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MODULES WHOSE CYCLIC SUBMODULES HAVE FINITE DIMENSION

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0. **Notation and introduction.** R denotes an associative ring with identity. Module means unitary right R-module. A module has finite Goldie dimension over R if it does not contain an infinite direct sum of nonzero submodules [6]. We say R has finite (right) dimension if it has finite dimension as a right R-module. We denote the fact that M has finite dimension by dim $(M)<\infty$.

A nonzero submodule N of a module M is large in M if N has nontrivial intersection with nonzero submodules of M [7]. In this case M is called an essential extension of N. $N\subseteq' M$ will denote N is essential (large) in M. If N has no proper essential extension in M, then N is closed in M. An injective essential extension of M, denoted $I(M)$, is called the injective hull of M.

For a module M, $Z(M)=\{m\in M \mid (0:m)\subseteq' R\}$ where $(0m:)=\{r\in R \mid mr=0\}$. $Z(M)$ is called the singular submodule of M. If $Z(M)=0$, M is said to be torsion-free.

We will study torsion-free modules whose cyclic submodules have finite dimension. These modules are characterized in several ways. Applying these results to the ring R will give some known results and several new characterizations of torsion-free rings with finite dimension.

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1. **Preliminaries.** Azumaya [1] defined a module X to be M-injective if given any map $f:K\to X$ where K is a submodule of M, there exist an extension $g:M\to X$ of f. In [10] the M-injective hull of a module has been defined to be

$$\bar{X} = X+\Sigma_{h:M\to I(X)}h(M).$$

Observe X is essential in $\bar{X}$ and that $\bar{X}$ need not be injective.

DEFINITION 1.1. *A submodule A of a module B is M-closed in B if A is closed in $A+\Sigma_{h:M\to B}h(M)$.*

We note that X is M-injective if and only if X is M-closed in its injective hull.

F. Cheatham [2] defines a submodule A of a module B to be M-pure if and only

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if for every positive integer k and finitely generated module N:
whenever the diagram

$$\begin{array}{ccccc} 0 & \to & N & \to & M^k \\ & & \downarrow & & \\ & & A & & \end{array}$$

can be completed to a commutative diagram

$$\begin{array}{ccccc} 0 & \to & N & \to & M^k \\ & & \downarrow & & \downarrow \\ & & A & \to & B \end{array}$$

then it also can be completed to a commutative diagram

$$\begin{array}{ccccc} O & \longrightarrow & N & \longrightarrow & M^k \\ & & \downarrow & \swarrow & \\ & & A & & \end{array}$$

Furthermore, a module is M-absolutely pure if and only if it is M-pure in each module which contains it. Since M-injective implies M-absolutely pure (see [2]), any direct sum of M-injective modules is M-absolutely pure. Such sums need not be M-injective (e.g. take $M=R$, R not Noetherian). If $M=R$, then M-pure becomes the purity of Cohn [4] which is equivalent to saying the inclusion $O\to A\to B$ is preserved after tensoring. The absolutely pure modules were studied in [8] and [9].

In the torsion-free case M-pure and M-closed are related for certain M.

Proposition 1.2. *Suppose each cyclic submodule of M has finite dimension. If A is a M-pure submodule of a torsion-free module B, then A is M-closed in B.*

Proof. Suppose $A\subseteq' A+[\Sigma h_i(m_i)]R=A+eR$. Let $(m_i)=(m_1,\ldots,m_n)\in\oplus m_iR$ and $\bigoplus h_i$ be the obvious map from $\bigoplus m_iR$ to B. Set $L'=[(\bigoplus h_i)^{-1}(A\cap eR)]\cap (m_i)R$. As $\dim(\bigoplus m_iR)<\infty$, by [11] we may choose a finitely generated submodule L of L' with $L\subseteq' L'$. We have:

$$\begin{array}{ccc} L & \subseteq' & (m_i)R \\ \downarrow(\bigoplus h_i)|_L & & \downarrow(\bigoplus h_i)|_{(m_i)R} \\ A\cap eR & \longrightarrow & eR \\ \downarrow & \swarrow & \\ A\subseteq B & & \end{array}$$

(Unspecified maps are inclusions.)

As A is M-pure in B, we have A is $(m_i)R$-pure in B. So there exist $f:(m_i)R\to A$ with $f|_L=(\bigoplus h_i)|_L$. Since $L\subseteq\ker(f-(\bigoplus h_i)|_L)$ and $Z(B)=0$, we have $f=(\bigoplus h_i)|_{(m_i)R}$. This says $e\in A$ and hence A is M-closed in B.

COROLLARY. 1.3. *Let each cyclic submodule of M have finite dimension. Then every torsion-free M-absolutely pure module is M-injective.*

The above corollary was proved in [2] for the case where $\dim(M)<\infty$.

COROLLARY. 1.4. *If $\dim(R)<\infty$, we have:*

(*a*) *Pure submodules of torsion-free modules are closed.*
(*b*) *Every torsion-free absolutely pure module is injective.*
And if $Z(R)=0$ also, then
(*c*) *Flat modules are torsion-free.*

2. **CFD Modules.**

DEFINITION 2.1. *A module is a CFD module if each of its cyclic submodules have finite dimension.*

Finite direct sums of finite dimensional modules and quotients of finite dimensional modules by closed submodules are finite dimensional [6]. By reducing to the cyclic case we get:

LEMMA 2.2.

(*a*) *If M_1 and M_2 are CFD modules, then $M_1 \oplus M_2$ is a CFD module.*
(*b*) *If M is a CFD module and K is closed in M, then M/K is a CFD module.*

COROLLARY. 2.3. *If $\{M_\alpha : \alpha \in A\}$, where A is an indexing set, is a collection of CFD modules, then $\bigoplus_{\alpha \in A} M_\alpha$ is a CFD module.*

If $\{M_\alpha : \alpha \in A\}$ is a collection of torsion-free modules and $\dim(R)<\infty$, then $\prod_{\alpha \in A} M_\alpha$ is a *CFD* module. A similar result follows.

PROPOSITION 2.4. *For any ring R, the following statements are equivalent:*
(*a*) *Any direct product of torsion-free CFD modules is a CFD module.*
(*b*) *R/I where $I=\bigcap \{L \subseteq R \mid Z(R/L)=0, \dim(R/L)<\infty\}$ is a torsion-free finite dimensional ring.*

Proof. (a) implies (b). Set $S=\{L \subseteq R \mid Z(R/L)=0 \text{ and } \dim(R/L)<\infty\}$. We have $0 \to R/I \to \prod_{L \in S} R/L$ and hence $I \in S$. If $x \in R/I$, then $\bar{x}R \simeq R/(I:x)$ where $\bar{x}$ is the image of x in R/I. But $(I:x) \supset I$; so $xI \subset I$. Observing that $\dim(R/I)$ and $Z(R/I)$ are the same over both of the rings finishes the implication.

(b) implies (a). Note that each torsion-free *CFD* module is a R/I-module.

We now characterize those *CFD* modules whose singular submodule is zero.

THEOREM 2.5. *Let $Z(M)=0$. The following statements are equivalent:*

(1) *M is a CFD module.*
(2) *The union of a directed set of M-closed submodules of a torsion-free module is a M-closed submodule.*

(3) *Every torsion-free M-absolutely pure module is M-injective.*
(4) *Any direct sum of torsion-free M-injective modules is M-injective*
(5) *The countable direct sum of torsion-free M-injective modules is M-injective.*
(6) *If A_R is a M-pure submodule of a torsion-free module B, then A is M-closed in B.*

Proof. (1) implies (6) is Proposition 1.2, while (6) implies (3) is Corollary 1.3. Now (3) implies (4) as M-injective always implies M-absolutely pure [2]. (4) implies (5) is immediate. (5) implies (1). Suppose $\bigoplus X_i \subset mR \subset M$.
Then

$$\begin{array}{c} 0 \to \bigoplus X_i \to mR \\ \downarrow \\ \bigoplus I(X_i) \end{array}$$

can be completed commutatively. Since mR is cyclic, the sum $\bigoplus X_i$ is finite. (1) implies (2). Let X be torsion-free and $\{S_\alpha : \alpha \in A\}$ a directed system of M-closed submodules of X. Let $e \in \Sigma_{h:M\to X} h(M)$ and $I \subseteq' R$ such that $eI \subseteq \bigcup S_\alpha$. As $\dim(eR) < \infty$, eI is finite dimensional. It follows that eI is in some S_α. But S_α being M-closed in X implies $e \in S_\alpha$. This makes $\bigcup S_\alpha$ M-closed in X. (2) implies (4) is clear and finishes the proof.

Letting $M=R$, we have:

COROLLARY 2.6. *Let $Z(R)=0$. The following statements are equivalent:*

(*a*) *$\dim(R) < \infty$.*
(*b*) *The union of a directed set of closed submodules of a torsion-free module is a closed submodule.*
(*c*) *Every torsion-free absolutely pure module is injective.*
(*d*) *Any direct sum of torsion-free injective modules is injective.*
(*e*) *Any countable direct sum of torsion-free injective modules is injective.*
(*f*) *Pure submodules of torsion-free modules are closed submodules.*

T. Cheatham showed (*a*) and (*c*) were equivalent. (*a*) equivalent to (*b*) is credited to Teply. The equivalence of (*a*) and (*f*) appears to be a new result.

M is quasi-injective if M is M-injective. For M quasi-injective we get a partial converse to Corollary 2.3.

THEOREM 2.7. *Suppose $Z(M)=0$ and M is quasi-injective. The following statements are equivalent:*

(*a*) *M is a CFD module.*
(*b*) *$M=\bigoplus_{\alpha\in A} M_\alpha$ where each M_α is indecomposable.*

Proof. (*b*) implies (*a*). As $M=\bigoplus M_\alpha$, each M_α is quasi-injective. If $0\neq S\subseteq M_\alpha$ the M_α-injective hull of S is a direct summand of M_α and hence equal to M_α. So $S\subseteq' M_\alpha$ or $\dim(M_\alpha)=1$. By Corollary 2.3 M is a *CFD* module. Conversely, let $mR\subseteq M$. The M-injective hull of mR is contained in M and is finite dimensional.

Thus, M contains indecomposable M-injectives. There exist a maximal collection of indecomposable M-injectives whose sum is direct. As this sum is M-injective it must be M.

If $M=I(M)$ we have:

COROLLARY 2.8. *Let $Z(M)=0$ and $M=I(M)$. The following are equivalent:*

(*a*) *M is a CFD module.*
(*b*) *M is a direct sum of indecomposable injectives.*

We now give a different characterization of finite dimensional torsion-free rings.

THEOREM 2.9. *Let $Z(R)=0$. The following are equivalent:*
(*a*) *$\dim(R)<\infty$.*
(*b*) *If M is torsion-free and quasi-injective, then M is a direct sum of indecomposable modules.*
(*c*) *There exist a cardinal number c such that each torsion-free quasi-injective is a direct sum of modules each generated by c elements.*

Proof. Theorem 2.7 gives (*a*) implies (*b*). (*b*) implies (*c*). We saw in the proof of Theorem 2.7 that every indecomposable quasi-injective is an essential extension of its cyclic submodules. Faith [5] has shown that each module has an unique (up to isomorphism) quasi-injective extension. So the collection of all isomorphism classes of torsion-free indecomposable quasi-injectives is a set. Choosing a set of generators for a member of each isomorphism class and summing the number of generators over the isomorphism classes give the cardinal number c. (*c*) implies (*a*) follows by restricting to the injective case which was proved by Teply [12].

Earlier we saw that if M is a *CFD* module and K is a closed submodule of M, then M/K is a *CFD* module. Now we will look at those torsion-free modules generated by certain *CFD* modules.

DEFINITION 3.0. *The class of modules generated by M is the collection of all homomorphic images of arbitrary direct sums of copies of M.*

PROPOSITION 3.1. *Let M be a torsion-free quasi-injective CFD module. Then every torsion-free module generated by M is a quasi-injective, M-injective, CFD module.*

Proof. Consider $\bigoplus_{\alpha\in A} M_\alpha, M_\alpha\cong M$. In [1] it was shown that if M is M_α-injective for each α, then M is $\bigoplus M_\alpha$-injective. Thus M is $\bigoplus M_\alpha$-injective and since $\bigoplus M_\alpha$ is a *CFD* module, we have $\bigoplus M_\alpha$ is quasi-injective. Also $\bigoplus M_\alpha$ is M-injective by Theorem 2.5. Observing that closed submodules of quasi-injective modules are direct summands completes the proof.

Essential extensions of finite dimensional modules are finite dimensional. The *CFD* property is not always assumed by essential extensions.

EXAMPLE Let F be a field and R an infinite direct product of copies of F. Then $\bigoplus F \subseteq' R$ and $\bigoplus F$ is a CFD module. However, R is not finite dimensional.

The following proposition and theorem give some conditions when the CFD property is taken on by essential extensions.

PROPOSITION 3.2. *Let $\{E_\alpha : \alpha \in A\}$ be a collection of torsion-free injective CFD modules. Then $\bigoplus E_\alpha$ is injective if and only if $I(\bigoplus E_\alpha)$ is a CFD module.*

Proof. If $I(\bigoplus E_\alpha)$ is a CFD module, then $\bigoplus E_\alpha$ is $I(\bigoplus E_\alpha)$-injective. So, $0 \to \bigoplus E_\alpha \subset I(\bigoplus E_\alpha)$ splits.

THEOREM 3.3. *Essential extensions of torsion-free CFD modules are CFD modules if and only if direct sums of torsion-free injective CFD modules are injective.*

Proof. Let $\{E_\alpha : \alpha \in A\}$ be a collection of torsion-free injective CFD modules. Then $I(\bigoplus E_\alpha)$ is by hypothesis a CFD module. So by Proposition 3.2 $I(\bigoplus E_\alpha) = \bigoplus E_\alpha$.

Conversely, let M be a torsion free CFD module. We have:

$$0 \to K \to \bigoplus_{m \in M} mR \to M \to 0$$

Choose T so that $K \oplus T \subseteq' \bigoplus mR$. Then

$$0 \to \bigoplus mR/K \to \bigoplus I(mR)/I(K) \approx I(T)$$

is exact. This makes $I(T)$ a CFD module. But, $M \approx \bigoplus mR/K \subset I(T)$ implies $I(M)$ is a CFD module.

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