] we know that K_n is finitely generated. Moreover, we may compute a generating set of K_n as follows: consider the generating set $g_1 = x$, $g_2 = x^{-1}$, $g_3 = y$, $g_4 = y^{-1}$ of F_2, and let $1, x, x^2, \ldots, x^{n-2}$ be representatives of the right cosets of $K_n \subset F_2$. For all i, j, there exist h_{ij} and some coset representative $x^{k(i,j)}$ such that we may write $x^i g_j = h_{ij} x^{k(i,j)}$. The elements h_{ij} form a generating set for K_n.

In our present setting, for $i < n-2$ we get

$$x^i g_1 = x^i \cdot x = 1 \cdot x^{i+1},$$

so that $h(i, 1) = 1$, and for $i = n-2$ we get $h(i, 1) = x^{n-1}$. Similarly, we compute for $i \geq 1$ that

$$x^i g_2 = x^i \cdot x^{-1} = 1 \cdot x^{i-1},$$

so that $h(i, 2) = 1$, and for $i = 0$ we compute $h(i, 2) = x^{-(n-1)}$.

Next, for all i, we compute

$$x^i y^{\pm 1} = x^i y^{\pm 1} x^{-i} \cdot x^i,$$

so that $h(i, 3) = h(i, 4)^{-1} = x^i y x^{-i}$. Eliminating inverses from this generating set yields the set

$$y,\, x^{n-1},\, xyx^{-1},\, x^2yx^{-2},\, \ldots,\, x^{n-2}yx^{-(n-2)}.$$

From [6, Proposition 3.9] we deduce that K_n is of rank n, and therefore the generating set above must provide a basis for K_n. Right-multiplying those generators of the form $x^i y x^{-i}$ by the generator x^{n-1} yields the desired generating set. □

Also important in the proof of Theorem 2.3 was the action of conjugation by σ_2. In order to generalize our theorem, we must make the following analysis.

Let F_2 be the free group on two generators x and y, and define an automorphism $\phi : F_2 \to F_2$ according to the formulas $\phi(x) = xy^{-1}x$ and $\phi(y) = xy^{-1}x^2$. Then the following lemma holds.

LEMMA 3.2. *Consider F_2 as the commutator subgroup $[B_3, B_3]$ with generators $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1^2\sigma_2^{-2}$. Then the automorphism ϕ of F_2 corresponds to conjugation of $[B_3, B_3]$ by the generator $\sigma_2 \in B_3$, so that $\phi(g) = \sigma_2^{-1} g \sigma_2$ for all $g \in F_2$.*

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PROOF. The proof is computational. First conjugating the generator x, we compute

$$\begin{aligned}\phi(x) &= xy^{-1}x \\ &= \sigma_1\sigma_2^{-1}\sigma_2^2\sigma_1^{-2}\sigma_1\sigma_2^{-1} \\ &= (\sigma_1\sigma_2\sigma_1^{-1})\sigma_2^{-1} \\ &= (\sigma_2^{-1}\sigma_1\sigma_2)\sigma_2^{-1} \\ &= \sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_2 \\ &= \sigma_2^{-1}x\sigma_2\end{aligned}$$

and

$$\begin{aligned}\phi(y) &= xy^{-1}x^2 \\ &= \sigma_1\sigma_2^{-1}\sigma_2^2\sigma_1^{-2}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \\ &= (\sigma_1\sigma_2\sigma_1^{-1})\sigma_2^{-1}\sigma_1\sigma_2^{-1} \\ &= (\sigma_2^{-1}\sigma_1\sigma_2)\sigma_2^{-1}\sigma_1\sigma_2^{-1} \\ &= \sigma_2^{-1}\sigma_1^2\sigma_2^{-1} \\ &= \sigma_2^{-1}\sigma_1^2\sigma_2^{-2}\sigma_2 \\ &= \sigma_2^{-1}y\sigma_2.\end{aligned}$$

This concludes the proof. □

This computation allows us to show that K_n is fixed by the conjugation action of σ_2^6 or σ_2^{-6} on the commutator subgroup $[B_3, B_3]$.

LEMMA 3.3. *Let $\phi : F_2 \to F_2$ be the map arising from conjugation of $[B_3, B_3]$ by σ_2, namely $\phi(x) = xy^{-1}x$ and $\phi(y) = xy^{-1}x^2$. Then, for all n, $\phi^6(K_n) = K_n$.*

PROOF. Consider the abelianization $F_2 \xrightarrow{ab} \mathbb{Z} \oplus \mathbb{Z}$. We find that ϕ descends to a map $\phi_* : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, and that relative to the basis $\{ab(x), ab(y)\}$ the map ϕ_* is represented by the matrix

$$\begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}.$$

The sixth power of this matrix is the identity. It follows that for any normal subgroup K such that F_2/K is abelian, we have $\phi^6(K) = K$. □

Lastly, note that any generator of K_n, when we substitute $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1^2\sigma_2^{-2}$, yields a product of braid group generators of the form

$$\sigma_1^{l_1}\sigma_2^{k_1}\sigma_1^{l_2} \cdots \sigma_2^{k_{m-1}}\sigma_1^{l_m}\sigma_2^{k_m},$$

where $k_i < 0$ and $l_i > 0$ for all i. Therefore, we require the following lemma in order to compare the generators to different braids in K_n.

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LEMMA 3.4. *Any braid represented by a word of the form*

$$\sigma_2^{k_1}\sigma_1^{l_1}\cdots\sigma_2^{k_m}\sigma_1^{l_m}\sigma_2^{n}\sigma_1,$$

where $k_i > 0$, $l_i < 0$ for all i, and $n > 1$, is 1-positive.

PROOF. We use induction on m, the length of the product. For $m = 0$, the claim is trivial. Assuming that the claim holds for those products of length $m-1$, we use the identities $\sigma_1^k\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^k$ and $\sigma_1^{-1}\sigma_2^k\sigma_1 = \sigma_2\sigma_1^k\sigma_2^{-1}$, and compute that

$$\begin{aligned}\sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}\sigma_1^{l_m}\sigma_2^{n}\sigma_1 &= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}\sigma_1^{l_m+1}(\sigma_1^{-1}\sigma_2^{n}\sigma_1)\\ &= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}\sigma_1^{l_m+1}(\sigma_2\sigma_1^{n}\sigma_2^{-1})\\ &= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}(\sigma_1^{l_m+1}\sigma_2\sigma_1)\sigma_1^{n-1}\sigma_2^{-1}\\ &= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m}(\sigma_2\sigma_1\sigma_2^{l_m+1})\sigma_1^{n-1}\sigma_2^{-1}\\ &= \sigma_2^{k_1}\sigma_1^{l_2}\cdots\sigma_2^{k_m+1}\sigma_1(\sigma_2^{l_m+1}\sigma_1^{n-1}\sigma_2^{-1}).\end{aligned}$$

The bracketed expression $\sigma_2^{l_m+1}\sigma_1^{n-1}\sigma_2^{-1}$ is 1-positive as $n > 1$, and the remaining terms in the product above are representative of a 1-positive braid, by assumption. By induction, the claim is proven. □

THEOREM 3.5. *Let $n > 2$. Then the restriction of the Dehornoy ordering to the subgroup $K_n \subset F_2 = [B_3, B_3]$ has no proper, nontrivial convex subgroups.*

PROOF. We proceed similarly to Theorem 2.3. Suppose that $C \subset K_n$ is a nontrivial, convex subgroup, and let $\beta \in C$ be a 1-positive braid. Denote the generators of K_n by $g_1, g_2, \ldots, g_n$; from Lemma 3.1 we know that $g_i > 1$ for all i. There are two cases to consider.

Case 1. The braid β commutes with σ_2. In this case, we proceed as in Case 1 of Theorem 2.3, to conclude that $\langle\beta\rangle$ must be cofinal in the Dehornoy ordering. Thus, we can find an integer k such that $\beta^k > g_i > 1$ for every generator g_i of K_n. Then $g_i \in C$ for all i, and we conclude that $C = K_n$.

Case 2. Suppose that β and σ_2 do not commute, and we proceed as in Case 2 of Theorem 2.3. Then, by the subword property of the Dehornoy ordering, we know that $\beta\sigma_2^k\beta^{-1} > 1$ for all $k > 0$, and hence $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k} > 1$ as well. We deduce that $1 < \beta\sigma_2^k\beta^{-1}\sigma_2^{-k} < \beta$ for all $k > 0$ as before. However, the braid $\beta\sigma_2^k\beta^{-1}\sigma_2^{-k}$ is not necessarily an element of K_n, but as conjugation by σ_2^6 preserves K_n by Lemma 3.3, we have $\beta\sigma_2^{6k}\beta^{-1}\sigma_2^{-6k} \in K_n$ for all $k > 0$. Hence, the inequality $1 < \beta\sigma_2^k\beta^{-1}\sigma_2^{-k} < \beta$ yields $\beta\sigma_2^{6k}\beta^{-1}\sigma_2^{-6k} \in C$ for all $k > 0$. We conclude that $\sigma_2^{6k}\beta^{-1}\sigma_2^{-6k} \in C$ for all $k > 0$.

Proceeding as in the proof of Theorem 2.3, we may conjugate β by an appropriate (sixth) power of σ_2 to conclude that the convex subgroup C in K_n contains a braid

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represented by a word of the form $\sigma_2^u \sigma_1 w$, where $u > 1$, and w is a 1-positive, 1-neutral or empty word. Then for each generator g_i of K_n, consider the braid represented by the word $g_i^{-1}\sigma_2^u\sigma_1 w$. As each g_i contains only positive powers of the braids $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1^2\sigma_2^{-2}$, we see that $g_i^{-1}\sigma_2^u\sigma_1$ represents a 1-positive braid, by Lemma 3.4. Therefore, the braid $g_i^{-1}\sigma_2^u\sigma_1 w$ is 1-positive, and we conclude that $1 < g_i < \sigma_2^u\sigma_1 w \in C$, hence $g_i \in C$ for all i, and $C = K_n$. □

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