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LIE DERIVATIONS ON SKEW ELEMENTS IN PRIME RINGS WITH INVOLUTION

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ABSTRACT. Let R be a prime ring with involution satisfying $x/2 \in R$ whenever $x \in R$. Assume that R has two nontrivial symmetric idempotents e_1, e_2 whose sum is not 1, and that the subrings determined by e_1, e_2, $1 - (e_1 + e_2)$ are not orders in simple rings of dimension at most 4 over their centers. Then if L is a Lie derivation of the skew elements K into R there exists a subring A of R, $A \subseteq \bar{K}$, a derivation $D:A \to RC$, the central closure of R, and a mapping $T:R \to C$, satisfying $L = D + T$ on K and $T[\overline{K \cap A}, A] = 0$.

Introduction. A Lie derivation on a ring R is a mapping $L:R \to R'$, R' a ring containing R, such that L is additive and $L[x,y] = [L(x),y] + [x,L(y)]$ for all x,y in R, where $[u,v] = uv - vu$ is the Lie product. A derivation on a ring R is a mapping $D:R \to R'$, R' a ring containing R, such that D is additive and $D(xy) = D(x)y + xD(y)$ for all x,y in R.

It is easily seen that if $D:R \to R'$ is a derivation and T is an additive mapping of R into the center of R' such that $T[R,R] = 0$, then $L = D + T$ is a Lie derivation on R. Martindale [3] has shown that if R is a primitive ring, not of characteristic 2, which contains a nontrivial idempotent, and has a Lie derivation $L:R \to R$, then there exists a primitive ring R' containing R, a derivation $D:R \to R'$ and an additive mapping $T:R \to Z(R')$, the center of R', such that $T[R,R] = 0$ and $L = D + T$. He has also noted (in conversation) that the same proof works for prime rings using the central closure for R'. Jacobs [2] has shown the following: Let R be a simple ring with involution, of characteristic not 2, with two nontrivial symmetric, orthogonal idempotents whose sum is not 1. Then if L is a Lie derivation of the skew elements K into R there exists a derivation $D:R \to R$ and an additive mapping $T:R \to Z(R)$ such that $T[R,R] = 0$ and $L = D + T$ on K, providing R is not isomorphic to the 4×4 matrices over a field on the 3×3 matrices over a field of characteristic 3.

We show the following: Let R be a prime ring with involution, with $x/2 \in R$ whenever $x \in R$, and with two symmetric orthogonal idempotents whose sum is not 1. Then if L is a Lie derivation of the skew elements K into R there exists a derivation $D:A \to RC$, where A is a subring of R, $A \subseteq \bar{K}$, RC is the central closure of R, and there

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exists a mapping $T:R \to C$ satisfying $L = D + T$ on K and $T([\overline{K \cap A}, A]) = 0$, providing the subrings determined by the idempotents are not orders in simple rings of dimensions at most 4 over their centers.

In what follows R will denote a prime ring with involution; $S = \{x : x^* = x\}$ is the set of symmetric elements of R; $K = \{x : x^* = -x\}$ is the set of skew elements of R; $R' = RC$ is the central closure of R; $\bar{B}$ is the subring of R generated by a subset B of R. Similar notation will be used for related rings.

PROPOSITION 1. (Erickson [1].) *Let R be a prime ring with involution. Then $\bar{K}$ contains a nonzero $*$-ideal of R unless R is an order in a simple ring which is at most 4-dimensional over its center.*

We now add the assumptions that $x/2 \in R$ whenever $x \in R$, and that R has two nontrivial symmetric orthogonal idempotents e_1, e_2 such that $e_1 + e_2 \neq 1$, and we set $e_3 = 1 - e_1 - e_2$ whether or not $1 \in R$.

Let $R_{ij} = e_iRe_j$, $i, j \in \{1, 2, 3\}$, and let $x_{ij} = e_ixe_j$, $x \in R$. Note that each R_{ii} is prime and $R = \Sigma \oplus R_{ij}$. Let K_i be the skew elements of R_{ii} and note that $K_i = e_iKe_i$. We also assume each $\bar{K}_i$ contains a nonzero $*$-ideal U_i. We use similar notation for R'. Note that $Z(R'_{ii}) = e_iC$, $i = 1, 2, 3$. Note that if R is a prime ring with involution such that $\bar{K}$ contains a nonzero ideal U then $x \in R$, $xK = 0$ or $Kx = 0$ implies $x = 0$, and $[x, K] = 0$ implies $x \in Z(R)$.

LEMMA 1. *Let $L:K \to R$ be a Lie derivation, and let $k_i \in K_i$, $i = 1, 2, 3$. Then with h, i, j distinct*

$$L(k_i) = a_i + a_{ij} + a_{ji} + a_{ih} + a_{hi} + z,$$

where

$$z \in C, \quad a_i = e_iL(k_i)e_i - ze_i \in R', \quad a_{mn} = e_mL(k_i)e_n, \quad m, n \in \{h, j\}, \quad m \neq n.$$

PROOF. Let $k_i \in K_i$, $k_j \in K_j$. Then $0 = [k_i, k_j]$ so

$$0 = L([k_i, k_j]) = [L(k_i), k_j] + [k_i, L(k_j)] = L(k_i)k_j - k_jL(k_i) + k_iL(k_j) - L(k_j)k_i.$$

Taking the R_{hj} component yields $e_hL(k_i)e_jk_j = 0$. Since k_j was arbitrary $e_hL(k_i)e_j = 0$. Similarly $e_jL(k_i)e_h = 0$. Taking the R_{jj} component yields

$$\begin{aligned} &e_jL(k_i)e_jk_j - k_je_jL(k_i)e_j = 0 &&\text{so } [e_jL(k_i)e_j, K_j] = 0, \\ &\text{so } e_jL(k_i)e_j \in Z(e_{jj}) \subseteq e_jC, &&\text{so } e_jL(k_i)e_j = e_jz_j \end{aligned}$$

for some $z_j \in C$.

Now let $0 \neq x_{jh} \in R_{jh}$, and let $x_{hj} = x_{jh}^*$. Then

$$\begin{aligned} 0 = L([k_i, x_{jh} - x_{hj}]) &= L(k_i)(x_{jh} - x_{hj}) - (x_{jh} - x_{hj})L(k_i) \\ &\quad + k_iL(x_{jh} - x_{hj}) - L(x_{jh} - x_{hj})k_i. \end{aligned}$$

Taking the R_{jh} component yields

$$0 = e_jL(k_i)x_{jh} - x_{jh}L(k_i)e_h = z_jx_{jh} - x_{jh}z_h = (z_j - z_k)x_{jh}.$$

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Since C is a field and $x_{jh} \neq 0$ we have $z_j = z_h$. Let $z = z_h = z_j$. Then $ze_j + ze_h = z(1 - e_i)$ and $L(k_i)$ has the stated form.

LEMMA 2. *Let $L:K \to R$ be a Lie derivation, and let $x_{ij} \in R_{ij}$. Then $e_hL(x_{ij} - x_{ij}^*)e_h = ze_h$ for some $z \in C, h, i, j$ distinct.*

PROOF. Let $X = x_{ij} - x_{ij}^*$ and let $k \in K_h$. Then

$$0 = L[X, k] = L(X)k - kL(X) + XL(k) - L(k)X.$$

Taking the R_{hh} component yields $0 = e_hL(X)e_hk - ke_hL(X)e_h$, so $e_hL(X)e_h \in Z(R_{hh})$, so $e_hL(X)e_h = ze_h$ for some $z \in C$.

DEFINITION 1. (a) *Define a mapping $D:K \to R'$ by D is additive,*

$$D(k_i) = L(k_i) - z,\ D(x_{ij} - x_{ij}^*) = L(x_{ij} - x_{ij}^*) - z,\ i \neq j,$$

where the z's are as in Lemmas 1 and 2. Note that

$$[D(k), r] = [L(k), r] \quad \text{for} \quad k \in K, \quad r \in R.$$

Also note that

$$e_iD(k)e_j = e_iL(k)e_j \quad \text{for} \quad k \in K, \quad i \neq j.$$

(b) *Define D on U_i by D is additive and*

$D(k\ell \ldots m) = D(k)\ell \ldots m + kD(\ell) \ldots m + \ldots + k\ell \ldots D(m),\ k, \ell, m \in K_i.$

(c) *Define D on $U_iR_{ij}U_j$ by D is additive and*

$$D(u_ix_{ij}u_j) = D(u_i)x_{ij}u_j + u_iD(x_{ij} - x_{ij}^*)u_j + u_ix_{ij}D(u_j),$$

where $u_i \in U_i$, $u_j \in U_j$, $x_{ij} \in R_{ij}$, $i \neq j$. Note $(x_{ij} - x_{ij}^*)u_j = x_{ij}u_j$ and $u_i(x_{ij} - x_{ij}^*) = u_ix_{ij}$.

(d) *Let $A = \Sigma_{i,j}(U_i + U_iR_{ij}U_j)$. Define D on A by the above with D additive.*

Note that A is a direct sum and is a subring of R. (In particular $(U_iR_{ij}U_j)(U_jR_{ji}U_i) \subseteq U_i$ since U_i is an ideal in R_{ii}.) Also note $A \subseteq \bar{K}$. We will show D is a derivation on A. We first need to show D is well defined on A.

LEMMA 3. *Let $m \in K_h$ and $n \in K_i$ or $n = x_{ij} - x_{ij}^*, h, i, j$ distinct. Then*

$$D(m)n + mD(n) = 0 = D(n)m + nD(m).$$

PROOF. $0 = [m, n]$ so

$$0 = L([m, n]) = [D(m), n] + [m, D(n)] = D(m)n - nD(m) + mD(n) - D(n)m.$$

This yields

$$D(m)n + mD(n) = nD(m) + D(n)m.$$

By Definition 1(a) and the lemmas we have

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$$(e_i + e_j)\{D(m)n + mD(n)\} = 0$$

hence

$$(e_i + e_j)\{nD(m) + D(n)m\} = 0$$

Also

$$e_h\{nD(m) + D(n)m\} = 0$$

hence

$$e_h\{D(m)n + mD(n)\} = 0.$$

Therefore

$$D(m)n + mD(n) = 0 = D(n)m + nD(m).$$

LEMMA 4. *Let*

$$K(i, j) = K_i \cup K_j \cup \{x_{ij} - x_{ij}^*: x \in R\}.$$

Define $P(y, k\ell \ldots m)$ *for*

$$y = e_h x e_i,\ x \in R, k, \ell, \ldots, m \in K(i, j)$$

by

$$P(y, k\ell \ldots m) = [[\ldots[[y - y^*, k]\ell], \ldots], m].$$

Then

(a) $$P(y, k\ell \ldots m) = yk\ell \ldots m - m^* \ldots \ell^* k^* y^*.$$

(b) $$e_h L\{P(y, k\ell \ldots m)\} = e_h D(y - y^*)k\ell \ldots m + yD(k\ell \ldots m) - e_h D(m^* \ldots \ell^* k^*)y^*.$$

LEMMA 5. *D is well defined on each* U_i.

PROOF. Assume $0 = \Sigma k\ell \ldots m$ with $k, \ell, \ldots, m \in U_i$, and let

$$r = \Sigma\{D(k)\ell \ldots m + kD(\ell) \ldots m + \ldots + k\ell \ldots D(m)\}.$$

Note that $r = \Sigma D(k\ell \ldots m)$ by Definition 1(b). For each $n \in K_j$, $j \neq i$, we have

$$rn = \Sigma k\ell \ldots D(m)n = \Sigma k\ell \ldots \{D(m)n + mD(n)\} = 0$$

using Lemma 3 and our assumption that $0 = \Sigma k\ell \ldots m$. Since $rn = 0$ for all $n \in K_j$ we have $re_j = 0$. Similarly $0 = re_h = e_h r = e_j r$. Also, for $x \in R$, $y = e_h x e_i$ we have

$$0 = \Sigma\{yk\ell \ldots m - m^* \ldots \ell^* k^* y^*\}$$

since $\Sigma k\ell \ldots m = 0$ implies $\Sigma m^* \ldots \ell^* k^* = 0$. Thus by Lemma 4(a) $\Sigma P(y, k\ell \ldots m) = 0$ so $\Sigma L\{P(y, k\ell \ldots m)\} = 0$. This implies $\Sigma yD(k\ell \ldots m)e_j = 0$ using Lemma 4(b), the assumption $\Sigma k\ell \ldots m = 0$ and $y^* e_j = 0$. Thus $0 = y\{\Sigma D(k\ell \ldots m)\}e_i = yre_i$. Since x was arbitrary $(e_h R e_i) r e_i = 0$ so $e_i r e_i = 0$. Therefore every component of r is 0 and

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D is well defined on each U_i.

LEMMA 6. *D is well defined on each $U_iR_{ij}U_j$, $i \neq j$.*

PROOF. Assume $0 = \Sigma uxv$ where $u \in U_i$, $x \in R_{ij}$, $v \in U_j$. Let $X = x - x^*$ and note that $uxv = uXv$. Let $r = \Sigma\{D(u)Xv + uD(X)v + uXD(v)\}$ and note that $r = \Sigma D(uxv)$ by Definition 1(c). Then for $k \in K_h$, $h \neq i, j$, we have

$$kr = \Sigma kD(u)Xv = \Sigma\{kD(u) + D(k)u\}Xv = 0$$

using Lemma 3 and our assumption $0 = \Sigma uXv$. Therefore $e_hr = 0$. Similarly $0 = e_jr = re_i = re_h$. Now let $z \in R$, $y = e_hze_i$. By assumption we have

$$0 = \Sigma(yuXv - v^*X^*u^*y^*).$$

Thus by Lemma 4(a) $\Sigma P(y, uXv) = 0$ so $0 = \Sigma L\{P(y, uXv)\}$. This implies $0 = \Sigma e_hyD(uXv)e_j$ using Lemma 4(b), the assumption $\Sigma uXv = 0$, and $y^*e_j = 0$. Therefore $e_hRe_ire_j = 0$ so $e_ire_j = 0$. Thus every component of R is 0 and D is well defined on each $U_iR_{ij}U_j$.

THEOREM 1. *D is well defined on $A = \Sigma_{i,j}(U_i + U_iR_{ij}U_j)$.*

PROOF. Definition 1 and Lemmas 1, 2, 5 and 6.

THEOREM 2. *$D{:}A \to R'$ is a derivation.*

PROOF. (a) It follows from Definition 1 that $D(xy) = D(x)y + xD(y)$ if $x, y \in U_i$.

(b) It follows from Definition 1 that $D(xy) = D(x)y + xD(y)$ if $x \in U_i$, $y \in U_iR_{ij}U_j$ or $x \in U_iR_{ij}U_j$, $y \in U_j$.

(c) If $x = \Sigma \ldots k_i \in U_i$ and $y = \Sigma\ell_j \ldots \in U_j$ then $xy = 0$ so $D(xy) = 0$.

$$D(x)y + xD(y) = \Sigma(0 + \{D(k_i)\ell_j + k_iD(\ell_j)\}) + 0$$

by Lemma 3, so $D(xy) = D(x)y + xD(y)$.

(d) Similarly $D(xy) = 0 = D(x)y + xD(y)$ if $x \in U_i$ and $y \in U_jR_{ji}U_i$, or $x \in U_i$, $y \in U_jR_{jh}U_h$, or $x \in U_iR_{ij}U_j$, $y \in U_i$ or $x \in U_iR_{ij}U_j$, $y \in U_h$, or $x \in U_iR_{ij}U_j$, $y \in U_iR_{ij}U_j$, *or* $x \in U_iR_{ij}U_j$, $y \in U_hR_{hi}U_i$, or $x \in U_iR_{ij}U_j$, $y \in U_hR_{hj}U_j$, or $x \in U_iR_{ij}U_j$, $y \in U_hR_{hi}U_i$ or $x \in U_iR_{ij}U_j$, $y \in U_iR_{ih}U_h$.

(e) Let $u_ix_{ij}v_j \in U_iR_{ij}U_j$ and $w_jy_{jh}z_h \in U_jR_{jh}U_h$, $x_{ji} = x_{ij}^*$, $y_{hj} = y_{jh}^*$. Then

$$\begin{aligned} D\{(u_ix_{ij}v_j)(w_jy_{jh}z_h)\} &= D\{u_i(x_{ij}v_jw_jy_{jh})z_h\} \\ &= D(u_i)x_{ij}v_jw_jy_{jh}z_h + u_iD(x_{ij}v_jw_jy_{jh} - y_{hj}w_jv_jx_{ji})z_h \\ &\quad + u_ix_{ij}v_jw_jy_{jh}D(z_h). \end{aligned}$$

Looking at the middle term we have

$$\begin{aligned} u_iD(x_{ij}v_jw_jy_{jh} - y_{hj}w_jv_jx_{ji})z_h &= u_iD\{(x_{ij}v_j - v_jx_{ji})(w_jy_{jh} - y_{hj}w_j) \\ &\quad - (w_jy_{jh} - y_{hj}w_j)(x_{ij}v_j - v_jx_{ji})\}z_h \\ &= u_iD[x_{ij}v_j - v_jx_{ji}, w_jy_{jh} - y_{hj}w_j]z_h \\ &= u_iD(x_{ij}v_j - v_jx_{ij})w_jy_{jh}z_h \\ &\quad + u_ix_{ij}v_jD(w_jy_{jh} - y_{hj}w_j)z_h \end{aligned}$$

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$$\begin{aligned} &= u_iD[x_{ij} - x_{ji}, v_j]w_jy_{jh}z_h \\ &\quad + u_ix_{ij}v_jD[w_j, y_{jh} - y_{hj}]z_h \\ &= u_iD(x_{ij} - x_{ji})v_jw_jy_{jh}z_h \\ &\quad + u_ix_{ij}D(v_j)w_jy_{jh}z_h + u_ix_{ij}v_jD(w_j)y_{jh}z_h \\ &\quad + u_ix_{ij}v_jw_jD(y_{jh} - y_{hj})z_h \end{aligned}$$

using the definition of D (with Lemmas 1 and 2.) Thus

$$D\{(u_ix_{ij}v_j)(w_jy_{jh}z_h)\} = D(u_ix_{ij}v_j)w_jy_jy_{jh}z_h + u_ix_{ij}v_jD(w_jy_{jh}z_h).$$

(f) Let $x \in U_iR_{ij}U_j$, $y \in U_jR_{ji}U_i$ and let $r = D(x)y + xD(y) - D(xy)$. For $k \in U_h$ we have

$$rk = xD(y)k + xyD(k) - xyD(k) - D(xy)k = xD(yk) - D\{(xy)k\} = 0$$

so $re_h = 0$. Similarly $0 = e_hr = e_jr = re_j$. Now let $z \in U_iR_{ih}U_h$. Then

$$\begin{aligned} rz &= D(x)yz + xD(y)z + xyD(z) - xyD(z) - D(xy)z \\ &= D(x)yz + xD(yz) - D\{(xy)z\} = D\{x(yz)\} - D\{(xy)z\} \\ &= 0 \end{aligned}$$

so $re_i = 0$. Similarly $e_ir = 0$ so $r = 0$.

Thus D is a derivation on A.

DEFINITION 2. *Define T on R by $T(x) = \frac{1}{2}\{L(x - x^*) - D(x - x^*)\}$.*

THEOREM 3. *$T:R \to C$ is additive, $L = D + T$ on K, and $T([\overline{K \cap A}, A]) = 0$.*

PROOF. T is additive since L and D are, and $T:R \to C$ by the definition of D.
If $x \in K$ then $x - x^* = 2x$. So $L = D + T$ on K.
Let $k, m \in K \cap A$. Then

$$L([k,m]) = [D(k), m] + [k, D(m)] = D([k,m])$$

using Definition 1(a) and Theorem 2. Thus $T([K \cap A, K \cap A]) = 0$. If $s = s^* \in S \cap A$ then $s - s^* = 0$ so $T(s) = \frac{1}{2}\{L(s - s^*) - D(s - s^*)\} = 0$. But $[K \cap A, S \cap A] \subseteq S \cap A$ so $T([K \cap A, S \cap A]) = 0$. Hence $T([K \cap A, A]) = 0$ since $A = S \cap A + K \cap A$. Assume $T([(K \cap A)^N, A]) = 0$. If $k_1, \ldots, k_{N+1} \in K \cap A$ and $r \in A$ then

$$\begin{aligned} [k_1 \ldots k_{N+1}, r] &= k_1 \ldots k_{N+1}r - rk_1 \ldots k_{N+1} = k_1(k_2 \ldots k_{N+1}r) \\ &\quad - (k_2 \ldots k_{N+1}r)k_1 + (k_2 \ldots k_{N+1})(rk_1) - (rk_1)(k_2 \ldots k_{N+1}) \\ &\in [K \cap A, A] + [(K \cap A)^N, A] \end{aligned}$$

This implies $T([(K \cap A)^N, A]) = 0$ for all N so $T([\overline{K \cap A}, A]) = 0$.
Putting the above together we have the following.

THEOREM. *Let R be a prime ring with involution with $x/2 \in R$ whenever $x \in R$. If R has two non-trivial symmetric orthogonal idempotents e_1, e_2 with $e_1 + e_2 \neq 1$, $e_3 = 1 - e_1 - e_2$, such that each $\bar{K}_i$, contains a nonzero $*$-ideal U_i, $i = 1, 2, 3$, then, letting $A = \Sigma(U_i + U_iR_{ij}U_j)$, for each Lie derivation $L:K \to R$, there is a derivation*

$D:A \to \overline{RC}$, and an additive mapping $T:R \to C$, with $L = D + T$ on K and $T([K \cap A, A]) = 0$.

If R is simple then $U_i = R_{ii}$ and $U_i R_{ij} U_j = R_{ii} R_{ij} R_{jj} = R_{ij}$ so $A = R$, and we have the existence of a derivation D on R and an additive mapping T on R with $L = D + T$ on K, $T:R \to Z$, $T([R,R]) = 0$, providing $\dim R_i/Z_i > 4$, $i = 1, 2, 3$. As noted earlier Jacobs has more complete results for R simple in his dissertation.

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