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HANKEL OPERATORS ON PSEUDOCONVEX DOMAINS OF FINITE TYPE IN $\mathbb{C}^2$

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ABSTRACT. The aim of this paper is to study small Hankel operators h on the Hardy space or on weighted Bergman spaces, where Ω is a finite type domain in $\mathbb{C}^2$ or a strictly pseudoconvex domain in $\mathbb{C}^n$. We give a sufficient condition on the symbol f so that h belongs to the Schatten class S_p, $1 \leq p < +\infty$.

1. **Introduction.** Let Ω a bounded pseudoconvex domain of finite type m in $\mathbb{C}^2$ given by $\Omega = \{z \in \mathbb{C}^2, r(z) < 0\}$, where r is a C^∞ function such that $|\nabla r(z)| = 1$ on $\partial\Omega = \{z\,;\, r(z) = 0\}$.

Let $\mathcal{H}(\Omega)$ be the space of holomorphic functions in Ω. We denote by S the Szegö projection: it is the orthogonal projection from $L^2(\partial\Omega)$ onto $L^2(\partial\Omega) \cap \mathcal{H}(\Omega)$, the subspace of holomorphic functions in Ω of which boundary values function is in $L^2(\partial\Omega)$. For g in $L^2(\partial\Omega)$, Sg has a holomorphic extension in Ω given by

$$\mathrm{Sg}(z) = \int_{\partial\Omega} S(z,\zeta) g(\zeta)\, d\sigma(\zeta), \quad z \in \Omega,$$

where $S(z,\zeta)$ is the Szegö kernel of Ω.

For f in $L^2(\partial\Omega) \cap \mathcal{H}(\Omega)$, the big Hankel operator H and the small Hankel operator h of symbol f are defined by

$$Hg = S(f\,\mathrm{Sg}) - f\,\mathrm{Sg}, \tag{1}$$

$$hg = S(f\overline{\mathrm{Sg}}), \quad g \in L^2(\partial\Omega). \tag{2}$$

Let $q > -1$ and $dV_q = \big(-r(z)\big)^q dV$, where dV is the Lebesgue measure of Ω. We denote by B_q the weighted Bergman projection: it is the orthogonal projection from $L^2(dV_q)$ onto the weighted Bergman space $A^2(dV_q) = L^2(dV_q) \cap \mathcal{H}(\Omega)$. Let g in $L^2(dV_q)$, then

$$B_q g(z) = \int_\Omega B_q(z,\zeta) g(\zeta)\, dV_q(\zeta),$$

where $B_q(z,\zeta)$ is the weighted Bergman kernel. We denote B_0 by B and $B_0(z,\zeta)$ by $B(z,\zeta)$.

For $f \in A^2(dV_q)$, the big Hankel operator H_q and the small Hankel operator h_q of symbol f are defined by

$$H_q g = B_q(f B_q g) - f B_q g, \tag{3}$$

$$h_q g = B_q(f\overline{B_q g}), \quad g \in L^2(dV_q). \tag{4}$$

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We denote h_0 by h and H_0 by H.

Hankel operators have been studied by many authors. A full characterization was established in the case of the disc [AFP], [CR], [Pell], [Zhu] .... It is well known that Hankel operators on Hardy spaces are bounded if and only if f is in BMO and compact if and only if f is in VMO. Concerning Hankel operators in Bergman spaces, they are bounded if and only if f is in the Bloch space and compact if and only if f is in the little Bloch space. These results have been extended to the unit ball of $\mathbb{C}^n$ by R. Coifman, R. Rochberg and G. Weiss [CRW] for Hankel operators on Hardy spaces and by Zheng [Zhe] for Hankel operators in Bergman spaces.

For other pseudoconvex domains in $\mathbb{C}^n$, the characterization of big Hankel operators in Bergman spaces is related to the study of the commutator $C_f g = fBg - B(fg)$. For strictly pseudoconvex domains a characterization of C_f and H has been obtained by F. Beatrous and S.-Y. Li [BL1], H. Li [L] and M. Peloso [Pelo]. For finite type domains in $\mathbb{C}^2$, a study of the commutator can be found in [BL1].

Concerning Hankel operators on Hardy spaces, S. Krantz and S.-Y. Li [KL2] proved the theorem of factorization of $H^1(\Omega)$ and deduced the characterization of h when Ω is a strictly pseudoconvex domain. For strictly pseudoconvex domains and finite type domains in $\mathbb{C}^2$, they studied the commutator $C_f = fSg - S(fg)$.

The characterization of symbols f such that these operators belong to the Schatten class S_p is an important question. For the unit disc, Hankel operators belong to S_p if and only if f is in the Besov space $B_p^{p,1/p}$, $1 \leq p < +\infty$. For small Hankel operators on the Hardy space, the result is still valid in the case of the unit ball of $\mathbb{C}^n$, $n \geq 2$, [FR] and [Zha].

The situation is different for big Hankel operators in Bergman spaces: for $p \geq 2n$, H_q is in S_p if and only if the symbol f is in $B_p^{p,1/p}$ but for $0 < p \leq 2n$, H_q is in S_p if and only if f is constant. The same cutoff phenomenon appears when Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ [L], [Pelo].

When Ω is a pseudoconvex domain of finite type in $\mathbb{C}^2$, it was proved in [KLR] that the big Hankel operator H on Bergman space is in S_p if and only if f is in some function space $Y_p(\Omega)$ when $p > 4$ and for $0 < p \leq 4$, H is in S_p if and only if f is constant this space $Y_p(\Omega)$ coincides with the analytic Besov spaces when Ω is a strictly pseudoconvex domain. In this paper, the case of ellipsoids is also considered.

The purpose if this paper is to extend the results of [S2] in which sufficient conditions are given on the symbol f so that small Hankel operators belongs to S_p when Ω is a complex ellipsoid.

Before stating our results, we recall the construction of the anisotropic pseudometric on $\partial\Omega$ which we shall use to define the anisotropic BMO($\partial\Omega$) and VMO($\partial\Omega$) spaces (see D. Catlin [Ca] and A. Nagel, E. Stein and S. Wainger [NSW]). Let $U = \{z, |r(z)| < \varepsilon\}$ a neighborhood of $\partial\Omega$. We consider the function $\tau(z,\delta)$ and the biholomorphic mapping Φ_z defined in [Ca]. Recall that $C_1\delta^{1/2} \leq \tau(z,\delta) \leq C_2\delta^{1/m}$. We denote by d_0 the anisotropic pseudometric on $\partial\Omega$ given by

$$d_0(z,\zeta) = \inf\big\{\delta > 0, \zeta \in Q(z,\delta)\big\},$$

where $Q(z,\delta) = \Phi_z\Big(\big\{(\zeta_1,\zeta_2), |\zeta_1| \le \tau(z,\delta) \text{ and } |\zeta_2| \le \delta\big\}\Big)$. For z on $\partial\Omega$ and $\delta > 0$, we denote by $B(z,\delta)$ the anisotropic ball $\{\zeta \in \partial\Omega, d_0(z,\zeta) < \delta\}$. It is well known that

$$\sigma\big(B(z,\delta)\big) \simeq \delta\tau^2(z,\delta),$$

where σ is the Lebesgue measure of $\partial\Omega$.

Let f in $L^1_{\text{loc}}(\partial\Omega)$. For z on $\partial\Omega$ and $\delta > 0$, we consider

$$m(f,z,\delta) = \frac{1}{\sigma\big(B(z,\delta)\big)} \int_{B(z,\delta)} f(\zeta)\, d\sigma(\zeta),$$

$$\operatorname{osc}(f,z,\delta) = \frac{1}{\sigma\big(B(z,\delta)\big)} \int_{B(z,\delta)} |f(\zeta) - m(f,z,\delta)|\, d\sigma(\zeta).$$

A function f in $L^1_{\text{loc}}(\partial\Omega)$ is in the anisotropic space $\mathrm{BMO}(\partial\Omega)$ if

$$\|f\|_{\mathrm{BMO}} = \sup_{z,\ \delta>0} \operatorname{osc}(f,z,\delta) < +\infty.$$

Let $f \in \mathrm{BMO}(\partial\Omega)$ and $0 < r < 1$. We note $M_r(f) = \sup \operatorname{osc}(f,z,\delta)$ where the supremum is considered for z on $\partial\Omega$ and $0 < \delta \le r$. The function f is in $\mathrm{VMO}(\partial\Omega)$ if $\lim_{r\longrightarrow 0} M_r(f) = 0$.

Let us recall the definition of S_p. If Θ is a compact operator in a Hilbert space H we can consider (s_i) the sequence of eigenvalues of $(\Theta^*\Theta)^{1/2}$. The s_i are called singular values of Θ. The operator Θ is said to belong to S_p if and only if (s_i) is in ℓ^p. The space S_p endowed with the norm $\|\Theta\|_{S_p} = (\sum_{i=0}^{\infty} s_i^p)^{1/p}$ is a Banach space when $1 \le p < +\infty$. The space S_1 is called the Trace Class of H and S_2 is the Hilbert Schmidt class [GK]. The following theorem holds :

THEOREM A. *Let f in $L^2(\partial\Omega)$ and h defined by (1).*

(i) If $f \in \mathrm{BMO}(\partial\Omega)$ then h is bounded,
(ii) if $f \in \mathrm{VMO}(\partial\Omega)$ then h is compact.
(iii) Let $1 \le p < +\infty$ and $l \in \mathbb{N}$ such that $lp > 2$, $f \in L^2(\partial\Omega) \cap H(\Omega)$ such that $\big(-r(z)\big)^l \nabla_z^l f \in L^p\big(B(z,z)dV(z)\big)$ then $h \in S_p$.

In the part (iii) of the theorem, the condition $lp > 2$ insures that the weight $\big(-r(z)\big)^{pl} B(z,z)$ is an integrable function. Let us remark that for l' in $\mathbb{N}$, $\big(-r(\zeta)\big)^l \nabla^l b$ in $L^p\big(\Omega, B(\zeta,\zeta)dV(\zeta)\big)$ if and only if $\big(-r(\zeta)\big)^{l+l'} \nabla^{l+l'} b$ in $L^p\big(\Omega, B(\zeta,\zeta)dV(\zeta)\big)$.

In Sections 2 and 3 we give the proof of the theorem A and in the section 4 we study small Hankel operators defined on weighted Bergman spaces h_q, $q \in \mathbb{N}$.

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2**. Boundedness and compactness.** The proof of (i) is classical. The Szegö projection is a singular integral operator with respect to the pseudometric d [CW]. We can consider $C_f g = S(fg) - f\,\mathrm{S}g$ the commutator associated to f. Let us remark that for g in $L^2(\partial\Omega)$,

$$hg = C_f\overline{\mathrm{S}g} + fS\overline{\mathrm{S}g}. \tag{5}$$

For f in BMO$(\partial\Omega)$, the proof of S. Janson [J] extends to this context to show that C_f is bounded. We prove now that $fS\overline{S}(\cdot)$ is a compact operator. We have only to prove that adjoint operator is bounded from $H^1(\partial\Omega)$ into $L^2(\partial\Omega)$ or $fS\overline{S}(\cdot)$ is bounded from $H^{-1}(\partial\Omega)$ in $L^2(\partial\Omega)$. Let g_1 in $H^{-1}(\partial\Omega)$ and g_2 in $L^2(\partial\Omega)$. Then,

$$\begin{aligned}\left\langle fS(\overline{S}g_1), g_2\right\rangle &= \left\langle \overline{S}g_1, S(\overline{f}g_2)\right\rangle\\ &= \int_{\partial\Omega} (\overline{S}g_1)(z)\overline{S(fg_2)}(z)\,d\sigma(z).\end{aligned}$$

Using a partition of the unity, we assume that $\frac{\partial r}{\partial z_1} \neq 0$. Then

$$\left\langle fS(\overline{S}g_1), g_2\right\rangle = \int_{\Omega} (\overline{S}g_1)(z)\overline{S(fg_2)}(z)\frac{\partial}{\partial z_1}\left(\frac{1}{\frac{\partial r}{\partial z_1}}\right) r(z)\,dV(z)$$

and

$$\begin{aligned}&\left|\left\langle fS(\overline{S}g_1), g_2\right\rangle\right|\\ &\quad\leq C\Big(\int_\Omega |\overline{S}g_1(z)|^2\big(-r(z)\big)\,dV(z)\Big)^{1/2}\Big(\int_\Omega |S(\overline{f}g_2)(z)|^2\big(-r(z)\big)\,dV(z)\Big)^{1/2}.\end{aligned}$$

Let us remark that, for a harmonic function F, we have

$$\Big(\int_\Omega |F(z)|^2\big(-r(z)\big)\,dV(z)\Big)^{1/2} \simeq \|F\|_{H^{-1/2}(\Omega)} \simeq \|F\|_{H^{-1}(\partial\Omega)}.$$

Since the operator S is bounded in $H^s(\partial\Omega)$ [B], we obtain

$$\left|\left\langle fS(\overline{S}g_1), g_2\right\rangle\right| \leq \|g_2\|_{H^{-1}(\partial\Omega)}\|\overline{f}g_1\|_{H^{-1}(\partial\Omega)}.$$

Let v in $H^1(\partial\Omega)$.

$$\Big|\int_{\partial\Omega} \overline{f}(z)g_1(z)v(z)\,d\sigma(z)\Big| \leq \|fv\|_{L^2(\partial\Omega)}\|g_1\|_{L^2(\partial\Omega)}.$$

By the Sobolev theorem, the function v is in $L^{2+\varepsilon}(\partial\Omega)$ and

$$\|fv\|_{L^2(\partial\Omega)} \leq C(f)\|v\|_{H^1(\partial\Omega)}.$$

This finishes the proof of the compactness of $fS\overline{S(\cdot)}$.

For the proof of the part (ii) of the theorem A, we use the relation (5). We have only to prove that the first operator is a limit of compact operators.

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Let $r > 0$ and $f_r(z) = m(f, z, r) = \frac{1}{\sigma(B(z,r))} \int_{B(z,r)} f(\zeta)\, d\sigma(\zeta)$. The function f_r is continuous on $\partial\Omega$, it is the uniform limit of f_n in $C^\infty(\partial\Omega)$. We then have

$$C_f = \left(C_f - C_{f_r}\right) + \left(C_{f_r} - C_{f_n}\right) + C_{f_n}.$$

Let (g_i) in $L^2(\partial\Omega)$ such that $g_i \longrightarrow 0$ weakly and let $\varepsilon > 0$. It follows from the theorem of Banach Steinhaus that there exists $M > 0$ such that $\|g_i\|_{L^2(\partial\Omega)} \le M$, $i \ge 0$.

For the unit ball of $\mathbb{C}^n$, R. Coifman, R. Rochberg and G. Weiss [CRW] proved that there exists $C > 0$ such that

$$\|(C_f - C_{f_r})g_i\|_{L^2(\partial\Omega)} \le CM_{Cr}(f)\|g_i\|_{L^2(\partial\Omega)}.$$

The result is still valid in the case of homogeneous domains. By definition of $\mathrm{VMO}(\partial\Omega)$, there exists $r > 0$ such that $CM_{Cr}(f) \le \varepsilon/3M$. Then

$$\|(C_f - C_{f_r})g_i\|_{L^2(\partial\Omega)} \le \varepsilon/3.$$

Let us remark that $(C_{f_r} - C_{f_n}) = C_{f_r - f_n}$. For g in $L^2(\partial\Omega)$

$$\|(C_{f_r - f_n})g\|_{L^2(\partial\Omega)} \le 2 \sup_{\zeta\in\partial\Omega} |f_r(\zeta) - f_n(\zeta)|\, \|g\|_{L^2(\partial\Omega)}.$$

Let n_0 such that, for $n \ge n_0$, $\sup_{\zeta\in\partial\Omega} |f_r(\zeta) - f_n(\zeta)| < \varepsilon/6M$, then

$$\|(C_{f_r} - C_{f_n})g_i\|_{L^2(\partial\Omega)} < \varepsilon/3.$$

For g in $L^2(\partial\Omega)$,

$$C_{f_n}g(z) = \int_{\partial\Omega} S(z,\zeta)\big(f_n(z) - f_n(\zeta)\big)g(\zeta)\, d\sigma(\zeta).$$

We use the pointwise estimates of the Szegö kernel to prove that C_{f_n} is an operator of order 1 in the sense of [NRSW]. Let

$$N_z = 4\left(\frac{\partial r}{\partial \bar z_1}\frac{\partial}{\partial z_1} + \frac{\partial r}{\partial \bar z_2}\frac{\partial}{\partial z_2}\right)$$

the complex normal direction such that $N_z r(z) = |\nabla r(z)|^2 = 1$ on $\partial\Omega$ and

$$L_z = \frac{\partial r}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2}$$

the complex tangential direction. The sequence f_n is in $C^\infty(\partial\Omega)$, then $|f_n(z) - f_n(\zeta)| \le C\tau\big(z, d(z,\zeta)\big)$ and

$$\Big|X_1 \cdots X_{k+l}\Big(S(z,\zeta)\big(f(z) - f(\zeta)\big)\Big)\Big| \le C\tau\big(z, d(z,\zeta)\big)\frac{\tau\big(z, d(z,\zeta)\big)^{-k-l}}{\sigma\Big(B\big(z, d(z,\zeta)\big)\Big)},$$

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when k of the X_j are L_z or $\overline{L}_z$ and l are L_ζ or $\overline{L}_\zeta$.

We recall now the definition of the anisotropic Sobolev spaces L^p_k. Define

$$L^k_p = \left\{f \in L^p(\partial\Omega)\,;\, L^j f \in L^p(\partial\Omega), 1 \le j \le k\right\}.$$

It was proved in [NRSW] that an operator of order 1 maps L^p into L^p_1, $1 < p < +\infty$. Then C_{f_n} is bounded from $L^2(\partial\Omega)$ into $L^2_1(\partial\Omega)$ and therefore it is a compact operator in $L^2(\partial\Omega)$. There exists i_0 such that, for $i \ge i_0$,

$$\|C_{f_n} g_i\|_{L^2(\partial\Omega)} \le \varepsilon/3.$$

Let us remark that the operator $S\overline{S}$ can be seen as a Friedrichs operator. It was proved in [KLLR] that such operators are Hilbert Schmidt operators.

3. **Schatten class.** If (e_i) and (f_i) are two orthonormal basis, a compact operator Θ in a Hilbert space H has the following Schmidt decomposition

$$\Theta = \Theta(\lambda) = \sum_{i=0}^{\infty} \lambda_i \langle e_j, \cdot\rangle f_j, \tag{6}$$

where $\langle\,,\,\rangle$ is the inner product in H. If Θ is given by (6), then $\lambda_j = s_j$ [GK]. We follow the method developed by R. Rochberg and S. Semmes [RS1] and [RS2]. We use a generalization of the Schmidt decomposition to approximate the singular values.

In the following, we shall consider domains $Q(z,\delta)$ for $z \in \Omega$, so we extend d_0 to $\mathbb{C}^2$ with the euclidian distance. Let $\psi \in C^\infty(\mathbb{C}^2)$ such that $\psi(z,\zeta) = 1$ when $|r(z)| \le \varepsilon/2$ and $|r(\zeta)| \le \varepsilon/2$ and $\psi(z,\zeta) = 0$ when $|r(z)| \ge \varepsilon$ or $|r(\zeta)| \ge \varepsilon$.

DEFINITION 3.1. *Let z and ζ in $\mathbb{C}^2$. Then,*

$$d(z,\zeta) = \psi(z,\zeta) d_0(z,\zeta) + \big(1 - \psi(z,\zeta)\big)|z - \zeta|.$$

Let

$$Q(z,\delta) = \left\{\zeta \in \mathbb{C}^2, d(z,\zeta) < \delta\right\}.$$

We consider a Whitney covering of Ω by domains $Q\big(w, \eta\delta(w)\big)$, $0 < \eta < 1$ and we denote by Q_j the ball $Q\big(w_j, \eta\delta(w_j)\big)$. We fix $C_0 > 0$ such that $\tilde{Q}_j \cap \tilde{Q}_{j'} = \emptyset$ if $j \ne j'$, where $\tilde{Q}_j = Q\big(w_j, \eta\delta(w_j)/C_0\big)$. Let $\pi(Q_j) = B_j$.

We use the Whitney covering to define the nearly weakly orthogonal (N.W.O.) family of elements of $L^2(\partial\Omega)$.

DEFINITION 3.2. *The family (e_j) in $L^2(\partial\Omega)$ is a N.W.O. family if and only if*

(i) $\|e_j\|_{L^2(\partial\Omega)} \simeq 1$,

(ii) the maximal operator T^* defined on $L^2(\partial\Omega)$ by

$$T^* f(z) = \sup_{z \in B_j} \frac{1}{\sigma(B_j)^{1/2}} |\langle f, e_j\rangle|$$

is bounded in $L^2(\partial\Omega)$.

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Such families allow us to prove that a compact operator belongs to the Schatten class S_p, $1 \le p < +\infty$.

THEOREM 3.3. *Let Θ be a compact operator on $L^2(\partial\Omega)$.*

(i) If Θ is given by (6), where (e_j) and (f_j) are two N.W.O. families and $(\lambda_j) \in \ell^p$, $1 \le p < +\infty$, then

$$\|\Theta\|_{S_p} \le C\Big(\sum_j |\lambda_j|^p\Big)^{1/p}.$$

(ii) Let $\Theta \in S_p$, $1 < p < +\infty$ and (e_j), (f_j) two N.W.O. families. Then

$$\Big(\sum_j |\langle e_j, \Theta f_j\rangle|^p\Big)^{1/p} \le C\|\Theta\|_{S_p}.$$

The following proposition provides the N.W.O. family that we shall use to study small Hankel operators.

PROPOSITION 3.4. *The family (e_j) defined by*

$$e_j(z) = \delta(w_j)\tau^2\big(w_j, \delta(w_j)\big)S(z, w_j)$$

is a N.W.O. family.

The proof of the theorem 3.3 and the proposition 3.4 can be found in [S2] in which they are given for complex ellipsoids in $\mathbb{C}^n$.

We shall prove that a small Hankel operator satisfies the relation (6) with (λ_j) in ℓ^p and e_j as above. This decomposition follows from a theorem of atomic decomposition of $N_z^l f$ in $L^p\Big(\big(-r(z)\big)^{lp}B(z,z)dV(z)\Big)$. The method is due to R. Coifman and R. Rochberg [CR] (see also [Co] and [S1]). In this case, the function $N_z^l f$ is not holomorphic, but we use the fact that f is holomorphic to prove an integral representation for $N_z^l f$. This representation is given with $N_z^l S(z,\zeta)$ and derivatives of f. This is done by Green formula and integration by parts. Then, following [CR], we use a η-lattice to approximate $N_z^l f$ with a Riemann sum. The theorem follows by iteration.

For g in $L^p\Big(\big(-r(z)\big)^{lp}B(z,z)dV(z)\Big)$ Let

$$\|g\|_{l,p} = \Big(\int_\Omega |g(z)|^p\big(-r(z)\big)^{lp}B(z,z)\,dV(z)\Big)^{1/p}.$$

In the following, we consider a function f in $L^2(\partial\Omega) \cap \mathcal{H}(\Omega)$ such that the function $N_z^l f$ is in $L^p\Big(\big(-r(z)\big)^{pl}B(z,z)dV(z)\Big)$, $lp > 2$ and a Whitney covering of Ω by domains $Q\big(w_j, \eta\delta(w_j)\big)$. We have the following result.

THEOREM 3.5. *There exists (λ_j) in ℓ^p such that*

$$f(z) = \sum_j \lambda_j \delta(w_j)\tau^2\big(w_j, \delta(w_j)\big)S(z, w_j).$$

PROOF. We begin to prove an integral representation for f.

PROPOSITION 3.6. *Let* $1 \le q \le l$. *There exists* f_q *in* $L^p\big(B(z,z)dV(z)\big)$ *such that*

(i) $\|f_q\|_{l,p} < +\infty$,

(ii) $f(z) = \int_\Omega S(z,\zeta) f_q(\zeta)\, dV_{q-1}(\zeta)$, $z \in \Omega$.

PROOF. Let $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$, $|\mathbf{k}| = k_1 + k_2$ and $D_{\mathbf{k}} = \frac{\partial^{k_1+k_2}}{\partial \zeta_1^{k_1} \partial \zeta_2^{k_2}}$. Let z in Ω. We use a construction of S. Bell [B1] and [B2]. Since $N_z r(z) = 1$ on $\partial\Omega$,

$$f(z) = 4 \sum_{i=1,2} \int_{\partial\Omega} S(z,\zeta) f(\zeta) \frac{\partial r}{\partial \zeta_i} \frac{\partial r}{\partial \bar\zeta_i}\, d\sigma(\zeta)$$

and the Green formula gives

$$f(z) = \int_\Omega S(z,\zeta)\big(\Delta r(\zeta) f(\zeta) + N_\zeta f(\zeta)\big)\, dV(\zeta).$$

Let $q \ge 1$. We suppose that the part (ii) of the proposition is true for $q-1$. Recall that there exist two functions $a(\cdot)$ and $b(\cdot)$ in $C^\infty(\bar\Omega)$ such that [S1]

$$1 = a(\zeta) N_\zeta r(\zeta) + b(\zeta)\big(-r(\zeta)\big). \tag{7}$$

Then,

$$\begin{aligned} f(z) &= \int_\Omega S(z,\zeta) f_{q-2}(\zeta) b(\zeta)\big(-r(\zeta)\big)^{q-1}\, dV(\zeta) \\ &\qquad - \frac{4}{q-1} \sum_{i=1,2} \int_\Omega S(z,\zeta) f_{q-2}(\zeta) a(\zeta) \frac{\partial r(\zeta)}{\partial \zeta_i} \frac{\partial}{\partial \bar\zeta_i}\Big(\big(-r(\zeta)\big)^{q-1}\Big)\, dV(\zeta) \\ &= I_1 + I_2. \end{aligned}$$

It remains to integrate I_2 by part with respect $\bar\zeta_1$ and $\bar\zeta_2$. By induction, we obtain functions $a_{\mathbf{k}}(\zeta)$ in $C^\infty(\bar\Omega)$, $|\mathbf{k}| \le q$ such that

$$f_q(\zeta) = \sum_{|\mathbf{k}| \le q} a_{\mathbf{k}}(\zeta) D_{\mathbf{k}} f(\zeta) \tag{8}$$

The part (i) of the proposition follows immediately from the preceding relation. ■

We use the integral formula to prove the theorem of atomic decomposition. We consider a Whitney covering of Ω with domains $Q\big(w, \eta\delta(w)\big)$, $\eta > 0$ small enough [CR]. The sequence (w_j) is called a η-lattice of Ω. The theorem follows from the proposition.

PROPOSITION 3.7. *There exists* G *in* $H(\Omega)$ *such that*

(i)

$$G(z) = \sum_j \nu_j \delta(w_j) \tau^2\big(w_j, \delta(w_j)\big) S(z, w_j),$$

with (ν_j) *is in* ℓ^p,

(ii)

$$\|N_z^l G\|_{l,p} \le C \sum_{|\mathbf{k}| \le l} \|D_{\mathbf{k}} f\|_{l,p},$$

(iii) $\|N_z^l f - N_z^l G\|_{l,p} \le \frac{1}{2} \sum_{|\mathbf{k}| \le l} \|D_{\mathbf{k}} f\|_{l,p}$.

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PROOF. It follows from the proposition 3.6 that

$$N_z^l f(z) = \int_\Omega N_z^l S(z,\zeta) f_l(\zeta)\, dV_{l-1}(\zeta), \quad z \in \Omega, \tag{9}$$

where f_l is given by the relation (8). We consider the domains E_j defined by

$$E_0 = Q_0 \setminus \Big(\bigcup_{j=1}^{+\infty} \tilde{Q}_i\Big) \quad \text{and} \quad E_j = Q_j \setminus \Big(\Big(\bigcup_{k=0}^{j-1} E_k\Big) \cup \Big(\bigcup_{k=j+1}^{+\infty} \tilde{Q}_k\Big)\Big).$$

Then $\tilde{Q}_j \subseteq E_j \subseteq Q_j$, $\bigcup_{j=0}^{+\infty} E_j = \Omega$ and $E_j \cap E_{j'} = \emptyset$ if $j \neq j'$. We use the fact that the domains E_j are mutually disjoint to approximate $N_z^l f$ by the function $N_z^l G$, where G is defined by

$$G(z) = \sum_j \mathrm{Vol}(E_j)\big(-r(w_j)\big)^{l-1} f_l(w_j) S(z, w_j).$$

In this case $\nu_j = \frac{f_l(w_j)\,\mathrm{Vol}(E_j)(-r(w_j))^{l-1}}{\delta(w_j)\tau^2(w_j,\delta(w_j))}$.

We begin to prove that (ν_j) is in ℓ^p. Let w_j be a point of the Whitney covering. By construction $\mathrm{Vol}(E_j) \simeq \delta^2(w_j)\tau^2\big(w_j, \delta(w_j)\big)$. Then

$$|\nu_j|^p \le C\delta(w_j)^{pl} |f_l(w_j)|^p \le C\delta(w_j)^{pl} \sum_{|\mathbf{k}|\le l} |a_{\mathbf{k}}(w_j)|^p |D_{\mathbf{k}} f(w_j)|^p.$$

Let us remark that $D_{\mathbf{k}} f$ is a holomorphic function, the subharmonicity of $|D_{\mathbf{k}} f|^p$ gives

$$|D_{\mathbf{k}} f(w_j)|^p \le \frac{C}{\mathrm{Vol}(\tilde{Q}_j)} \int_{\tilde{Q}_j} |D_{\mathbf{k}} f(w)|^p\, dV(w).$$

If w is in $\tilde{Q}_j$, $B(w,w) \simeq \mathrm{Vol}(\tilde{Q}_j)^{-1}$ [Ca] and

$$|D_{\mathbf{k}} f(w_j)|^p \le C \int_{\tilde{Q}_j} |D_{\mathbf{k}} f(w)|^p B(w,w)\, dV(w). \tag{10}$$

We use the relation (10) and the fact that $\delta(w) \simeq \delta(w_j)$ in $\tilde{Q}_j$ to obtain

$$\begin{aligned}\sum_j |\nu_j|^p &\le C \sum_j \int_{\tilde{Q}_j} \sum_{|\mathbf{k}|\le l} |D_{\mathbf{k}} f(\zeta)|^p \delta(\zeta)^{pl} B(\zeta,\zeta)\, dV(\zeta)\\ &\le C \sum_{|\mathbf{k}|\le l} \|D_{\mathbf{k}} f\|_{l,p}^p < +\infty.\end{aligned}$$

For the proof of (ii) and (iii), we consider the kernel function $C_{l-1}(z,\zeta)$ defined by

$$\begin{aligned} C_{l-1}(z,\zeta) &= \tau\big(z, D(z,\zeta)\big)^{-2} D(z,\zeta)^{-1-l} && \text{if } z \text{ and } \zeta \text{ in } U\cap\Omega\\ C_{l-1}(z,\zeta) &= 1 && \text{otherwise}\end{aligned}$$

and we consider the family of functions in $L^p\big(\delta(z)^{lp} B(z,z) dV(z)\big)$

$$L_j(z) = \delta(w_j)\tau^2\big(w_j, \delta(w_j)\big) C_{l-1}(z, w_j).$$

We use the auxiliary result.

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PROPOSITION 3.8. *Let L_j as above.*
(i) Let $1 \le p < +\infty$ and (λ_j) in ℓ^p. There exists $C > 0$ such that

$$\Big\|\sum_j \lambda_j L_j\Big\|_{l,p} \le C\Big(\sum_j |\lambda_j|^p\Big)^{1/p}.$$

(ii) There exists $\gamma(\Omega) = \gamma \ge 1$ such that, for z in Ω,

$$|N_z^l f(z) - N_z^l G(z)| \le C\eta^{1/m} \sum_j \Big(\int_{\hat{Q}_j} \sum_{|\mathbf{k}|\le l} |D_{\mathbf{k}} f(\zeta)|\big(-r(\zeta)\big)^{lp} B(\zeta,\zeta)\, dV(\zeta)\Big)^{1/p} L_j(z),$$

where $\hat{Q}_j = Q\big(w_j, \gamma\delta(w_j)\big)$.

Notice that the domains $\hat{Q}_j$ are almost disjoint, we have the following result:

COROLLARY 3.9. *There exists $C > 0$ such that*

$$\|N_z^l f - N_z^l G\|_{l,p} \le C\eta^{1/m} \sum_{|\mathbf{k}|\le l} \|D_{\mathbf{k}} f\|_{l,p}.$$

The proposition 3.7 follows if η is small enough such that $C\eta^{1/m} < 1/2$.

PROOF. We follow the method of [CR] to prove the part (i) of the proposition 3.8. We consider the function

$$k(z) = \sum_j |\lambda_j| \delta(w_j)^{-l} \chi_{E_j}(z).$$

Let us remark that $\|k\|_{l,p} \simeq (\sum_j |\lambda_j|^p)^{1/p}$ and there exists $C > 0$ such that

$$\Big|\sum_j \lambda_j L_j(z)\Big| \le \sum_j |\lambda_j| \delta(w_j) \tau\big(w_j, \delta(w_j)\big)^2 C_{l-1}(z, w_j) \le C T_{l-1} k(z),$$

where T_{l-1} is the integral operator associated to C_{l-1} and defined by

$$T_{l-1} g(z) = \int_\Omega C_{l-1}(z,\zeta) g(\zeta)\, dV_{l-1}(\zeta).$$

We only have to prove that T_{l-1} is bounded in $L^p\big(\delta(z)^{pl} B(z,z) dV(z)\big)$.

If $p = 1$. $B(z,z) \simeq \delta(z)^{-2}\tau\big(z, \delta(z)\big)^{-2}$, we have for ζ in Ω [BCG],

$$\int_\Omega C_{l-1}(z,\zeta)\delta(z)^l B(z,z)\, dV(z) \le C\delta(\zeta)^{-1}\tau\big(\zeta,\delta(\zeta)\big)^{-2},$$

therefore

$$\begin{aligned}
\|T_{l-1} g\|_{l,1} &\le \int_\Omega |g(\zeta)| \Big(\int_\Omega C_{l-1}(z,\zeta)\delta(z)^l B(z,z)\, dV(z)\Big) \delta(\zeta)^{l-1}\, dV(\zeta)\\
&\le C \int_\Omega |g(\zeta)| \delta(\zeta)^{l-2} \tau\big(\zeta,\delta(\zeta)\big)^{-2}\, dV(\zeta)\\
&\le C\|g\|_{l,1}.
\end{aligned}$$

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If $1 < p < +\infty$. We denote by s the function such that $s(x) = 2$ if $x < 0$ and $s(x) = m$ if $x > 0$. It is well known that T_{l-1} is bounded in $L^p\Big(\delta(z)^\alpha \tau\big(z, \delta(z)\big)^\beta dV(z)\Big)$ for α and β such that $0 < 1 + \alpha + \frac{\beta}{s(\beta)}$ and $1 + \alpha + \frac{\beta}{s(-\beta)} < lp$ [S1]. The choice $\alpha = lp - 2$ and $\beta = -2$ allows us to show that T_{l-1} is bounded in $L^p\big(\delta(z)^{lp} B(z,z) dV(z)\big)$. ∎

For the part (ii) of the proposition 3.8, we consider z in Ω. Then,

$$\begin{aligned} |N_z^l f(z) - N_z^l G(z)| \;\le\; & \sum_j |N_z^l S(z, w_j)| \int_{E_i} |f_l(\zeta) - f_l(w_j)|\, dV_{l-1}(\zeta) \\ & + \sum_j \int_{E_j} |f_l(\zeta)|\, |N_z^l S(z, w_j) - N_z^l S(z, \zeta)|\, dV_{l-1}(\zeta). \end{aligned}$$

We use the technical result.

LEMMA 3.10. *Let $\theta > 0$ such that $Q\big(z, \theta\delta(z)\big) \subset \Omega$. Let z in U, $w \in Q\big(z, \theta\delta(z)\big)$ and η small enough so that $Q\big(w, \eta\delta(w)\big) \subset Q\big(z, \theta\delta(z)\big)$. There exist $\gamma > 0$ and $C > 0$ such that*

(i)

$$\sup_{\zeta \in Q(w, \eta\delta(w))} |N_z^l S(z, \zeta) - N_z^l S(z, w)| \le C\eta^{1/m} D(z, w)^{-(1+l)} \tau^{-2}\big(z, D(z, w)\big),$$

(ii)

$$\sup_{\zeta \in Q(w, \eta\delta(w))} |f_l(\zeta) - f_l(w)|^p \le C\eta^{p/m} \int_{\hat{Q}} \sum_{|\mathbf{k}| \le l} |D_{\mathbf{k}} f(\zeta)|^p B(\zeta, \zeta)\, dV(\zeta),$$

where $\hat{Q} = Q\big(z, \gamma\delta(z)\big)$.

PROOF. The proof of the part (i) of the lemma is given in [S1] for the Bergman kernel, the method is the same for the Szegö kernel.

The part (ii) follows from the subharmonicity of $|D_{\mathbf{k}} f|^p$. Let ζ in $Q\big(w, \eta\delta(w)\big)$ and w in Ω. From (8),

$$\begin{aligned} |f_l(\zeta) - f_l(w)|^p \;\le\; & C \sum_{|\mathbf{k}| \le l} |a_{\mathbf{k}}(w)|^p |D_{\mathbf{k}} f(\zeta) - D_{\mathbf{k}} f(w)|^p \\ & + C \sum_{|\mathbf{k}| \le l} |a_{\mathbf{k}}(\zeta) - a_{\mathbf{k}}(w)|^p |D_{\mathbf{k}} f(\zeta)|^p. \end{aligned}$$

Let us remark that $|a_{\mathbf{k}}(\zeta) - a_{\mathbf{k}}(w)| \le C\eta^{1/m}$ if ζ in $Q\big(w, \eta\delta(w)\big)$. Then,

$$|f_l(\zeta) - f_l(w)|^p \le C \sum_{|\mathbf{k}| \le l} |D_{\mathbf{k}} f(\zeta) - D_{\mathbf{k}} f(w)|^p + C\eta^{p/m} \sum_{|\mathbf{k}| \le l} |D_{\mathbf{k}} f(\zeta)|^p.$$

The lemma follows from [S1] for the first sum and from the relation (10) for the second. ∎

We use the proposition 3.6 to finish the proof of the theorem by iteration. It remains to prove that the integral formula (9) is true when f is replaced by G. We denote by T the integral operator associated to the Kernel $|N_z^l S(z, \zeta)|$ and defined by

$$Tg(z) = \int_\Omega |N_z^l S(z, \zeta)| g(\zeta)\, dV_{l-1}(\zeta).$$

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Let j_0 in $\mathbb{N}$. We denote by $G_{j_0}(z)$ the truncated function

$$G_{j_0}(z) = \sum_{j\leq j_0} \nu_j \delta(w_j)\tau^2\big(w_j, \delta(w_j)\big)S(z, w_j).$$

Then,

$$N_z^l G_{j_0}(z) = \int_\Omega N_z^l S(z,\zeta) G_{l,j_0}(\zeta)\, dV_{l-1}(\zeta),$$

where G_{l,j_0} is given by the relation (8). Let us remark that there exists $C > 0$ such that $|N_z^l S(z,\zeta)| \leq CC_{l-1}(z,\zeta)$. By Proposition 3.8, the operator T is bounded in $L^p\big(\delta(z)^{lp}B(z,z)dV(z)\big)$, $1 \leq p < +\infty$, then the relation (9) is true for G.

Let G^i be the function associated to $f - \sum_{k=0}^{i-1} G^k$. It follows from the proposition that

$$\Big\| N_z^l f - \sum_{k=0}^{i-1} N_z^l G^k \Big\|_{l,p} \leq 2^{-i} \sum_{\mathbf{k}\leq l} \|D_{\mathbf{k}} f\|_{l,p}.$$

Then $N_z^l f = \sum_{i=0}^{\infty} N_z^l G^i$. ∎

The theorem 3.3 allows us to prove that h is in S_p. Let f in $L^2(\partial\Omega)$ such that $\big(-r(z)\big)^l \nabla_z^l f \in L^p\big(B(z,z)dV(z)\big)$. There exists (λ_j) in ℓ^p such that

$$f(z) = \sum_j \lambda_j \delta(w_j)\tau^2\big(w_j, \delta(w_j)\big)S(z, w_j).$$

Let g in $L^2(\partial\Omega)$. Then,

$$\begin{aligned} hg(z) &= \int_{\partial\Omega} S(z,\zeta) f(\zeta)\overline{\mathrm{Sg}}(\zeta)\, d\sigma(\zeta) \\ &= \sum_j \lambda_j \delta(w_j)\tau^2\big(w_j, \delta(w_j)\big) \int_{\partial\Omega} S(\zeta, w_j) S(z,\zeta)\overline{\mathrm{Sg}}(\zeta)\, d\sigma(\zeta) \\ &= \sum_j \lambda_j \delta(w_j)\tau^2\big(w_j, \delta(w_j)\big) S(z, w_j)\overline{\mathrm{Sg}}(w_j). \end{aligned}$$

Let us remark that $\overline{\mathrm{Sg}}(w_j) = \int_{\partial\Omega} S(\zeta, w_j)\bar{g}(\zeta)\, d\sigma(\zeta)$. Then

$$hg = \sum_{j=0}^{\infty} \lambda_j \langle e_j, g\rangle e_j,$$

where (e_j) is a N.W.O. family and (λ_j) is in ℓ^p. By Theorem 3.3, h is in S_p. ∎

4. **Hankel operators in Bergman spaces.** In this section, we study small Hankel operators defined on weighted Bergman spaces. Recall that the Bloch space is defined by:

$$\tilde{B} = \Big\{ f \in C^1(\Omega), \sup_z |r(z)\nabla f(z)| < +\infty \Big\}$$

and $B = \tilde{B} \cap H(\Omega)$. It is well known that for a function f in B, there exists $C = C(f) > 0$ such that $|f(\zeta)| \leq C\Big|\ln\big(-r(\zeta)\big)\Big|$, $\zeta \in \Omega$.

The little Bloch space is the subspace of B defined by:

$$B_0 = \Big\{ f \in B, \lim_{z\to\partial\Omega} |r(z)\nabla f(z)| = 0 \Big\}.$$

The following theorem holds :

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THEOREM B. *Let q in $\mathbb{N}$, f in $A^2(dV_q)$ and h_q defined by (3). Then,*

(i) If $f \in \mathcal{B}$ then h_q is bounded,

(ii) if $f \in \mathcal{B}_0$ then h_q is compact,

(iii) Let $1 \le p < +\infty$ and $l \in \mathbb{N}$ such that $lp > 2$, if f in $A^2(dV_q)$ such that $\big(-r(z)\big)^l \nabla_z^l f \in L^p\big(B(z,z)dV(z)\big)$ then $h_q \in S_p$.

PROOF. For the part (i), we consider f in $\mathcal{B}$ and $g \in L^2(dV_q)$. Let us remark that $\zeta \longrightarrow B_q(z,\zeta)\overline{B_q g}(\zeta)$ is an antiholomorphic function, the relation (7) gives

$$
\begin{aligned}
hg(z) &= \int_\Omega B_q(z,\zeta) f(\zeta)\overline{B_q g}(\zeta)\, dV_q(\zeta)\\
&= \int_\Omega B_q(z,\zeta)F(\zeta)\overline{B_q g}(\zeta)\, dV_{q+1}(\zeta),
\end{aligned}
\tag{11}
$$

where $F(\zeta) = N_\zeta f(\zeta) + \Big(-b(\zeta) + \frac{N_\zeta a(\zeta)+a(\zeta)\Delta r(\zeta)}{1+q}\Big) f(\zeta)$. The function f is in $\mathcal{B}$, then

$$
\sup_{\zeta\in\Omega}\big(-r(\zeta)\big)|F(\zeta)| \le \sup_{\zeta\in\Omega} C\big(-r(\zeta)\big)\Big(|\nabla f(\zeta)| + \big|\ln\big(-r(\zeta)\big)\big|\Big) < +\infty \tag{12}
$$

Let $G(\zeta) = \big(-r(\zeta)\big)F(\zeta)\overline{B_q g}(\zeta)$. The function G is in $L^2(dV_q)$ and $\|G\|_{L^2(dV_q)} \le C\|g\|_{L^2(dV_q)}$. We then have

$$
\|hg\|_{L^2(dV_q)} \le \|B_q G\|_{L^2(dV_q)} \le \|G\|_{L^2(dV_q)} \le C\|g\|_{L^2(dV_q)}.
$$

Let f in $\mathcal{B}_0$. Let $\delta > 0$ and $\Omega_\delta = \{z \in \Omega, -\delta < r(z) < 0\}$. Let φ_δ defined on Ω by $\varphi_\delta(\zeta) = 1$ if $\zeta \in \Omega_\delta$ and 0 otherwise. For g in $L^2(dV_q)$ and z in Ω, it follows from (11) that

$$
\begin{aligned}
hg(z) &= \int_{\Omega_\delta} B_q(z,\zeta)\overline{B_q g}(\zeta)F(\zeta)\varphi_\delta(\zeta)\, dV_{q+1}(\zeta)\\
&\qquad + \int_\Omega B_q(z,\zeta)F(\zeta)\overline{B_q g}(\zeta)\big(1-\varphi_\delta(\zeta)\big)\, dV_{q+1}(\zeta)\\
&= h_1(\overline{B_q g})(z) + h_2(\overline{B_q g})(z).
\end{aligned}
$$

Let $\varepsilon > 0$ and g' in $L^2(dV_q)$. Then,

$$
|h_1 g'(z)| \le \sup_{\delta(\zeta)<\delta}\big(-r(\zeta)\big)|F(\zeta)| \int_\Omega |B_q(z,\zeta)|\,|g'(\zeta)|\, dV_q(\zeta)
$$

and $\|h_1 g'\|_{2,q} \le C' \sup_{\delta(\zeta)<\delta} |F(\zeta)|\, \|g'\|_{2,q}$. If $\delta > 0$ is small enough, from relation (12), $\sup_{\delta(\zeta)<\delta} |F(\zeta)| \le \varepsilon / C'$ and

$$
\|h_1 g'\|_{2,q} \le \varepsilon \|g'\|_{2,q}.
$$

It remains to prove that h_2 is a compact operator. This operator is an integral operator with kernel $B_q(z,\zeta)\big(1-\varphi_\delta(\zeta)\big)\big(-r(\zeta)\big)F(\zeta)$. Let us remark that for ζ in Ω,

$$
\int_\Omega B_q(z,\zeta)B_q(\zeta,z)\, dV_q(z) = B_q(\zeta,\zeta).
$$

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The function f is in the little Bloch space, there exists $C = C(\delta) > 0$ such that

$$\int_\Omega \int_\Omega |B_q(z,\zeta)\varphi_\delta(\zeta)|^2 \, dV_q(z)\, dV_q(\zeta) \le C \int_{\Omega_\delta} B_q(\zeta,\zeta)\, dV_q(\zeta) \le C(\delta).$$

Then h_2 is a Hilbert Schmidt type operator and hence a compact operator.

For the part (iii), we approximate h by finite rank operators defined with the sequence $\delta(w_J)^{2+q}\tau\big(w_j,\delta(w_j)\big)^2 B_q(z,w_j)$ which is a N.W.O. family of elements of $A^2(dV_q)$. ∎

5**. Remarks and problems.** Theorems A and B are still valid when Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$. Concerning the necessary conditions, the part (i) and (ii) of the theorem A have been obtained by S. Krantz and S.-Y. Li [KL1] when Ω is a strictly pseudoconvex domain and a proof of the part (iii) can be found in [BPS1]. In this paper, the case of some ellipsoids is also considered and [BPS2] deals with the case of general ellipsoids and some classes of pseudoconvex domains of finite type in $\mathbb{C}^2$. The case of general pseudoconvex domains of finite type in $\mathbb{C}^2$ remains an open problem.

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