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A New Form of the Segal-Bargmann Transform for Lie Groups of Compact Type

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Abstract. I consider a two-parameter family $B_{s,t}$ of unitary transforms mapping an L^2-space over a Lie group of compact type onto a holomorphic L^2-space over the complexified group. These were studied using infinite-dimensional analysis in joint work with B. Driver, but are treated here by finite-dimensional means. These transforms interpolate between two previously known transforms, and all should be thought of as generalizations of the classical Segal-Bargmann transform. I consider also the limiting cases $s \to \infty$ and $s \to t/2$.

In [H1] I introduced an analog on an arbitrary connected compact Lie group K of the classical Segal-Bargmann transform. In fact [H1] gives two versions of the transform, B_t and C_t, where t is a positive parameter. Let $K_{\mathbb{C}}$ be the complexification of K, a certain complex Lie group that contains K as a subgroup, let ρ_t be the heat kernel measure at the identity on K, and let μ_t be the heat kernel measure at the identity on $K_{\mathbb{C}}$. The map B_t is a unitary map of $L^2(K, \rho_t)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_t)$, where $\mathcal{H}L^2$ denotes the space of square-integrable holomorphic functions. The map B_t is given by applying the time t heat operator $e^{t\Delta_K/2}$ to a function f on K, and then analytically continuing $e^{t\Delta_K/2} f$ to $K_{\mathbb{C}}$. Meanwhile, C_t is a unitary map of $L^2(K, dx)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, where dx is Haar measure on K and ν_t is obtained from μ_t by integrating over the action of K. The prescription for C_t is precisely the same as for B_t: $e^{t\Delta_K/2}$ followed by analytic continuation. B. Driver extended the results of [H1] to Lie groups of compact type, a class that includes both compact Lie groups and $\mathbb{R}^d$, thus allowing the classical transform and the compact-group transform to be treated in a unified way. These results were motivated by work [G1], [G2], [G3] of L. Gross—see [D], [Hi1], [Hi2] for more information. See also [GM], [H2], [H3], [H4], [HS], [St] for additional results. See [B] and the lecture notes [H5] for information and motivation concerning the classical Segal-Bargmann transform (*i.e.*, for $\mathbb{R}^d$).

The purpose of this paper is to study a two-parameter family $B_{s,t}$ of unitary transforms on a Lie group K of compact type. Here s and t are positive parameters with $s > t/2$. These transforms were introduced in joint work [DH] with Driver, where their isometricity was proved using techniques of stochastic analysis and their definition was motivated by the study of quantized Yang-Mills theory on a space-time cylinder. Here I give a self-contained and purely finite-dimensional account, and I consider the limit $s \to t/2$ in addition to the limit $s \to \infty$ considered in [DH]. I give here a new, simple method of proving isometricity of the transforms (Theorem 1.2), and a new method of proving surjectivity, based on a parabolic Harnack inequality (Theorem 2.4).

Section 1 treats the generic case, $t/2 < s < \infty$. The transform $B_{s,t}$ is a unitary map of $L^2(K, \rho_s)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t})$, where $\mu_{s,t}$ is a certain heat kernel measure on $K_{\mathbb{C}}$. The

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transform in all cases is given by $f \to$ analytic continuation of $e^{t\Delta_K/2}f$. Note that the transform depends only on t; the only s-dependence is in the measures. When $s = t$, $B_{s,t}$ coincides with the transform B_t of [H1], [D].

Section 2 treats the limiting cases $s \to \infty$ and $s \to t/2$. The $s \to \infty$ limit yields the C_t version of the transform, which means that as s varies from t to ∞, $B_{s,t}$ interpolates between the two previously known versions of the transform. The $s \to \infty$ limit is a crucial ingredient in the quantization scheme of [DH]. The $s \to t/2$ limit is more complicated. In case $K = \mathbb{R}^d$ this limit yields the finite-dimensional version of the Fourier-Wiener transform. In this case, the measure $\mu_{s,t}$ collapses as $s \to t/2$ onto the imaginary axis in $\mathbb{C}^d = (\mathbb{R}^d)_{\mathbb{C}}$ and the space $\mathcal{H}L^2(\mathbb{C}^d, \mu_{s,t})$ turns into an ordinary (non-holomorphic) L^2-space. At the opposite extreme, in case K is compact and semisimple, the limiting measure $\mu_{t/2,t}$ is still absolutely continuous with respect to Haar measure on $K_{\mathbb{C}}$ and the transform is an isometry *onto* $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{t/2,t})$. Finally, in case K is compact and abelian, we have intermediate behavior. Here the measure collapses onto the imaginary axis as in the $\mathbb{R}^d$ case, and $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{t/2,t})$ is not a closed subspace of $L^2(K_{\mathbb{C}}, \mu_{t/2,t})$. Nevertheless, the image of the limiting transform is a nice Hilbert space of holomorphic functions—it simply does not contain all L^2 holomorphic functions.

Sections 3 shows that in the $\mathbb{R}^d$ case, all the transforms are in a certain sense equivalent; in particular, the properties of $B_{s,t}$ can in this case be deduced from those of the "standard" transform B_t. No such equivalence holds for more general groups.

I thank Bruce Driver for many useful conversations, especially about the surjectivity problem in the $s = t/2$ case.

1 The Generic Case: $t/2 < s < \infty$

Definition 1.1 *A Lie group is of* compact type *if it is locally isomorphic to some compact group.*

Thus $\mathbb{R}^d$ is of compact type, since it is locally isomorphic to a d-torus. Of course, compact groups are also of compact type. It can be shown [D, Cor. 2.2] that every connected Lie group of compact type is the product of a compact group and $\mathbb{R}^d$. See, for example, [He], [Vara] for standard results about Lie groups of compact type.

Let K be a connected Lie group of compact type, and let $\mathfrak{k}$ be the Lie algebra of K. Fix once and for all an inner product on $\mathfrak{k}$ which is invariant under the adjoint action of K. (A Lie group admits such an inner product if and only if it is of compact type.) Let $X_1, \ldots, X_n$ be an orthonormal basis for $\mathfrak{k}$, and view each X_k as a left-invariant vector field. Let

$$\Delta_K = \sum_{k=1}^{n} X_k^2.$$

This is a bi-invariant differential operator on K and is independent of the choice of orthonormal basis.

Let $\rho_s(x)$ be the solution to the equation

$$\frac{d\rho}{ds} = \frac{1}{2}\Delta_K \rho_s,$$

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subject to the initial condition

$$\rho_s \to \delta_e \quad \text{as} \quad s \to 0.$$

This function is called the heat kernel (at the identity) for K. It is known that ρ_s exists and is unique, and is a strictly positive, C^∞ function on K. With little danger of confusion, we will also let ρ_s denote the associated measure

$$d\rho_s = \rho_s(x)\,dx,$$

where dx denotes Haar measure on K. (Of course, dx is defined only up to a constant. The definition of the heat kernel *function* depends on the normalization of Haar measure, but the heat kernel *measure* is always a probability measure.)

Let $K_{\mathbb{C}}$ be the complexification of K, which is a certain connected complex Lie group whose Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is the complexification of $\mathfrak{k}$, and which contains K as a subgroup. (See [H1, Sect. 3] or [Ho] for the definition.) For example, if $K = \mathbb{R}^d$ then $K_{\mathbb{C}} = \mathbb{C}^d$, and if $K = \mathrm{SU}(2)$, then $K_{\mathbb{C}} = \mathrm{SL}(2;\mathbb{C})$. The key property of the complexified group $K_{\mathbb{C}}$ is that every finite-dimensional representation of K has a unique analytic continuation to a holomorphic representation of $K_{\mathbb{C}}$. As a consequence of this property, it may be shown [H1], [D] that the heat kernel function $\rho_t(x)$ has a unique analytic continuation to $K_{\mathbb{C}}$.

Consider now the Hilbert space $L^2(K,\rho_s)$. For $t < 2s$ define

$$B_{s,t}\colon L^2\bigl(K,\rho_s(x)\,dx\bigr) \to \mathcal{H}(K_{\mathbb{C}})$$

by

$$B_{s,t}f(g) = \int_K \rho_t(gx^{-1})f(x)\,dx, \quad g \in K_{\mathbb{C}}. \tag{1.1}$$

Here $\rho_t(gx^{-1})$ refers to the analytically continued heat kernel and $\mathcal{H}(K_{\mathbb{C}})$ to the space of holomorphic functions on $K_{\mathbb{C}}$. Note that the formula for $B_{s,t}$ involves the time t heat kernel, whereas the domain Hilbert space involves the time s heat kernel. We will prove that the integral is convergent and that $B_{s,t}f$ is in fact holomorphic. Unless K is compact, convergence fails when $t \geq 2s$—see Theorem 2.2 in Section 2.

Now consider the operator $A_{s,t}$ on $K_{\mathbb{C}}$ given by

$$A_{s,t} = \left(s - \frac{t}{2}\right)\sum_{k=1}^{n} X_k^2 + \frac{t}{2}\sum_{k=1}^{n} JX_k^2. \tag{1.2}$$

Here J denotes the complex structure on $\mathfrak{k}_{\mathbb{C}}$, and the X_k's and JX_k's are viewed as left-invariant vector fields on $K_{\mathbb{C}}$. For example, if $K = \mathbb{R}^d$ and $K_{\mathbb{C}} = \mathbb{C}^d$, then $X_k = \partial/\partial x_k$ and $JX_k = \partial/\partial y_k$. For $t < 2s$ (*i.e.*, $s > t/2$) $A_{s,t}$ is a second-order, left-invariant, elliptic differential operator on $K_{\mathbb{C}}$. Let $\mu_{s,t}$ be the function on $K_{\mathbb{C}}$ given formally by

$$\mu_{s,t} = e^{A_{s,t}/2}(\delta_e).$$

More precisely, let $\mu_{s,t,r}$ be the solution to the equation

$$\frac{d\mu_{s,t,r}}{dr} = \frac{1}{2}A_{s,t}\mu_{s,t,r}$$
$$\lim_{r\to 0}\mu_{s,t,r} = \delta_e.$$

Then $\mu_{s,t} = \mu_{s,t,1}$. Existence and properties are shown, for example, in [Ro]. In particular $\mu_{s,t}$ is a C^∞, strictly positive function with rapid decay at infinity.

We will let $\mu_{s,t}$ also denote the associated probability measure, $d\mu_{s,t} = \mu_{s,t}(g)\,dg$, where dg is Haar measure on $K_{\mathbb{C}}$. Finally we will set

$$\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t}) = \mathcal{H}(K_{\mathbb{C}}) \cap L^2(K_{\mathbb{C}}, \mu_{s,t}).$$

We are now ready to state the main result. This was proved previously in [DH, Thm. 5.3] for the compact case using methods of stochastic analysis.

Theorem 1.2 *Let s and t be positive numbers with $s > t/2$. Then for all $f \in L^2(K, \rho_s)$, the integral*

$$B_{s,t}f(g) = \int_K \rho_t(gx^{-1})f(x)\,dx$$

is absolutely convergent for all $g \in K_{\mathbb{C}}$ and depends holomorphically on g. The map $B_{s,t}$ is an isometric isomorphism of $L^2(K, \rho_s)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t})$.

Proof We will consider separately the compact and $\mathbb{R}^d$ cases. The general case follows by reduction to those cases.

Case 1: *K* ***is compact.*** Although the method of [H1] can be used with little change, I wish to give a different argument which at least formally does not involve matrix entries. (The matrix entries still play an important technical role.) The argument uses a method proposed, in a more general setting, by T. Thiemann [T, Sect. 2.3].

According to [H1, Prop. 1], the heat kernel ρ_t admits a unique analytic continuation to $K_{\mathbb{C}}$. Although it is possible to use the transform itself to deduce this, as in [D], it is not difficult to obtain the result directly, as in [H1]. In this the compact case there is no trouble with the convergence of the integral that defines $B_{s,t}$ or with the holomorphicity of $B_{s,t}f$. So we need only address isometricity and surjectivity.

Let us first give a heuristic argument for the isometricity of $B_{s,t}$, and then show that this can be made rigorous. Since $\rho_s(x^{-1}) = \rho_s(x)$ [H1], integrating a function ϕ with respect to the measure $\rho_s(x)\,dx$ gives the same result as computing $e^{s\Delta_K/2}(\phi)$ and then evaluating at the identity. Similarly (see [D, Thm. 2.7]), integrating a function on $K_{\mathbb{C}}$ against $\mu_{s,t}(g)\,dg$ gives the same result as applying $e^{A_{s,t}/2}$ and then evaluating at the identity. Thus the isometricity of $B_{s,t}$ amounts to the statement

$$e^{s\Delta_K/2}(\overline{f_1}f_2)(e) = e^{A_{s,t}/2}(\overline{e^{t\Delta_K/2}f_1}e^{t\Delta_K/2}f_2)(e). \tag{1.3}$$

Let us assume that f_1 and f_2 themselves admit an analytic continuation to $K_{\mathbb{C}}$. (The space of such f's is dense.) Now, $\Delta_K = \sum X_k^2$, regarded as a left-invariant differential operator on $K_{\mathbb{C}}$, commutes with complex conjugation and with analytic continuation. Thus

$$\overline{e^{t\Delta_K/2} f_1} = e^{t\Delta_K/2}\overline{f_1}.$$

Note that on the left, we are first applying $e^{t\Delta_K/2}$, then analytically continuing, and then taking the complex conjugate. On the right we are first analytically continuing, then taking the complex conjugate and then applying $e^{t\Delta_K/2}$.

Next consider the operators

$$Z_k = \frac{1}{2}(X_k - iJX_k)$$
$$\bar{Z}_k = \frac{1}{2}(X_k + iJX_k),$$

which reduce in the case $K_{\mathbb{C}} = \mathbb{C}^d$ to $\partial/\partial z_k$ and $\partial/\partial \bar{z}_k$. On the holomorphic function f_2 we have $Z_k f_2 = X_k f_2$ and $\bar{Z}_k f_2 = 0$, and on the anti-holomorphic function $\overline{f_1}$, $Z_k \overline{f_1} = 0$ and $\bar{Z}_k f_1 = X_k f_1$. It follows that

$$\overline{e^{t\Delta_K/2} f_1} e^{t\Delta_K/2} f_2 = e^{t\sum Z_k^2/2} e^{t\sum \bar{Z}_k^2/2}(\overline{f_1} f_2).$$

So the desired norm equality becomes

$$e^{s\Delta_K/2}(\overline{f_1} f_2)(e) = e^{A_{s,t}/2} e^{t\sum Z_k^2/2} e^{t\sum \bar{Z}_k^2/2}(\overline{f_1} f_2)(e). \tag{1.4}$$

Now, the holomorphic vector field Z_k automatically commutes with the anti-holomorphic vector field $\bar{Z}_l$ (or calculate this directly). Thus the second and third exponents on the right of (1.4) may be combined. The exponent that results is

$$\frac{t}{2}\sum_{k=1}^{n}(Z_k^2 + \bar{Z}_k^2) = \frac{t}{4}\sum_{k=1}^{n}(X_k^2 - JX_k^2).$$

This is a constant times the Casimir operator for $K_{\mathbb{C}}$, which is bi-invariant and therefore commutes with the left-invariant operator $A_{s,t}$. So in the end all three exponents on the right in (1.4) may be combined. It thus suffices to have the sum of the three exponents on the right in (1.4) equal to the exponent on the left. Recalling the definition (1.2) of $A_{s,t}$ and multiplying both sides by 2, we need

$$s\sum X_k^2 = \left(s - \frac{t}{2}\right)\sum X_k^2 + \frac{t}{2}\sum JX_k^2 + \frac{t}{2}\sum(X_k^2 - JX_k^2), \tag{1.5}$$

which is true!

Let us now make this argument rigorous. Let $\mathcal{F}$ denote the space of finite linear combinations of matrix entries for finite-dimensional irreducible representations of K. Since K is compact, $\rho_s(x)$ is bounded and bounded away from zero. Thus $L^2(K, \rho_s)$ is the same space of functions as $L^2(K, dx)$, with a different but equivalent norm. So by the Peter-Weyl theorem, $\mathcal{F}$ is dense in $L^2(K, \rho_s)$. On $\mathcal{F}$ there is little trouble in justifying the above arguments. Indeed, [H1, Lem. 2, Lem. 8] shows that when integrating matrix entries against heat kernels, one may expand everything in power series. Thus $B_{s,t}$ is isometric on $\mathcal{F}$ and so also by a straightforward passage-to-the-limit on $L^2(K, \rho_s)$.

For surjectivity of the transform, we note that the surjectivity proof of [H1] (for the $s = t$ case) applies with only the most obvious of changes. In particular, the proof of the "averaging lemma" [H1, Lem. 11], which is the key to the proof of surjectivity, applies with μ_t replaced by $\mu_{s,t}$. The proof of Theorem 2.4 gives another way to prove surjectivity here. Although that theorem requires K to be semisimple, in the case $s > t/2$ the operator $A_{s,t}$ is always elliptic and semisimplicity is not needed. In the case $s > t/2$, we could use the Li-Yau parabolic Harnack inequality (*e.g.*, [Da, Thm. 5.3.5]) in place of the "subelliptic" inequality of [Varo], [S-C].

Case 2: $K = \mathbb{R}^d$**.** In this case the theorem can be reduced to the standard ($s = t$) case—see Section 3. However, it is just as easy to give a direct proof. See [DH, Sect. 3] for more details.

Since $\mathbb{R}^d$ is non-compact, we need to address the convergence of the integral in Theorem 1.2. That integral will be absolutely convergent for all f provided that (in additive notation) for all $z \in \mathbb{C}^d$,

$$\frac{\rho_t(z - x)}{\rho_s(x)} \in L^2(\mathbb{R}^d, \rho_s).$$

Using the explicit formula $\rho_t(x) = (2\pi t)^{-d/2} e^{-x^2/2t}$, it is easily verified that this holds precisely when $s > t/2$. When $s > t/2$, Morera's Theorem will show that the integral is holomorphic as a function of $z \in \mathbb{C}^d$.

The formal argument for isometricity is precisely the same as in the compact case; we merely need a dense subspace on which it can be made rigorous. Among the possibilities for such a subspace are the space of polynomials and the space of finite linear combinations of exponentials.

To prove surjectivity, I claim it suffices to prove that the holomorphic polynomials are dense in $\mathcal{H}L^2(\mathbb{C}^d, \mu_{s,t})$. After all, the operator $e^{t\Delta/2}$ is invertible on the space of polynomials on $\mathbb{R}^d$ of degree at most n, with inverse $e^{-t\Delta/2}$ given by a terminating power series. So every polynomial p on $\mathbb{R}^d$ is of the form $e^{t\Delta/2}q$ for some polynomial q, and so every holomorphic polynomial $p_{\mathbb{C}}$ on $\mathbb{C}^d$ is the analytic continuation of $e^{t\Delta/2}q$, where the $p = e^{-t\Delta/2}q$ is the restriction of $p_{\mathbb{C}}$ to $\mathbb{R}^d$.

So it remains only to prove the density of holomorphic polynomials. I repeat here the argument of [DH, Sect. 3.2]. In the case $s = t$ this is conventionally done using Taylor series [B, Sect. 1b]. For $s \neq t$ we use instead the holomorphic version of the Hermite expansion—this coincides with the Taylor expansion when $s = t$. By a variant of a well known result, $L^2(\mathbb{C}^d, \mu_{s,t})$ is the orthogonal direct sum of the subspaces $\mathcal{F}_{n,s,t}$ given by

$$\mathcal{F}_{n,s,t} = e^{-A_{s,t}/2} \text{ (homogeneous polynomials of degree } n\text{)}.$$

Furthermore, the expansion of any $F \in L^2(\mathbb{C}^d, \mu_{s,t})$ as a (L^2-convergent, orthogonal) sum of elements of $\mathcal{F}_{n,s,t}$ may be accomplished as follows: Apply $e^{A_{s,t}/2}$ to F, then expand $e^{A_{s,t}/2}F$

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as a sum of homogeneous polynomials of degree n (the Taylor series of $e^{A_{s,t}/2}F$), and then apply $e^{-A_{s,t}/2}$ term-by-term to that series. Here $e^{A_{s,t}/2}F$ is computed as the convolution of F with $\mu_{s,t}$. Note that it is at least formally clear that this prescription gives an expansion of F in terms of elements of $\mathcal{F}_{n,s,t}$. This expansion of F is called the Hermite expansion.

The key observation is this: If $F \in L^2(\mathbb{C}^d, \mu_{s,t})$ is holomorphic, then so is $e^{A_{s,t}/2}F$, and so also each term, say p_n, in the Taylor expansion of $e^{A_{s,t}/2}F$. Thus the terms $e^{-A_{s,t}/2}p_n$ in the Hermite expansion of F are holomorphic polynomials. (It is straightforward to verify that if $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_{s,t})$ then the convolution of F with $\mu_{s,t}$ exists and is holomorphic. Furthermore, $A_{s,t}$ commutes with $\partial/\partial \bar{z}_k$ so if p_n is holomorphic then so is $e^{-A_{s,t}/2}p_n$.) All of this shows that if $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_{s,t})$ then F is the L^2-convergent sum of a series of holomorphic polynomials. Thus holomorphic polynomials are dense.

Note that if $s = t$ then $A_{s,t}$ is zero on all holomorphic functions. In that case the above expansion of $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_{s,t})$ is nothing other than the Taylor expansion of F.

Case 3: $K = \mathbb{R}^d \times G$***, with*** G ***compact and semisimple.*** Then $K_{\mathbb{C}} = \mathbb{C}^d \times G_{\mathbb{C}}$ and the Lie algebra decomposition $\mathfrak{k} = \mathbb{R}^d \oplus \mathfrak{g}$ is automatically orthogonal with respect to any Ad-invariant inner product. It follows that the heat kernels on both the real side and the complex side will factor. Now (suppressing the measures) we have the standard result

$$L^2(\mathbb{R}^d) \otimes L^2(G) \cong L^2(\mathbb{R}^d \times G),$$

where the isomorphism takes $f_1 \otimes f_2$ to the product function $f_1(x)f_2(y)$. Meanwhile on the complex side, we have

$$\mathcal{H}L^2(\mathbb{C}^d) \otimes \mathcal{H}L^2(G_{\mathbb{C}}) \cong \mathcal{H}L^2(\mathbb{C}^d \times G_{\mathbb{C}}),$$

with $F_1 \otimes F_2$ mapping to $F_1(z)F_2(g)$. This result requires a small proof which is given in the appendix.

Now, since the heat kernel on $\mathbb{R}^d \times G$ factors, the transform applied to a product function will be just the product of the separate transforms. That is, the transform for the product group is (under the above isomorphisms) just the tensor product of the separate transforms. So all the desired properties of the transform for K follow from the corresponding properties of the transforms for $\mathbb{R}^d$ and for G.

Case 4: The General Case Although every connected Lie group of compact type is a product of a compact group with $\mathbb{R}^d$ [D, Cor. 2.2], the Lie algebras of the two factors need not be orthogonal, since the compact factor need not be semisimple. Thus the method of Case 3 is not sufficiently general. But for any K which is connected and of compact type, the universal cover of K is of the form $\tilde{K} = \mathbb{R}^d \times G$, where G is compact and simply connected, hence semisimple. (See [He, Chap. II, Cor. 6.5 and Thm. 6.9].) Thus by Case 3, the transform for $\tilde{K}$ is well-defined, isometric, and surjective.

Now, K itself is $\tilde{K}/N$, where N is a discrete central subgroup, and

$$\rho_s(x) = \sum_{n \in N} \tilde{\rho}_s(\tilde{x}n) \tag{1.6}$$

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where ρ_s is the heat kernel for K, $\tilde{\rho}_s$ is the heat kernel for $\tilde{K}$, and $\tilde{x}$ is any point in $\tilde{K}$ which projects to x. Note that N is finitely-generated (since it is $\pi_1(K)$) and abelian, hence a product of cyclic groups, and each generator of an infinite cyclic factor must have non-zero $\mathbb{R}^d$-component. So there is no problem with convergence of the sum. It follows that

$$L^2(K, \rho_s) = L^2(\tilde{K}, \tilde{\rho}_s)^N. \tag{1.7}$$

Here a superscript N indicates the functions which are invariant under the right (or equivalently left) action of N.

The complexification of K is $K_{\mathbb{C}} = \tilde{K}_{\mathbb{C}}/N$, where $\tilde{K}_{\mathbb{C}}$ means the complexification of the universal cover of K, which is the same as the universal cover of the complexification. (Recall that $\tilde{K} \subset \tilde{K}_{\mathbb{C}}$.) We have, by analogy to (1.7)

$$\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t}) = \mathcal{H}L^2(\tilde{K}_{\mathbb{C}}, \tilde{\mu}_{s,t})^N. \tag{1.8}$$

By construction, the transform for $\tilde{K}$ commutes with the action of N, and so it maps $L^2(\tilde{K}, \tilde{\rho}_s)^N$ into $\mathcal{H}L^2(\tilde{K}_{\mathbb{C}}, \tilde{\mu}_{s,t})^N$. Furthermore, in light of (1.6), the transform for $\tilde{K}$, restricted to the N-invariant subspace, coincides with the transform for K. So convergence and isometricity of the transform for K follow from the corresponding properties for $\tilde{K}$.

To establish surjectivity for the transform for K, we must prove that the transform for $\tilde{K}$ maps the N-invariant subspace of $L^2(\tilde{K}, \tilde{\rho}_s)$ onto the N-invariant subspace of $\mathcal{H}L^2(\tilde{K}_{\mathbb{C}}, \tilde{\mu}_{s,t})$. So suppose $F \in \mathcal{H}L^2(\tilde{K}_{\mathbb{C}}, \tilde{\mu}_{s,t})^N$. Then by the surjectivity of the transform for $\tilde{K}$, there exists $f \in L^2(\tilde{K}, \tilde{\rho}_s)$ such that

$$F(g) = \int_{\tilde{K}} \tilde{\rho}_t(gx^{-1}) f(x)\, dx.$$

Since $F(g) = F(gn)$, we have

$$\begin{aligned} \int_{\tilde{K}} \tilde{\rho}_t(gx^{-1}) f(x)\, dx &= \int_{\tilde{K}} \tilde{\rho}_t(gnx^{-1}) f(x)\, dx \\ &= \int_{\tilde{K}} \tilde{\rho}(gx^{-1}) f(xn)\, dx. \end{aligned} \tag{1.9}$$

Now, if $f(x) \in L^2(\tilde{K}, \tilde{\rho}_s)$, then both $f(x)$ and $f(xn)$ are in $L^2(\tilde{K}, \tilde{\rho}_r)$ for all $r < s$. (There is no trouble on the G factor, and on the $\mathbb{R}^d$ factor this is an easy calculation.) If we take $t/2 < r < s$, then (1.9) says that $B_{r,t}$ applied to $f(x)$ is the same as $B_{r,t}$ applied to $f(xn)$. But $B_{r,t}$ is isometric and therefore injective, and so we conclude that $f(x) = f(xn)$ for almost every x. Thus f is in the N-invariant subspace, and $B_{s,t}$ is surjective. This completes the proof of Theorem 1.2. ∎

2 The Limiting Cases: $s \to t/2$ and $s \to \infty$

In the case $s = t$, Theorem 1.2 reduces to the results of [H1], [D]. In particular $\mu_{t,t} = \mu_t$, where μ_t is the measure in those papers. This case is special because in this case, and only in this case, the operator $A_{s,t} = A_{t,t}$ annihilates holomorphic functions. This property is central to the development of the "$J^{\perp}$" expansion in [D], [DG]. The case $s = t$ is also more amenable to passing to the infinite-dimensional limit, because one has in this case dimension-independent bounds. See [D, Cor. 5.5 and 5.9] and [HS, Thm. 4]. One can still take the infinite-dimensional limit when $s \neq t$ (see [DH]), but the "restriction map" which plays a crucial role in [HS] is unavailable in this case, and so different methods are needed.

In this section we will stay within the finite-dimensional realm, but will examine what happens in the limits $s \to t/2$ and $s \to \infty$. Recall that in every case our transform takes f to the analytic continuation of $e^{t\Delta_K/2} f$, but that we are varying the inner product (*i.e.*, the measure) on the domain and range.

Theorem 2.1 *Let K be compact and normalize Haar measure on K to have mass one. Then $f \in L^2(K, \rho_s)$ if and only if $f \in L^2(K, dx)$, and*

$$\|f\|_{L^2(K,dx)} = \lim_{s\to\infty} \|f\|_{L^2(K,\rho_s)}.$$

The function

$$\nu_{s,t}(g) = \int_K \mu_{s,t}(xg)\, dx, \quad g \in K_{\mathbb{C}}$$

is independent of s and will be denoted $\nu_t(g)$. For all $F \in \mathcal{H}(K_{\mathbb{C}})$, $F \in L^2(K_{\mathbb{C}}, \mu_{s,t})$ if and only if $F \in L^2(K_{\mathbb{C}}, \nu_t)$, and

$$\|F\|_{L^2(K_{\mathbb{C}},\nu_t)} = \lim_{s\to\infty} \|F\|_{L^2(K_{\mathbb{C}},\mu_{s,t})}.$$

Thus the map

$$f \to \text{ analytic continuation } (e^{t\Delta_K/2} f)$$

is an isometric isomorphism of $L^2(K, dx)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$.

This last isometric isomorphism was verified in [H1, Thm. 2] and was denoted C_t. Note that I am using here a different normalization of Haar measure on K, and therefore a different normalization of ν_t, than in [H3]. In that paper I normalize Haar measure on K to coincide with the Riemannian volume measure, so that K need not have volume one.

Proof Since we are assuming that K is compact, $\rho_s(x)$ will be bounded and bounded away from zero. Thus the L^2 norm with respect to $\rho_s(x)\, dx$ is finite if and only if the L^2 norm with respect to Haar measure is finite. It is an easy and standard result that $\rho_s(x)$ converges uniformly to the constant function 1, which establishes the first limit in the theorem.

Now, let δ_K denote Haar measure on K, viewed as a measure on $K_{\mathbb{C}}$. Then since $A_{s,t}$ is a left-invariant operator, we have formally

$$\nu_{s,t} = e^{A_{s,t}}(\delta_K).$$

But the two terms in the definition of $A_{s,t}$ commute, so

$$\nu_{s,t} = e^{t/2 \sum JX_k^2} e^{(s-t/2)\sum X_k^2}(\delta_K).$$

Since δ_K is K-invariant, the exponential involving $\sum X_k^2$ has no effect, and the s-dependence vanishes.

The equivalence of square-integrability with respect to $\mu_{s,t}$ and ν_t is implied by the "averaging lemma" [H1, Lem. 11]. This is stated in [H1] for the case $s = t$, but the same proof applies in general. Using commutativity again we have for $s > t$

$$\mu_{s,t} = e^{(s-t)\sum X_k^2} e^{t/2 \sum JX_k^2} e^{t/2 \sum X_k^2}(\delta_e).$$

Thus

$$\mu_{s,t}(g) = \int_K \mu_{t,t}(gx^{-1})\rho_{s-t}(x)\,dx,$$

from which it follows that $\lim_{s\to\infty} \mu_{s,t}(g) = \nu_t(g)$ for all g. Furthermore, applying the averaging lemma to $\mu_{t,t}$ we see that for all $s > t$, $\mu_{s,t}(g)$ is dominated by a constant (independent of s) times $\nu_t(g)$. So Dominated Convergence gives the second limit in the theorem. The methods of [H1] are sufficient to make all of this rigorous. ∎

Thus we obtain the unitary transform C_t as the $s \to \infty$ limit of $B_{s,t}$, in the compact case. In the general case there is undoubtedly a unitary map similar to C_t—certainly this is so if $K = \mathbb{R}^d$. However, it is not as easy to obtain it as a limit of the $B_{s,t}$'s. First, to get convergence of the measures ρ_s and $\mu_{s,t}$ as $s \to \infty$, we need to multiply them by an appropriate function of s, $(2\pi s)^{d/2}$ in the $\mathbb{R}^d$ case. Second, if K is not compact, then $L^2(K, \rho_s)$ and $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t})$ are different spaces of functions for different values of s, so one would have to be careful about how a limiting theorem is stated. I will not pursue this matter here.

We now turn to the opposite extreme, the limit $s \to t/2$. In the case $K = \mathbb{R}^d$ the measure $\mu_{s,t}$ collapses as $s \to t/2$ onto the imaginary axis, and our transform becomes a finite-dimensional version of the Fourier-Wiener transform. In the case K compact, we still have when $s = t/2$ an isometric map *into* an L^2-space of holomorphic functions, and it is *onto* the holomorphic L^2-space if K is semisimple.

Theorem 2.2 *Let $K = \mathbb{R}^d$ so that $K_{\mathbb{C}} = \mathbb{C}^d$. As s tends to $t/2$, the measures $\mu_{s,t}$ on $\mathbb{C}^d$ converge in the weak-$*$ topology to the Gaussian measure*

$$(\pi t)^{-n/2} \exp(-y^2/t)\,dy = \rho_{t/2}(y)\,dy$$

on the "imaginary axis" $i\mathbb{R}^d$. The transform $B_{t/2,t}$ makes sense, say, on polynomials and is given by

$$\begin{aligned} B_{t/2,t} f(iy) &= \int_{\mathbb{R}^d} \rho_t(iy - x) f(x)\,dx \\ &= (2\pi t)^{-n/2} e^{y^2/2t} \int_{\mathbb{R}^d} e^{iyx/t} e^{-x^2/2t} f(x)\,dx. \end{aligned}$$

This transform maps the space of polynomials isometrically into $L^2(i\mathbb{R}^d, \rho_{t/2})$. Extending continuously we obtain an isometric isomorphism of $L^2(\mathbb{R}^d, \rho_{t/2})$ onto $L^2(i\mathbb{R}^d, \rho_{t/2})$.

Note that $B_{t/2,t}$ is essentially the Fourier transform, unitary from $L^2(\mathbb{R}^d, dx)$ to itself, disguised by conversion to Gaussian measure on both sides—that is, the Fourier-Wiener transform (*e.g.* [BSZ, Chap. 1.7]).

Note also that in the limit $s \to t/2$, the analyticity on the range is lost. That is, the range of $B_{t/2,t}$ is not an L^2-space of holomorphic functions, but an ordinary L^2-space, the elements of which need not even be continuous. This loss of analyticity coincides with a loss of convergence in the integral that defines $B_{t/2,t}$.

Proof Convergence of the measure follows from the explicit formula

$$\mu_{s,t}(x+iy) = \big(2\pi(s-t/2)\big)^{n/2} e^{-x^2/2(s-t/2)} (\pi t)^{-n/2} e^{-y^2/t}.$$

If f is a polynomial, then $F = e^{t\Delta/2} f$ is also a polynomial. So there is no difficulty in letting $s \to t/2$ to obtain

$$\|f\|_{L^2(\mathbb{R}^d,\rho_{t/2})} = \lim_{s\to t/2} \|f\|_{L^2(\mathbb{R}^d,\rho_s)} = \lim_{s\to t/2} \|F\|_{L^2(\mathbb{C}^d,\mu_{s,t})} = \|F\|_{L^2(i\mathbb{R}^d,\rho_{t/2})}.$$

So $B_{t/2,t}$ is isometric on polynomials. But $e^{t\Delta/2}$ is invertible on the space of polynomials of degree at most k, and so every polynomial is in the image of $B_{t/2,t}$. Since polynomials are dense in $L^2(\mathbb{R}^d, \rho_{t/2})$, $B_{t/2,t}$ is densely defined and has dense image. ∎

Theorem 2.3 *Let K be compact. Then as $s \to t/2$ the measures $\mu_{s,t}$ on $K_\mathbb{C}$ converge in the weak-$*$ topology to a probability measure, denoted $\mu_{t/2,t}$. If K is semisimple, $\mu_{t/2,t}$ is absolutely continuous with respect to Haar measure on $K_\mathbb{C}$. For any compact K, the transform $B_{s,t}$ makes sense with $s = t/2$, and $B_{t/2,t}$ is an isometry of $L^2(K, \rho_{t/2})$ into $\mathcal{H}L^2(K_\mathbb{C}, \mu_{t/2,t})$.*

Proof Every connected compact Lie group is of the form $K = (G \times T)/N$, where G is compact and semisimple, T is a torus, and N is a finite central subgroup [BtD, Thm. 8.1]. In the semisimple case, the function $\mu_{s,t}$ on $G_\mathbb{C}$ exists and is strictly positive [Ro, IV.4] even when $s = t/2$, because in that case the operator $\sum JX_k^2$ is subelliptic and therefore hypoelliptic. Furthermore, as in the proof of Theorem 2.2, we have for $s > t/2$

$$\mu_{s,t}(g) = \int_G \mu_{t/2,t}(gx^{-1}) \rho_{s-t/2}(x)\, dx.$$

It follows that the weak-$*$ limit of $\mu_{s,t}$ as s approaches $t/2$ is $\mu_{t/2,t}$. Furthermore,

$$\int_G \mu_{t/2,t}(gx)\, dx = \nu_t(g) = \int_G \mu_{s,t}(gx)\, dx \tag{2.1}$$

where $\nu_t(g)$ is as in Theorem 2.2 and decays rapidly at infinity. If f is in the space $\mathcal{F}$ (as in proof of Theorem 1.2), then the function $F =$ analytic continuation of $e^{t\Delta_K/2} f$ grows only exponentially at infinity. Using (2.1) we may control the integral of F near infinity uniformly in s, giving

$$\lim_{s\to t/2} \|F\|_{L^2(K_\mathbb{C},\mu_{s,t})} = \|F\|_{L^2(K_\mathbb{C},\mu_{t/2,t})}.$$

Thus, $B_{t/2,t}$ is isometric on $\mathcal{F}$ and so also on $L^2(G, \rho_{t/2})$.

In the torus case T, we have $T_{\mathbb{C}} = T \times \mathbb{R}^d$, and the measure $\mu_{s,t}$ is the product of the heat kernel measure $\rho_{(s-t/2)/2}$ on T and the Gaussian measure $\rho_{t/2}$ on $\mathbb{R}^d$. So again there is no trouble in letting s tend to $t/2$. Taking products and periodizing over N pose no problem. ∎

Theorem 2.4 *Let K be compact and semisimple. Then the transform $B_{t/2,t}$ of Theorem 2.3 maps* onto $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{t/2,t})$.

Proof We will use a parabolic Harnack inequality to derive a weak form of the averaging lemma. The weak averaging lemma will show that if $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \mu_{t/2,t})$ then for all $r < t$ there exists f_r such that F is the analytic continuation of $e^{r\Delta_K/2} f$. The isometricity of the transform $B_{r/2,r}$ together with another application of the parabolic Harnack inequality will show that the norm of f_r remains bounded as $r \to t^-$, from which it follows that $\lim_{r\to t^-} f_r$ exists. This limiting function f_t is the pre-image of F under $B_{t/2,t}$.

Since K is semisimple, the elements of the form JX, $X \in \mathfrak{k}$, together with their commutators, span the whole Lie algebra $\mathfrak{k}_{\mathbb{C}}$ ($[J\mathfrak{k}, J\mathfrak{k}] = [\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$). It follows that the operator $A_{t/2,t}$ is subelliptic and thus that the definition of $\mu_{s,t}$ makes sense even when $s = t/2$. The function $\mu_{t/2,t}$ is a strictly positive C^∞ function. We will use the following lemmas concerning $\mu_{t/2,t}$. (Compare Lemma 2.5 to [H1, Lem. 11].)

Lemma 2.5 **(Weak Averaging Lemma)** *Assume K is compact and semisimple and let $\nu_t(g) = \int_K \mu_{t/2,t}(gx)\,dx$. Then for all r and t with $0 < r < t$ there exists a constant C_1 such that*

$$\mu_{t/2,t}(g) \geq C_1 \nu_r(g)$$

for all g in $K_{\mathbb{C}}$.

Lemma 2.6 *Assume K is compact and semisimple. Then for all $t > 0$ there exists a constant C_2 such that for all $r \in (3t/4, t)$*

$$\mu_{r/2,r}(g) \leq C_2 \mu_{t/2,t}(g)$$

for all g in $K_{\mathbb{C}}$.

Proof of Lemma 2.5 There is a natural left-invariant metric on $K_{\mathbb{C}}$ associated to the operator ΣJX_k^2, namely, $d(x, y) = \inf_\gamma \int |\dot\gamma(t)|\,dt$, where the infimum is over all piecewise smooth curves for which $\dot\gamma(t)$ is in the span of the vector fields JX_k for all t, and where if $\dot\gamma = \Sigma a_k JX_k$, then $|\dot\gamma|^2 = \Sigma a_k^2$. If K is semisimple, then such a curve exists for any x and y, and d is a continuous metric on $K_{\mathbb{C}}$ (*e.g.* [VSC, Chap. III.4]). This metric is described in a different but equivalent way in [S-C, Sect. 2]. A parabolic Harnack inequality in this setting seems to have first been proved by Varopoulos [Varo]; I will refer to the form given on the bottom of p. 439 of [S-C] (using Thm. 2.1 and Sect. 8.3). Since K is compact and

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d is continuous, $d(e,x)$ is bounded for $x \in K$. The left-invariance of the metric then implies that $d(g,gx)$ is bounded for $g \in K_{\mathbb{C}}$, $x \in K$. So as a consequence of [S-C] we have a constant C such that

$$\mu_{r/2,r}(gx) \leq C\mu_{t/2,t}(g)$$

for all $g \in K_{\mathbb{C}}$ and $x \in K$. Thus

$$\begin{aligned}\mu_{t/2,t}(g) &\geq \frac{1}{C}\sup_{x\in K}\mu_{r/2,r}(gx)\\ &\geq \frac{1}{C}\int_K \mu_{r/2,r}(gx)\,dx = \frac{1}{C}\nu_r(g).\end{aligned}$$ ∎

Proof of Lemma 2.6 This follows from the same inequality in [S-C], choosing the r there so that the lower time interval in the inequality contains our r and the upper time interval contains our t. ∎

As in Theorem 2.2 and noting that $\mu_{t/2,t}$ is Ad-K-invariant, ν_r is the same as the K-averaged heat kernel in [H1]. According to the Weak Averaging Lemma, if F is square-integrable with respect to $\mu_{t/2,t}$, then it is square-integrable with respect to ν_r for all $r < t$. Thus by [H1, Thm. 2] or Theorem 2.2, there exists for each $r < t$ a function $f_r \in L^2(K,dx)$ such that F is the analytic continuation of $e^{r\Delta_K/2} f_r$. But then applying the isometry $B_{r/2,r}$ to the function f_r and applying Lemma 2.6

$$\|f_r\|_{L^2(K,\rho_{r/2})} = \|F\|_{L^2(K_{\mathbb{C}},\mu_{r/2,r})} \leq C_2\|F\|_{L^2(K_{\mathbb{C}},\mu_{t/2,t})}.$$

Thus the norm of f_r in $L^2(K,\rho_{r/2})$, and so also in $L^2(K,dx)$, remains bounded as r increases to t. But now the restriction of F to K has an expansion in terms of an orthonormal basis for $L^2(K,dx)$ of eigenvectors ψ_n for $-\Delta_K$ with eigenvalues $\lambda_n \geq 0$:

$$F = \sum a_n\psi_n.$$

So

$$f_r = \sum_{\pi} e^{r\lambda_n/2} a_n\psi_n,$$

with $\sum e^{r\lambda_n}|a_n|^2$ bounded as r increases to t. By Monotone Convergence, $\sum e^{t\lambda_n}|a_n|^2 < \infty$ and so the sum $f_t := \sum e^{t\lambda_n/2}a_n\psi_n$ converges in L^2. Then F is the analytic continuation of $e^{t\Delta_K/2} f_t$. This proves Theorem 2.4. ∎

Theorem 2.7 *Let $K = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, so that $K_{\mathbb{C}} = \mathbb{C}/2\pi\mathbb{Z}$. Then the image of the map $B_{t/2,t}$ is precisely the space of functions F satisfying:*

1. *F is holomorphic on $\mathbb{C}$ and satisfies $F(z+2\pi) = F(z)$ for all $z \in \mathbb{C}$.*

2. *F is square-integrable with respect to* $\mu_{t/2,t}$; *that is,*

$$(\pi t)^{-1/2} \int_{\mathbb{R}} |F(iy)|^2 e^{-y^2/t}\, dy < \infty.$$

3. *There exists constants a, b, and c, with* $c < 1/2$ *such that*

$$|F(x + iy)| \le a \exp(be^{c|y|})$$

for all x, y.

If F is in the image of $B_{t/2,t}$, *then for some constant d*

$$|F(x + iy)| \le de^{y^2/2t}$$

for all x, y.

We may not take $c = 1/2$ in Condition 3, as the function

$$F(z) = \cos(e^{iz/2})$$

demonstrates. (Observe that F is 2π-periodic, even though $e^{iz/2}$ is 2π-anti-periodic.) After all, $|F(iy)| \le 1$, which means that F is square-integrable, but $F(\pi + iy) = \cosh(e^{-y/2})$, which grows too rapidly as $y \to -\infty$ for F to be in the image of $B_{t/2,t}$. Note that in the presence of Conditions 1 and 2 of the theorem, the very weak bounds of Condition 3 imply the stronger bounds given in the last part of the theorem. This is a consequence of the Phragmen-Lindelöf method (*e.g.*, [Ru, Chap. 12]), which allows the values of a holomorphic function in the strip $0 < \operatorname{Re} z < 2\pi$ to be controlled by its values on the boundary, given Condition 3.

Proof If F is $B_{t/2}, tf$ for some f, then clearly F will satisfy Condition 1, and by Theorem 2.3, F will satisfy Condition 2. Meanwhile,

$$B_{t/2,t} f(x + iy) = \int_0^{2\pi} \rho_t(x + iy - x') f(x')\, dx'$$

where

$$\begin{aligned} \rho_t(x + iy - x') &= \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \exp\{-(x - x' - 2\pi n + iy)^2/2t\} \\ &= \frac{1}{\sqrt{2\pi t}} e^{y^2/2t} \sum_{n=-\infty}^{\infty} e^{-i(x-x'-2\pi n)y/t} e^{-(x-x'-2\pi n)^2/2t}. \end{aligned} \tag{2.2}$$

But the second sum in (2.2) is easily seen to be bounded uniformly in x, x', and y which gives the estimate in the last part of the theorem, and so certainly Condition 3.

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Conversely, suppose we have a function F which satisfies Conditions 1, 2, and 3. Then choose α with $c < \alpha < 1/2$, and let

$$h_\varepsilon(z) = \exp\{-\varepsilon \cos \alpha(z-\pi)\}.$$

For $x \in [0, 2\pi]$, $h_\varepsilon(x+iy)$ tends to zero very rapidly as y tends to infinity. Let us apply Cauchy's Formula to the function $F(z)e^{z^2/2t}h_\varepsilon(z)$, over the rectangle $0 < \operatorname{Re} z < 2\pi$, $-A < \operatorname{Im} z < A$. The presence of h_ε together with Condition 3 allow us to let $A \to \infty$, giving

$$F(z)e^{z^2/2t}h_\varepsilon(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(2\pi+iy)e^{2\pi^2/t}e^{2\pi iy/t}e^{-y^2/2t}h_\varepsilon(2\pi+iy)}{2\pi+iy-z}\,i\,dy$$
$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(iy)e^{-y^2/2t}h_\varepsilon(iy)}{iy-z}\,i\,dy.$$

Now, $F(2\pi+iy) = F(iy)$ is square-integrable with respect to $e^{-y^2/t}\,dy$, which implies that $F(iy)e^{-y^2/2t}$ is square-integrable dy. For $0 < \operatorname{Re} z < 2\pi$, $1/(2\pi+iy-z)$ is also square-integrable dy, which means the product of the two is integrable dy. Moreover, $|h_\varepsilon(iy)| \leq 1$ and $|h_\varepsilon(2\pi+iy)| \leq 1$ for all y, so by dominated convergence we may interchange the limit $\varepsilon \to 0$ with the integrals. So

$$F(z)e^{z^2/2t} = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(iy)e^{2\pi^2/t}e^{2\pi iy/t}e^{-y^2/2t}}{2\pi+iy-z}\,i\,dy$$
$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(iy)e^{-y^2/2t}}{iy-z}\,i\,dy. \tag{2.3}$$

Putting the $e^{z^2/2t}$ onto the other side and estimating the integrals using the square-integrability of F gives

$$|F(x+iy)| \leq ce^{y^2/2t}\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{2\pi-x}}\right)$$

for $0 < x < 2\pi$. These estimates would apply to any holomorphic function on this strip that satisfied Condition 3 and was square-integrable over the two boundary lines. Given these estimates and the periodicity of F, we may run a similar argument on the strip $0 < \operatorname{Re} z < 4\pi$. The square-root singularity in our estimates for F causes no problem and we obtain a formula similar to (2.3) but with $4\pi+iy-z$ in the denominator of the first term. This gives good estimates on $F(z)$ for $\operatorname{Re} z$ near 2π, hence also near $\operatorname{Re} z = 0$. So finally

$$|F(x+iy)| \leq ce^{y^2/2t}. \tag{2.4}$$

The estimate (2.4) implies that F is square-integrable with respect to the measure ν_r for any $r < t$. Since the transform $C_r = \lim_{s\to\infty} B_{s,r}$ is surjective, there exists f_r such that F is the analytic continuation of $e^{r\Delta/2}f_r$. We now wish to allow r to increase to t as in the proof of Theorem 2.4. We may directly verify that square-integrability with respect to $\mu_{t/2,t}$ implies square-integrability with respect to $\mu_{r/2,r}$, with uniform bounds as r increases to t. The rest of the argument is as in Theorem 2.4. ∎

3 Reduction of the $\mathbb{R}^d$ Case to Standard Form

In this section we will see how to deduce the properties of $B_{s,t}$ in the $\mathbb{R}^d$ case from the properties of the "classical" Segal-Bargmann transform B_t $(= B_{t,t})$, thus showing that $B_{s,t}$ is, in the $\mathbb{R}^d$ case, just a disguised form of B_t. To understand why this is so, imagine making a change of measure on $\mathbb{R}^d$ from ρ_s to Lebesgue measure. Our transform will then be an integral operator with some kernel $k_{s,t}(x,z)$. The functions $k_{s,t}(\cdot,z)$, with z fixed, are the "coherent states". No matter the values of s and t, these coherent states will be the usual Gaussian wave packets, that is, functions of the form $a\exp[-(x-b)^2/2\sigma]\exp ic\cdot x$. Here the width σ depends on s and t but not z, and as z varies over $\mathbb{C}^d$ with s and t fixed, all possible values of (b,c) are achieved. All that changes with s and t is the distance scale σ, the normalization constants a, and the labeling of the coherent states. Of course, nothing so simple occurs on groups other than $\mathbb{R}^d$. In the $\mathbb{R}^d$ case, the C_t version of the transform can also be reduced to the B_t version by a similar method—see [H1, Eq. (A.18)].

So now let $r = 2(s - t/2)$, so that $r > 0$. Recalling the definition of $A_{s,t}$ we may compute $\mu_{s,t}$ explicitly:

$$\mu_{s,t}(x+iy) = (\pi r)^{-d/2}(\pi t)^{-d/2}e^{-x^2/r}e^{-y^2/t}.$$

Now consider the map U defined on $\mathcal{H}L^2(\mathbb{C}^d,\mu_{s,t})$ given by

$$UF(z) = e^{az^2}F(z).$$

Since $\exp az^2$ is holomorphic and nowhere zero, U is a unitary map of $\mathcal{H}L^2(\mathbb{C}^d,\mu_{s,t})$ onto the Hilbert space

$$\mathcal{H}L^2(\mathbb{C}^d, |e^{az^2}|^{-2}\mu_{s,t}).$$

Choosing $a = (r-t)/4rt$ gives

$$|e^{az^2}|^{-2}\mu_{s,t} = c_1 e^{-|z|^2/\tau}$$

where $\tau = 2rt/(r+t)$ and the value of c_1 is immaterial.

The composite transform $UB_{s,t}$ is

$$UB_{s,t}f(z) = c\int_{\mathbb{R}^d} e^{az^2}e^{-(z-x)^2/2t}f(x)\,dx.$$

By combining the exponents and completing the square we obtain

$$\begin{aligned}UB_{s,t}f(z) = c\int_{\mathbb{R}^d}&\exp\left\{-\frac{1}{2\tau}\left(z-\frac{2r}{r+t}x\right)^2\right\}\\ &\times f(x)\exp\left\{\frac{r-t}{2t(r+t)}x^2\right\}\,dx.\end{aligned}\tag{3.1}$$

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This is just the time-τ heat operator, combined with a dilation and applied to the function

$$g(x) = f(x)\exp\left\{\frac{r-t}{2t(r+t)}x^2\right\}.$$

Now let $\widetilde{g}(y) = g\big((r+t)y/2r\big)$. Making the change-of-variable $y = 2rx/(r+t)$ in (3.1) gives

$$UB_{s,t}f(z) = c_2\int_{\mathbb{R}^d}\exp\left\{-\frac{1}{2\tau}(z-y)^2\right\}\widetilde{g}(y)\,dy. \tag{3.2}$$

Furthermore, a tedious but straightforward calculation shows that the norm of f in $L^2(\mathbb{R}^d,\rho_s)$ is the same (up to a constant) as the norm of $\widetilde{g}$ in $L^2(\mathbb{R}^d,\rho_\tau)$. Note that the right side of (3.2) is just the standard Segal-Bargmann transform B_τ, isometric from $L^2(\mathbb{R}^d,\rho_\tau)$ onto $\mathcal{H}L^2(\mathbb{C}^d,\mu_\tau)$, applied to $\tilde{g}$. Undoing all of these steps shows that $B_{s,t}$ is, up to a constant, an isometric isomorphism of $L^2(\mathbb{R}^d,\rho_s)$ onto $\mathcal{H}L^2(\mathbb{C}^d,\mu_{s,t})$. By considering the case $f=1$ we see that the constant is one. ■

4 Appendix

Let X and Y be two complex manifolds with measures that are given in local coordinates by smooth strictly positive densities with respect to Lebesgue measure. I wish to show that $\mathcal{H}L^2(X)\otimes\mathcal{H}L^2(Y)$ is isomorphic to $\mathcal{H}L^2(X\times Y)$, with the isomorphism taking $F_1\otimes F_2$ to the product function $F_1(x)F_2(y)$. Now, the conditions on the measures guarantee that in all three cases, the holomorphic subspace is closed in L^2 [D, Lem. 3.2]. Since also $F_1(x)F_2(y)$ is holomorphic if F_1 and F_2 are, it follows that under the usual isomorphism of $L^2(X)\otimes L^2(Y)$ with $L^2(X\times Y)$, $\mathcal{H}L^2(X)\otimes\mathcal{H}L^2(Y)$ maps *into* $\mathcal{H}L^2(X\times Y)$.

We may show that the image is all of $\mathcal{H}L^2(X\times Y)$ as follows. We know [D] that the pointwise evaluation map $F\to F(g)$ is bounded on each of the $\mathcal{H}L^2$-spaces. Thus for each $x\in X$ there exists $k_x\in\mathcal{H}L^2(X)$ such that $\int k_x(x')F(x')\,dx' = F(x)$ for all $F\in\mathcal{H}L^2(X)$, and similarly for Y. So now suppose $F(z,g)$ is holomorphic and square-integrable on $X\times Y$, and that F is orthogonal to the image of $\mathcal{H}L^2(X)\otimes\mathcal{H}L^2(Y)$. Then for all $x\in X$, $y\in Y$

$$\begin{aligned}0 &= \int_{X\times Y}F(x',y')k_x(x')k_y(y')\,dx'\,dy'\\ &= \int_Y\left[\int_X F(x',y')k_x(x')\,dx'\right]k_y(y')\,dy'\\ &= \int_Y F(x,y')k_y(y')\,dy'.\end{aligned}$$

Here we have used that $F(x',y')$ is holomorphic and square-integrable in x' for almost every y'. But now $F(x,y')$ is holomorphic and square-integrable in y' for almost every x, so for almost every x

$$F(x,y) = 0$$

for every y. Since F is continuous and the measure on $X \times Y$ is "nice", this implies that $F \equiv 0$.

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