ON THE INFERENCE OF MAGNETIC FIELD VECTORS FROM STOKES PROFILES: A GENERALIZATION OF THE WEAK-FIELD APPROXIMATION

ZHONG-QUAN QU, YOU-JI DING, JIA-YU XUAN AND SHI-HUI YE
Yunnan Observatory, Kunming 650011, China

ABSTRACT A generalization of Jefferies et al.’s weak-field approximation for the inference of the strength and polar angle of magnetic field vectors from Stokes profiles is obtained while the inference of azimuthal angle follows a relationship derived by Landi Degl’Innocenti’s perturbative solutions to the transfer equations in which the magneto-optical effect is taken account. It is found that the weak-field condition ($\Delta \lambda_H / \Delta \lambda_D \leq 1$) is not necessary for the new method when the fitting is done in the line wings.

I. INTRODUCTION

Among the methods for inference of magnetic field vectors from Stokes $I, Q, U, V$ profiles, the weak-field approximation (WFA) proposed by Jefferies et al. (1989, hereafter JLS, 1991) is one of the most effective and the simplest. Under the assumption that the Zeeman splitting $\Delta \lambda_H$ is a fraction of the Doppler width $\Delta \lambda_D$, they represented the Voigt function $H(a, v \pm v_b)$ (where $v_b = \Delta \lambda_H / \Delta \lambda_D$, $v = \Delta \lambda / \Delta \lambda_D$ and $\Delta \lambda$ is the distance from line center.) by the low-order terms of Taylor series as

$$H(a, v \pm v_b) = H(a, v) \pm v_b H'(a, v) + \frac{v_b^2 H''(a, v)}{2} \pm \frac{v_b^3 H'''(a, v)}{6} + \ldots .$$

Approximate expressions for absorption coefficients $\eta_I$, $\eta_Q$, $\eta_U$ and $\eta_V$ were obtained by keeping only the former three terms of the series. Substituting these expressions into the transfer equations they found corresponding simplified equations. When considering the requirement for consistency between the simplified equations and making some assumptions, they obtained useful relationships to infer the magnetic field vectors from Stokes profiles (see JLS).

In the following, we use the Taylor series (1) for $H(a, v)$ up to the third order. An expression for $v_b$ and the relationship between $\tan 2\chi$ and $U/Q$ are obtained. All of these will be used to extract the magnetic field vectors from Stokes profiles.

II. THEORETICAL DERIVATION
We take the former four terms of the Taylor expansion for the Voigt function and insert it in the expressions for \( \eta_I, \eta_Q, \eta_U \) and \( \eta_V \). This gives the same expressions as JLS' for \( \eta_I, \eta_Q \) and \( \eta_U \) while

\[
\eta_V = -\eta_0 \cos \gamma \left[ v_b H'(a, v) + \left( \frac{v_b^3}{6} \right) H'''(a, v) \right] + 0(v_b^5). \tag{2}
\]

According to Landi Degl'Innocenti and Landi Degl'Innocenti (1973, hereafter LL), the magneto-optical effect is only of order \( x^3(x = v_L/\Delta v_D, v_L \) is the Larmor frequency) in \( Q \) and \( U \), but has no influence until the fourth order of \( x \) on \( I \) and \( V \). \( v_b \) and \( \gamma \) are dominated by \( I \) and \( V \), so the magneto-optical effect can be ignored. The transfer equations thus can be approximately written (the significance of symbols in this paper is the same as that of JLS) as

\[
\mu \frac{dI}{d\tau} = \left[ 1 + \eta_0 H(a, v) + (1 + \cos^2 \gamma) H''(a, v) \left( \frac{v_b}{2} \right)^2 \right] (I - B)
\]

\[
-\eta_0 \cos \gamma \left[ v_b H'(a, v) + \left( \frac{v_b^3}{6} \right) H'''(a, v) \right] V
\]

\[
-\eta_0 \left( \frac{v_b \sin \gamma}{2} \right)^2 H''(a, v) \left[ Q \cos 2\chi + U \sin 2\chi \right], \tag{3a}
\]

\[
\mu \frac{dQ}{d\tau} = \left[ 1 + \eta_0 H(a, v) \right] Q - \eta_0 \left( \frac{v_b \sin \gamma}{2} \right)^2 H''(a, v) (I - B) \cos 2\chi, \tag{3b}
\]

\[
\mu \frac{dU}{d\tau} = \left[ 1 + \eta_0 H(a, v) \right] U - \eta_0 \left( \frac{v_b \sin \gamma}{2} \right)^2 H''(a, v) (I - B) \sin 2\chi, \tag{3c}
\]

\[
\mu \frac{dV}{d\tau} = \left[ 1 + \eta_0 H(a, v) \right] V + \left[ \eta_0 v_b \cos \gamma H'(a, v) + \eta_0 \left( \frac{v_b^3}{6} \right) \cos \gamma H'''(a, v) \right] (I - B). \tag{3d}
\]

Comparing the magnitudes of the right-hand terms in the first equation, we can find that all the terms except the first one can be dropped, especially in the line wings where the magnetic field vectors are determined. Thus the first and last equations may be written

\[
\mu \frac{dI^*}{d\tau} = (1 + \eta_0 H) I^* - \mu \frac{dB}{d\tau}, \tag{4a}
\]

\[
\mu \frac{dV}{d\tau} = (1 + \eta_0 H) V - \eta_0 v_b \cos \gamma H'(1 + \frac{v_b^2}{6} H''') I^* \tag{4b}
\]

where

\[
I^* = I - B. \tag{5}
\]
We use the same method as LL and JLS. Eqs. (4) require for consistency
(set $\partial B/\partial v=0$)

$$V(v, \tau) = -v_b \cos \gamma (1 + \frac{v_b^2 H'''}{6 H'}) \frac{\partial I(a, v)}{\partial v}$$

(6)

provided

$$\mu \frac{d}{d\tau} \left[ \ln (v_b \cos \gamma H' + \frac{v_b^3}{6 \cos \gamma H''}) \right] \ll 1 + \eta_b H(a, v).$$

(7)

The same expressions of $Q$ and $U$ of JLS and the corresponding assumption can be employed with good approximation for deriving $v_b \sin \gamma$. We rewrite Eq.(6) and the expressions of $Q$ and $U$ of JLS in the following way:

$$v_b \cos \gamma (1 + \frac{v_b^2 H'''}{6 H'}) = -\frac{V(v, \tau)}{\partial I/\partial v} \equiv SS1(v, \tau),$$

(8a)

$$\left(\frac{v_b \sin \gamma}{2}\right)^2 = \left| \frac{H'(a, v)}{H''(a, v)} \left( \frac{Q^2 + U^2}{Q^2 + U^2} \right)^{1/2} \right| \equiv SS2(v, \tau).$$

(8b)

Neglecting the second order and the higher ones of $H'''/H'$, we can obtain the following expressions through several algebraic steps from Eqs.(8)

$$v_b = \left[ \frac{SS1^2 + 4SS2}{1 + (SS1^2 H'''/3H')^2} \right]^{1/2}$$

(9a)

$$\sin \gamma = \left( \frac{4SS2}{v_b} \right)^{1/2}.$$  

(9b)

For deriving the relationship between $\tan 2\chi$ and $U/Q$, we make use of LL’s perturbative solutions, but some of their symbols are changed in order to keep agreement with JLS’. It is not difficult to find that LL and JLS are compatible with each other. In fact $V$, $Q$ and $U$ of JLS are equivalent to $V_1$, $Q_2$ and $U_2$ of LL respectively. if the varying with depth of the third orders of $Q$ or $U$ is neglected, i.e.

$$\frac{dQ_3}{d\tau} = \frac{dU_3}{d\tau} = 0,$$

(10)

one can obtain expressions for $Q_3$ and $U_3$ from LL , with

$$Q_0 = U_0 = 0, \ Q_1 = U_1 = 0,$$

(11)

and

$$\frac{U}{Q} = \frac{U_0 + U_1 + U_2 + U_3 + \cdots}{Q_0 + Q_1 + Q_2 + Q_3 + \cdots},$$

(12)
We reach after several algebraic steps

\[
\tan 2\chi = \frac{U/Q + v_b \cos \gamma SS3(v, \tau)}{1 - (U/Q)v_b \cos \gamma SS3(v, \tau)}, \tag{13}
\]

where

\[
SS3(v, \tau) \equiv \frac{2\eta_0}{(1 + \eta_0 H)H''[F'H'' - F''H' + \frac{v_b^2 H'''}{6H}]]. \tag{14}
\]

It is obvious that when \(v_b\) is small enough or the magneto-optical effect is neglected or \(\gamma = \pi /2\), Eq.(13) returns to the more familiar relationship

\[
\tan 2\chi = \frac{U}{Q}. \tag{15}
\]

Eqs.(9) and Eq.(13) are the basic formulae we need for inferring the magnetic field vectors from Stokes profiles. Finally a useful variable which represents the difference between the accurate and approximate expressions of the Voigt functions is defined as

\[
D_{\text{eff}} = \left| \frac{H(a, v + v_b) - H(a, v - v_b)}{2} \right| - \left| v_b H'(a, v) + \frac{v_b^3}{6} H''''(a, v) \right|, \tag{16}
\]

where \(v_b\) is the value determined by Eqs.(9a). \(D_{\text{eff}}\) varies with wavelength within the line. We would expect that the less this variable is at one wavelength, the more reliable the fit parameters at the same wavelength are, i.e., a good fit has to be self-consistent.

REFERENCES