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Anisotropic area-preserving flows for plane curves and entropy inequalities

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Abstract This paper describes two anisotropic area-preserving flows for plane curves, both of which are considered to deform one convex curve into another. Different monotonic entropy functions are identified under these flows, which can be utilized to derive two significant entropy inequalities: the log-Minkowski inequality and the curvature entropy inequality, as well as the Brunn-Minkowski inequality.

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Key words: Anisotropic area-preserving flow, Curvature entropy inequality, Log-Minkowski inequality, Yau's problem

1 Introduction

Curve evolution problems serve as a framework for simulating a variety of phenomena [9], such as crystal growth, phase interface dynamics, and flame propagation. For an embedded curve γ, its evolution problem can be expressed by

$$\frac{\partial \gamma}{\partial t} = \beta N, \tag{1.1}$$

where N is the unit inward normal vector of γ, and β is some geometric quantity related to γ.

By [15, (1.3) and (1.4)] or [12, (1.18) and (1.19)], the evolution equations for the length and enclosed area of the evolving curve γ are given by

$$\frac{dl}{dt} = -\int_\gamma \beta\kappa ds, \tag{1.2}$$

$$\frac{dA}{dt} = -\int_\gamma \beta ds, \tag{1.3}$$

where s is the arclength parameter. The most famous model in (1.1) is the curve shortening flow for $\beta = \kappa$. According to (1.2) and (1.3), this flow acts as a gradient flow for the length of the evolving curve, and both the length and enclosed area of the curve are non-increasing. It was systematically studied by Gage and Hamilton [17] and Grayson [23], and the well-known Gage-Hamilton-Grayson theorem (cf. [2, 12]) states that any embedded plane curve evolves into a convex curve and eventually shrinks to a round point under the curve shortening flow.

Professor S.T. Yau proposed an interesting problem concerning curve evolution, specifically *whether one can find a parabolic curvature flow that evolves a curve into another one in finite time*

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or infinite time. To research this problem, Lin and Tsai [42] introduced an impressive length-preserving flow, which states that a convex curve (i.e., a closed, embedded, and smooth curve with positive curvature everywhere) can be deformed into another curve if the two curves have the same length and the curvature of the evolving curve has a uniform upper bound. Inspired by the "Curvature Difference Flow" suggested by Yau, Gao and Zhang [21] utilized an area-preserving flow, which is a revised version of the "Curvature Difference Flow", and it evolves a convex curve into another one without extra conditions. For the centro-affine case, Ivaki [34] showed a centro-affine curvature flow that deforms a convex curve into an ellipse. In higher dimensions, Stancu [56] studied a similar question of evolving a smooth closed convex hypersurface into another hypersurface with the same centro-affine curvature. Further recent developments addressing Yau's problem are available in [19] and [43].

Another notable category in curve evolution problems is anisotropic curvature flow, which concerns the planar L_p Minkowski problem; some work on this topic can be found in [18, 33, 54, 55, 67], among others. For insightful discussions on the L_p Minkowski problem [44] and its generalizations from the perspective of curvature flows, it is highly recommended to refer to [6, 8, 11, 13, 27, 39, 40], and the literature therein. The logarithmic Minkowski problem [5] ($p = 0$) is a significant one which is related to the cone-volume measure. For an n-dimensional convex body L, its cone-volume measure is defined by $dV_L = \frac{1}{n} h_L dS_L$, where h_L and dS_L are the support function and the surface area measure of L, respectively. The log-Minkowski inequality is pivotal in establishing the uniqueness of the logarithmic Minkowski problem, which remains unsolved. Böröczky et al. [4] conjectured that the log-Minkowski inequality

$$\int_{S^{n-1}} \log \frac{h_K}{h_L} d\bar{V}_L \geq \frac{1}{n} \log \frac{V(K)}{V(L)} \tag{1.4}$$

holds for two origin symmetric convex bodies K and L, where $d\bar{V}_L$ is the cone-volume probability measure of L. They also showed that this is equivalent to the log-Brunn-Minkowski inequality

$$V\big(\lambda \cdot K +_\circ (1-\lambda) \cdot L\big) \geq V(K)^\lambda V(L)^{1-\lambda} \tag{1.5}$$

holds for $\lambda \in [0, 1]$, where the log Minkowski addition $\lambda \cdot K +_o (1-\lambda) \cdot L$ is defined as

$$\lambda \cdot K +_o (1-\lambda) \cdot L = \bigcap_{u \in S^{n-1}} \left\{x \in \mathbb{R}^n \mid x \cdot u \leq h_K(u)^\lambda h_L(u)^{1-\lambda}\right\}.$$

Böröczky et al. [4] researched the planar case for (1.4) and (1.5) and showed that the equalities hold in (1.4) and (1.5) if and only if K and L are dilates or K and L are parallelograms with parallel sides. Xi and Leng [61] introduced the conception of dilation position, and extended the log-Brunn-Minkowski and log-Minkowski inequalities to the nonsymmetric case. They solved Dar's conjecture in the plane and established the relationship between the log-Brunn-Minkowksi inequality and Dar's conjecture in the plane when convex bodies are at a dilation position. Saroglou [52] established the inequality (1.5) with its equality cases for pairs of convex bodies that are both unconditional with respect to some orthonormal basis. Kolesnikov and Livshyts [35] made significant progress towards the L_p Brunn-Minkowski inequality by establishing a local version of this inequality for $p \in [1 - \frac{c}{n^{3/2}}, 1)$, where c is a constant independent of n. They also obtained the local uniqueness

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of the associated L_p Minkowski problem. Chen et al. [10] extended the local uniqueness of the L_p Minkowski problem in [35] to the global uniqueness for $p \in (p_0, 1)$ and $p_0 < 1$ by PDE methods. Recently, a more comprehensive account of various aspects of the L_p Brunn-Minkowski and L_p Minkowski inequalities can be found in [3, 28, 29, 30, 36, 45, 47, 49, 50, 57, 64, 66, 68], and the literature therein.

Before delving into the models of this paper, we first introduce the correlation between concepts in convex geometry and anisotropic curvature flows.

Let K be a planar convex body and γ be its boundary curve with the support function h. If K is smooth, then it can be characterized by the curvature and the support function, with respect to the tangent angle θ of the curve γ.

Meanwhile, the curvature κ of γ can be expressed in terms of its support function h, that is,

$$\kappa = \frac{1}{h_{\theta\theta} + h}. \tag{1.6}$$

Let L denote another planar convex body, $\tilde{\gamma}$ the boundary curve of L with support function $\tilde{h}$ and curvature $\tilde{\kappa}$. The areas of K and L, as well as the mixed area of K with respect to L, are given by (cf. [24])

$$V(K) = \frac{1}{2}\int_0^{2\pi} \frac{h}{\kappa} d\theta = \frac{1}{2}\int_0^{2\pi} \left(h^2 - h_\theta^2\right) d\theta, \tag{1.7}$$

$$V(L) = \frac{1}{2}\int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} d\theta = \frac{1}{2}\int_0^{2\pi} \left(\tilde{h}^2 - \tilde{h}_\theta^2\right) d\theta, \tag{1.8}$$

$$V(K, L) = \int_0^{2\pi} \frac{h}{\tilde{\kappa}} d\theta = \int_0^{2\pi} \frac{\tilde{h}}{\kappa} d\theta = \int_0^{2\pi} \left(h\tilde{h} - h_\theta \tilde{h}_\theta\right) d\theta. \tag{1.9}$$

When L is origin symmetric and is an isoperimetrix, we adopt the following notation appearing in Gage's anisotropic curvature flow [16, p.454], that is,

$$\mathscr{A} = 2V(K),$$
$$\alpha = 2V(L),$$
$$\mathscr{L} = V(K, L).$$

Clearly, $\mathscr{A}$ and α represent twice the areas of K and L, respectively, and $\mathscr{L}$ is called the Minkowski length of γ with respect to $\tilde{\gamma}$.

The first aim of this paper is to investigate Yau's problem through the "Curvature Ratio Flow" inspired by Stancu's centro-affine flow in [56]. Let γ and $\tilde{\gamma}$ be two origin symmetric planar curves, with $\tilde{\gamma}$ being fixed. We consider the following two evolution problems:

$$\frac{\partial \gamma}{\partial t} = \left(\frac{\kappa}{\tilde{\kappa}} - \Lambda_1(t)\right) hN, \tag{1.10a}$$

$$\frac{\partial \gamma}{\partial t} = \left(\frac{\kappa}{\tilde{\kappa}} - \Lambda_2(t)\right) \tilde{h}N, \tag{1.10b}$$

for $\Lambda_1(t) = \frac{\mathscr{L}}{\mathscr{A}}$ and $\Lambda_2(t) = \frac{\alpha}{\mathscr{L}}$, where N is the unit inward normal vector of γ, and h and $\tilde{h}$ are the support functions of γ and $\tilde{\gamma}$, respectively. From (1.3), we have known that both of these flows are

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area-preserving. The area-preserving flow was first introduced by Gage [15] to better understand the curve shrinking flow, and other recent impressive work on area-preserving flows can be found in [14, 20, 21, 22, 60], among others.

The main theorem regarding the evolution problem is as follows.

Theorem 1.1. *Let γ_0 be an origin symmetric convex curve, and $\tilde{\gamma}$ be a fixed origin symmetric convex curve. Then the flow (1.10a) or (1.10b) exists in $[0, \infty)$. Under this flow, the evolving curve remains symmetric and convex, and keeps its enclosed area. Finally, it converges smoothly to a fixed curve γ_∞ (congruent to $\sqrt{\frac{\mathscr{A}}{\alpha}}\tilde{\gamma}$) in the C^∞ sense as time goes to infinity.*

The second aim of this paper is to investigate some entropy inequalities using area-preserving flows (1.10a) and (1.10b). Entropy plays an important role in both physics and information theory. In physics, entropy is a measure of the degree of disorder in a system, while in information theory, it quantifies the uncertainty of information or the uncertainty of a random variable. Different types of entropy, such as Shannon entropy and relative entropy, establish a direct connection to optimal transportation, Monge-Ampère equations, and geometric inequalities in convex geometry [59].

As applications of the flows (1.10a) and (1.10b), we will show the log-Minkowski inequality and the curvature entropy inequality, which are proven respectively in references [5] and [45], under the appropriate entropy associated with the flows. The Brunn-Minkowski inequality can also be obtained from the flows (1.10a) and (1.10b). The "Gauss Curvature Ration" flow, similar to (1.10a), may be a beneficial attempt to address the unsolved log-Minkowski inequality in higher dimensions, which is also a problem that we are considering.

Theorem 1.2 (Log-Minkowski inequality)**.** *Let K and L be two origin symmetric, smooth planar convex bodies. Then*

$$\int_{S^1} \log \frac{h_K}{h_L} dV_L \geq \frac{V(L)}{2} \log \frac{V(K)}{V(L)}, \tag{1.11}$$

where the equality holds in (1.11) if and only if K and L are homothetic.

Remark 1.3. *From the equivalence of the log-Minkowski inequality (1.4) and the log-Brunn-Minkowski inequality (1.5), we can derive the log-Brunn-Minkowski inequality in the plane. Equality also holds in the inequalities (1.4) and (1.5) when K and L are parallelograms with parallel sides in the discrete case.*

Theorem 1.4 (Curvature entropy inequality)**.** *Let K and L be two origin symmetric, smooth planar convex bodies. Then*

$$\int_{S^1} \log \frac{\kappa_K}{\kappa_L} dV_L \geq \frac{V(L)}{2} \log \frac{V(L)}{V(K)}, \tag{1.12}$$

where the equality holds in (1.12) if and only if K and L are homothetic.

Remark 1.5. *The curvature entropy inequality (1.12) still holds without any symmetry condition but with a dilation position condition for K and L (see [65]). This case can also be proved by flow (1.10b) when $\tilde{\gamma}$ is a convex curve with positive support function, the necessary illustration can be found in Remark 2.14 and Remark 3.3, in which the key step is inspired by impressive work in Li-Wang [41].*

By the log-Minkowski inequality (1.11) or the curvature entropy inequality (1.12), we can discuss the uniqueness for the cone-volume measure in the smooth setting for origin symmetric planar convex bodies, which is viewed as the uniqueness for the logarithmic Minkowski problem in the smooth setting for origin symmetric planar convex bodies. Specifically, when K and L are origin symmetric, smooth planar convex bodies with same cone-volume measures, then from (1.11), we have

$$\int_{S^1} \log h_K dV_K = \int_{S^1} \log h_K dV_L \geq \int_{S^1} \log h_L dV_L = \int_{S^1} \log h_L dV_K \geq \int_{S^1} \log h_K dV_K,$$

and by (1.12),

$$\int_{S^1} \log \kappa_K dV_K = \int_{S^1} \log \kappa_K dV_L \geq \int_{S^1} \log \kappa_L dV_L = \int_{S^1} \log \kappa_L dV_K \geq \int_{S^1} \log \kappa_K dV_K.$$

From the equality case in (1.11) or (1.12), it yields $K = L$.

Since the speed of flow (1.10a) contains the support function of the evolving curve, it is a locally constrained flow, which has proven to be a powerful tool for exploring geometric inequalities in recent years, especially for Alexandrov-Fenchel inequalities. Notably, recent research has increasingly focused on locally constrained inverse curvature flows, as evidenced by works such as [7, 25, 26, 31, 32, 38, 46, 51, 53, 62, 63], among others.

The paper is structured as follows. In Section 2, we investigate two anisotropic area-preserving flows (1.10a) and (1.10b) for convex curves and establish short time existence, long-term existence, and convergence. In Section 3, by examining the monotonicity of exact monotonic geometric quantities under flows (1.10a) and (1.10b), we derive some entropy inequalities and the Brunn-Minkowski inequality applicable to convex curves.

2 Anisotropic area-preserving flows

In this section, we first show the short time existence for flows (1.10a) and (1.10b) by linearization of the evolution equations for the support function of the evolving curve. In order to demonstrate the long-term existence and convergence for flows (1.10a) and (1.10b), we prove that the support function, curvature, and the higher derivatives of both the support function and curvature are uniformly bounded. Finally, we obtain the final shape for flows (1.10a) and (1.10b) through two monotonic geometric quantities.

2.1 Short time existence

To study the flows (1.10a) and (1.10b), we provide some basic evolution equations based on some basic computations [12] or [17].

Lemma 2.1. *For flow (1.10a), the evolution equations for the support function h and the curvature κ are*

$$h_t = -\left(\frac{\kappa}{\tilde{\kappa}} - \Lambda_1(t)\right) h, \tag{2.1a}$$

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$$\kappa_t = \kappa^2 \left(\left(\frac{\kappa}{\tilde{\kappa}} h \right)_{\theta\theta} + \left(\frac{\kappa}{\tilde{\kappa}} h \right) - \frac{\Lambda_1(t)}{\kappa} \right), \tag{2.1b}$$

and that for flow (1.10b) *are*

$$h_t = -\left(\frac{\kappa}{\tilde{\kappa}} - \Lambda_2(t) \right) \tilde{h}, \tag{2.2a}$$

$$\kappa_t = \kappa^2 \left(\left(\frac{\kappa}{\tilde{\kappa}} \tilde{h} \right)_{\theta\theta} + \left(\frac{\kappa}{\tilde{\kappa}} \tilde{h} \right) - \frac{\Lambda_2(t)}{\tilde{\kappa}} \right). \tag{2.2b}$$

In order to show the short time existences for the flows (1.10a) and (1.10b), we need to consider the linearization for equations (2.1a) and (2.2a). Actually, we first claim that the support function h is periodic with period 2π. Since the initial curve γ_0 and $\tilde{\gamma}$ are convex, they satisfy the closing condition

$$\int_0^{2\pi} \frac{e^{i\theta}}{\kappa_0} d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \frac{e^{i\theta}}{\tilde{\kappa}} d\theta = 0.$$

Together with the evolution equations of the curvature (2.1b) and (2.2b), one can deduce that

$$\int_0^{2\pi} \frac{e^{i\theta}}{\kappa} d\theta = 0,$$

which implies that the evolving curve is periodic with period 2π. Then, by (1.6), we can get that the support function h is also periodic with period 2π.

Next, we deal with the linearization for equations (2.1a) and (2.2a). For (2.1a), by (1.6), (1.7) and (1.9), the functional

$$F_1(h) = h_t + \frac{h}{(h_{\theta\theta} + h)\tilde{\kappa}} - \frac{\int_0^{2\pi} \frac{h}{\tilde{\kappa}} d\theta}{\int_0^{2\pi} (h^2 - h_\theta^2) d\theta} h \tag{2.3}$$

is considered. In fact, F_1 is an operator from $C^{2,\beta}(S^1 \times [0, \omega))$ to $C^\alpha(S^1 \times [0, \omega))$ for $0 < \alpha < \beta \leq 1$. This deduces that the Fréchet derivative of F_1 at some point $h_0 > 0$ is

$$\begin{aligned} DF_1(h_0)[h] =& h_t - \frac{h_0}{((h_0)_{\theta\theta} + h_0)^2 \tilde{\kappa}} h_{\theta\theta} + \left(\frac{(h_0)_{\theta\theta}}{((h_0)_{\theta\theta} + h_0)^2 \tilde{\kappa}} - \frac{\mathscr{L}(0)}{\mathscr{A}(0)} \right) h \\ &+ \frac{h_0}{\mathscr{A}(0)^2} \left(2\mathscr{L}(0) \int_0^{2\pi} h_0 h d\theta - 2\mathscr{L}(0) \int_0^{2\pi} (h_0)_\theta h_\theta d\theta - \mathscr{A}(0) \int_0^{2\pi} \frac{h}{\tilde{\kappa}} d\theta \right) \end{aligned}$$

Thus, the linearization of the evolution equation of the support function h for (2.1a) at h_0 is

$$\begin{aligned} h_t =& \frac{h_0}{((h_0)_{\theta\theta} + h_0)^2 \tilde{\kappa}} h_{\theta\theta} - \left(\frac{(h_0)_{\theta\theta}}{((h_0)_{\theta\theta} + h_0)^2 \tilde{\kappa}} - \frac{\mathscr{L}(0)}{\mathscr{A}(0)} \right) h \\ &- \frac{h_0}{\mathscr{A}(0)^2} \left(2\mathscr{L}(0) \int_0^{2\pi} h_0 h d\theta - 2\mathscr{L}(0) \int_0^{2\pi} (h_0)_\theta h_\theta d\theta - \mathscr{A}(0) \int_0^{2\pi} \frac{h}{\tilde{\kappa}} d\theta \right) \end{aligned} \tag{2.4}$$

Similarly, for (2.2a), we consider the functional

$$F_2(h) = h_t + \frac{\tilde{h}}{(h_{\theta\theta} + h)\tilde{\kappa}} - \frac{\int_0^{2\pi} (\tilde{h}^2 - \tilde{h}_\theta^2) d\theta}{\int_0^{2\pi} \frac{h}{\tilde{\kappa}} d\theta} \tilde{h}.$$

The linearization of the evolution equation of the support function h for (2.2a) at h_0 is

$$h_t = \frac{\tilde{h}}{((h_0)_{\theta\theta} + h_0)^2 \tilde{\kappa}} h_{\theta\theta} + \frac{\tilde{h}}{((h_0)_{\theta\theta} + h_0)^2 \tilde{\kappa}} h - \frac{\tilde{h}\alpha}{\mathscr{L}(0)^2} \int_0^{2\pi} \frac{h}{\tilde{\kappa}} d\theta \tag{2.5}$$

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The equations (2.4) and (2.5) are uniformly parabolic with smooth coefficients. It follows from the implicit function theorem of Banach spaces that equations (2.1a) and (2.2a) both have a unique smooth solution over some small time interval (cf. Chou-Zhu [12, Sec.1.2]). This implies the short time existence of the flows (1.10a) and (1.10b).

Theorem 2.2. *The flows (1.10a) and (1.10b) have a unique smooth solution on $S^1 \times [0, T)$ for some $T > 0$.*

2.2 Long-term existence

Proposition 2.3. *The curvature $\kappa(\cdot, t)$ of the evolving curve has a uniform upper bound as long as the flow (1.10a) or (1.10b) exists.*

To do this proposition, some necessary lemmas are given.

Lemma 2.4. *If an initial curve γ_0 evolves under flow (1.10a) or (1.10b), its Minkowski length $\mathscr{L}(t)$ is decreasing.*

Proof. For flow (1.10a), from (1.7), (1.9) and (2.1a), it has

$$\begin{aligned} \frac{d\mathscr{L}}{dt} &= \int_0^{2\pi} \frac{h_t}{\tilde{\kappa}} d\theta \\ &= -\int_0^{2\pi} \left(\frac{\kappa}{\tilde{\kappa}} - \Lambda_1(t) \right) \frac{h}{\tilde{\kappa}} d\theta \\ &= -\int_{\tilde{\gamma}} \frac{\kappa h}{\tilde{\kappa}} d\tilde{s} + \frac{\left(\int_{\tilde{\gamma}} h d\tilde{s} \right)^2}{\int_{\tilde{\gamma}} \frac{h\tilde{\kappa}}{\kappa} d\tilde{s}}, \end{aligned}$$

where $d\tilde{s} = \frac{1}{\tilde{\kappa}} d\theta$. Due to the Cauchy-Schwarz inequality, $\frac{d\mathscr{L}}{dt} \leq 0$ and its equality holds if and only if the quantity $\frac{\kappa}{\tilde{\kappa}}$ is a positive constant.

Similarly, for flow (1.10b), we have

$$\begin{aligned} \frac{d\mathscr{L}}{dt} &= \int_0^{2\pi} \frac{h_t}{\tilde{\kappa}} d\theta \\ &= -\int_0^{2\pi} \left(\frac{\kappa}{\tilde{\kappa}} - \Lambda_2(t) \right) \frac{\tilde{h}}{\tilde{\kappa}} d\theta \\ &= -\int_0^{2\pi} \frac{\kappa \tilde{h}}{\tilde{\kappa}^2} d\theta + \frac{\alpha^2}{\mathscr{L}} \\ &= -\int_0^{2\pi} \frac{\kappa \tilde{h}}{\tilde{\kappa}^2} d\theta + \frac{\left(\int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} d\theta \right)^2}{\int_0^{2\pi} \frac{\tilde{h}}{\kappa} d\theta}. \end{aligned}$$

From the Cauchy-Schwarz inequality, $\frac{d\mathscr{L}}{dt} \leq 0$ with equality holds if and only if the quantity $\frac{\kappa}{\tilde{\kappa}}$ is a positive constant. □

Lemma 2.5. *Suppose that γ_0 is an origin symmetric convex curve. Then the evolving curve $\gamma(\cdot, t)$ keeps symmetry as long as the flow (1.10a) or (1.10b) exists.*

Proof. Because γ_0 is an origin symmetric convex curve, $\gamma(\cdot+\pi,0)=\gamma(\cdot,0)$. As we know, $\gamma(\cdot,t)$ satisfies (1.10a) (or (1.10b)) with initial $\gamma(\cdot,0)$, and $\gamma(\cdot+\pi,t)$ satisfies (1.10a) (or (1.10b)) with initial $\gamma(\cdot+\pi,0)$. Due to Theorem 2.2, $\gamma(\cdot+\pi,t)=\gamma(\cdot,t)$ as long as the flow (1.10a) (or (1.10b)) exists. □

Theorem 2.6. *Let γ_0 be an origin symmetric convex curve. If γ_0 evolves according to the flow* (1.10a) *or* (1.10b)*, then it is always convex.*

Proof. Suppose that the flow exists in the time interval $[0,\omega)$, and $t_0=\inf\{t \mid \kappa(\theta,t)=0, \theta\in S^1\}<\omega$. Recall that flow (1.10a) and (1.10b) are area-preserving. Then, by Lemma 2.5, the support function of the evolving curve $h(\theta,t)>0$ for $(\theta,t)\in S^1\times[0,t_0]$. Otherwise, if the evolving curve had a point where its support function vanishes, the symmetry would imply the existence of nearby points with both positive and negative curvature.

For flow (1.10a), set $f_1=\frac{\kappa}{\tilde{\kappa}}he^{\mu_1 t}$, where μ_1 is large enough. From (2.1b), it yields that

$$(f_1)_t=\left(\frac{\kappa^2}{\tilde{\kappa}}h\right)(f_1)_{\theta\theta}+Q_1(\kappa)f_1. \tag{2.6}$$

where $Q_1(\kappa)=\frac{\kappa^2}{\tilde{\kappa}}h-\frac{\kappa}{\tilde{\kappa}}+\mu_1$. The quadratic polynomial $Q_1(\kappa)$ is positive when μ_1 is large enough. Let $(f_1)_{\min}(t)=\min\{f(\theta,t)\mid\theta\in S^1\}$. We claim that $(f_1)_{\min}(t)$ is non-decreasing. If not, there exists an η, where $0<\eta<(f_1)_{\min}(0)$, such that at some time $t>0$, $(f_1)_{\min}(t)=\eta$. Set $t_1^*=\inf\{t\mid(f_1)_{\min}(t)=\eta\}$. The continuity of f_1 ensures that this minimum η is achieved for the first time at point (θ_1^*,t_1^*). Then, at this point, we have

$$(f_1)_t\le 0,\quad (f_1)_{\theta\theta}\ge 0,\quad f_1=\eta>0.$$

This contradicts the fact that f_1 satisfies (2.6), and thus $(f_1)_{\min}(t)\ge(f_1)_{\min}(0)$, which implies that $\kappa\ge\frac{\tilde{\kappa}}{h}(f_1)_{\min}(0)e^{-\mu_1 t}>0$ for all $t\in[0,t_0]$. It contradicts the hypothesis of t_0.

For flow (1.10b), set $f_2=\frac{\kappa}{\tilde{\kappa}}\tilde{h}e^{\mu_2 t}$, where μ_2 is large enough. By (2.2b), we have

$$(f_2)_t=\left(\frac{\kappa^2}{\tilde{\kappa}}\tilde{h}\right)(f_2)_{\theta\theta}+Q_2(\kappa)f_2,$$

where $Q_2(\kappa)=\frac{\kappa^2}{\tilde{\kappa}}\tilde{h}-\frac{\kappa}{\tilde{\kappa}}\Lambda_2+\mu_2$. It follows from the Minkowski inequality $\mathscr{L}^2\ge\mathscr{A}\alpha$ (see [16]) and Lemma 2.4 that $\frac{\alpha}{\mathscr{L}_0}\le\Lambda_2\le\frac{\mathscr{L}_0}{\mathscr{A}}$. Hence, the quadratic polynomial $Q_2(\kappa)$ is positive when μ_2 is large enough. For f_2, by a similar discussion as for f_1, we can achieve $\kappa\ge\frac{\tilde{\kappa}}{\tilde{h}}(f_2)_{\min}(0)e^{-\mu_2 t}>0$ for all $t\in[0,t_0]$. This contradicts the hypothesis of t_0.

Hence, the evolving curve under flow (1.10a) or (1.10b) is always convex. □

To get the upper and lower bounds for support function h, we set

$$h_{\max}(t)=\max_{\theta\in S^1}h(\theta,t)\quad\text{and}\quad h_{\min}(t)=\min_{\theta\in S^1}h(\theta,t).$$

Lemma 2.7. *If $\gamma(\cdot,t)$ is an origin symmetric convex solution to flow* (1.10a) *or* (1.10b)*, then its support function h has uniform upper and lower bounds.*

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Proof. Let $\rho(\cdot,t)$ be the radial function of $\gamma(\cdot,t)$ and $\rho_{\max}(t)=\max_{\eta\in S^1}\rho(\eta,t)$. The radial function $\rho(\cdot,t)$ and the support function $h(\cdot,t)$ have the relation

$$\rho(\cdot,t)=\sqrt{h(\cdot,t)^2+h_\theta(\cdot,t)^2}. \tag{2.7}$$

If $h(\theta,t)$ reaches its maximum with respect to θ at the point (θ_1,t), then $h_\theta(\theta_1,t)=0$. It follows from (2.7) that

$$h_{\max}(t)\le\rho_{\max}(t).$$

Similarly, $h_{\min}(t)\ge\rho_{\min}(t)$, where $\rho_{\min}(t)=\min_{\eta\in S^1}\rho(\eta,t)$. If $\eta_{\max}\in S^1$ is a value satisfying $\rho(\eta_{\max},t)=\rho_{\max}(t)$, then

$$h(\theta,t)\ge\rho_{\max}(t)|\cos(\theta-\eta_{\max})|,$$

derived from the definition of support function and Lemma 2.5. This together with Lemma 2.4 shows that

$$\mathscr{L}(0)\ge\mathscr{L}(t)=\int_0^{2\pi}\frac{h(\theta,t)}{\tilde\kappa(\theta)}d\theta\ge\rho_{\max}(t)\int_0^{2\pi}\frac{|\cos(\theta-\eta_{\max})|}{\tilde\kappa(\theta)}d\theta,$$

which deduces that

$$\rho_{\max}(t)\le C_1=\frac{\mathscr{L}(0)}{\int_0^{2\pi}\frac{|\cos(\theta-\eta_{\max})|}{\tilde\kappa(\theta)}d\theta}, \tag{2.8}$$

here C_1 is a constant independent of t. This yields that

$$h(\theta,t)\le h_{\max}(t)\le\rho_{\max}(t)\le C_1.$$

By the facts that $\gamma(\cdot,t)$ is origin symmetric and $\frac{dA}{dt}=0$, we have

$$\mathscr{A}(0)=\mathscr{A}(t)\le4\pi\rho_{\min}(t)\rho_{\max}(t)\le4\pi\rho_{\min}(t)C_1.$$

This tells us

$$\rho_{\min}(t)\ge C_2=\frac{\mathscr{A}(0)}{4\pi C_1}. \tag{2.9}$$

Hence, $h(\theta,t)\ge h_{\min}(t)\ge\rho_{\min}(t)\ge C_2$. □

Lemma 2.8. *Under flow (1.10a) or (1.10b), the derivation of the support function h for the evolving curve has uniform upper and lower bounds.*

Proof. From (2.8) and (2.9), we have known that $\rho(\cdot,t)$ has uniform upper and lower bounds. This together with (2.7) yields uniform bounds for $h_\theta(\cdot,t)$. □

In order to obtain the upper bound of the curvature $\kappa(\cdot,t)$, we need to follow Chou's technique introduced firstly in [58] by discussing the auxiliary quantities. For flow (1.10a), consider

$$\Phi_1(\theta,t)=\frac{\kappa(\theta,t)h(\theta,t)}{(h(\theta,t)-m_0)\tilde\kappa(\theta)}$$

and for flow (1.10b), consider

$$\Phi_2(\theta,t)=\frac{\kappa(\theta,t)\tilde h(\theta)}{(h(\theta,t)-m_0)\tilde\kappa(\theta)},$$

where $m_0=\frac{C_2}{2}$.

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Proof of Proposition 2.3. **Case 1.** For flow (1.10a)

Directly dealing with Φ_1 leads to complicated calculations since Φ_1 contains $\kappa(\cdot,t)$ and $h(\cdot,\theta)$. To simplify the calculations and make these computation generality, by (2.7), we have the following equivalent form

$$\Phi_1(\theta,t)=\frac{\kappa(\theta,t)h(\theta,t)}{(h(\theta,t)-m_0)\tilde{\kappa}(\theta)}=\frac{h(\theta,t)\Lambda_1(t)-h_t}{h(\theta,t)-m_0},\tag{2.10}$$

where $m_0=\frac{C_2}{2}$. Let $(\Phi_1)_{\max}(t)=\max_{\theta\in S^1}\Phi_1(\theta,t)=\Phi_1(\theta_2,t)$. By Lemma 2.7, the upper bound for κ is clear when Φ_1 achieves its maximum at $t=0$. If $\Phi_1(\theta,t)$ attains its maximum at point (θ_2,t), where $t>0$, then we have

$$(\Phi_1)_\theta=0\quad\text{and}\quad(\Phi_1)_{\theta\theta}\le 0.\tag{2.11}$$

By (2.10), the first and the second derivatives for Φ_1 with respect to θ are

$$(\Phi_1)_\theta=\frac{h_\theta\Lambda_1-h_{t\theta}}{h-m_0}-\frac{(h\Lambda_1-h_t)h_\theta}{(h-m_0)^2}$$

and

$$\begin{aligned}(\Phi_1)_{\theta\theta}&=\frac{h_{\theta\theta}\Lambda_1-h_{t\theta\theta}}{h-m_0}-\frac{2(h_\theta\Lambda_1-h_{t\theta})h_\theta+(h\Lambda_1-h_t)h_{\theta\theta}}{(h-m_0)^2}+\frac{2(h\Lambda_1-h_t)h_\theta^2}{(h-m_0)^3}\\&=\frac{h_{\theta\theta}\Lambda_1-h_{t\theta\theta}}{h-m_0}-\frac{2h_\theta}{h-m_0}(\Phi_1)_\theta-\frac{(h\Lambda_1-h_t)h_{\theta\theta}}{(h-m_0)^2}.\end{aligned}$$

Combining with (2.11), we can get

$$h_{\theta\theta}\Lambda_1-h_{t\theta\theta}\le\frac{h\Lambda_1-h_t}{h-m_0}h_{\theta\theta}=\Phi_1h_{\theta\theta}.\tag{2.12}$$

By equality (1.6), it yields

$$\kappa_t=-\kappa^2(h_t+h_{t\theta\theta}).$$

This together with (2.1a) deduces that

$$\begin{aligned}h_{tt}&=\left(h\Lambda_1-\frac{\kappa h}{\tilde{\kappa}}\right)_t\\&=h_t\Lambda_1+h\frac{d\Lambda_1}{dt}-\frac{1}{\tilde{\kappa}}(\kappa h)_t\\&=h_t\Lambda_1+h\frac{d\Lambda_1}{dt}-\frac{1}{\tilde{\kappa}}(\kappa h_t+\kappa_t h)\\&=h_t\Lambda_1+h\frac{d\Lambda_1}{dt}-\frac{1}{\tilde{\kappa}}(\kappa h_t-\kappa^2(h_t+h_{t\theta\theta})h).\end{aligned}\tag{2.13}$$

Then, at point (θ_2,t), from (2.12) and (2.13), we have

$$\begin{aligned}(\Phi_1)_t=&\frac{h_t\Lambda_1+h\frac{d\Lambda_1}{dt}-h_{tt}}{h-m_0}-\frac{(h\Lambda_1-h_t)h_t}{(h-m_0)^2}\\=&\frac{\kappa h_t}{(h-m_0)\tilde{\kappa}}-\frac{\kappa^2hh_t}{(h-m_0)\tilde{\kappa}}-\frac{\kappa^2hh_{t\theta\theta}}{(h-m_0)\tilde{\kappa}}-\frac{h_t}{h-m_0}\Phi_1\\\le&\frac{h_t}{h}\Phi_1-h_t\kappa\Phi_1+h_{\theta\theta}\kappa(\Phi_1-\Lambda_1)\Phi_1-\frac{h_t}{h-m_0}\Phi_1\end{aligned}$$

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$$
\begin{aligned}
&= \frac{h_t}{h}\Phi_1 - h_t\kappa\Phi_1 + \left(\frac{1}{\kappa} - h\right)\kappa(\Phi_1 - \Lambda_1)\Phi_1 - \frac{h_t}{h-m_0}\Phi_1 \\
&= -\Lambda_1\left(\frac{h}{h-m_0}\right)\Phi_1 + \left(1 + \frac{m_0}{h}\right)\Phi_1^2 - m_0\kappa\Phi_1^2.
\end{aligned}
$$

By (2.10), $\frac{\Phi_1}{C} \leq \kappa \leq C\Phi_1$ holds for a positive constant C. Recall that $\Lambda_1(t) = \frac{\mathscr{L}(t)}{\mathscr{A}(t)} \leq \frac{\mathscr{L}(0)}{\mathscr{A}(0)}$. Without loss of generality, assume that $\Phi_1 \gg 1$, we finally get the following estimate

$$
\begin{aligned}
(\Phi_1)_t &\leq \frac{\mathscr{L}(0)}{\mathscr{A}(0)}\left(\frac{h}{h-m_0}\right)\Phi_1 + \left(1 + \frac{m_0}{h}\right)\Phi_1^2 - m_0\kappa\Phi_1^2 \\
&\leq \frac{\mathscr{L}(0)}{\mathscr{A}(0)}\left(\frac{h}{h-m_0}\right)\Phi_1^2 + \left(1 + \frac{m_0}{h}\right)\Phi_1^2 - \frac{m_0}{C}\Phi_1^3 \\
&\leq C_3\Phi_1^2 - \frac{m_0}{C}\Phi_1^3.
\end{aligned}
$$

This deduces that there exists a positive constant C_4 independent of t such that $\Phi_1(\theta_2, t) \leq C_4$. From Lemma 2.7, we have

$$
\kappa(\theta,t) = \frac{\Phi_1(\theta,t)(h(\theta,t)-m_0)\tilde{\kappa}(\theta)}{h(\theta,t)} \leq \frac{\Phi_1(\theta_2,t)(h(\theta,t)-m_0)\tilde{\kappa}(\theta)}{h(\theta,t)} \leq C_4\tilde{\kappa}_{\max} := C_5,
$$

for any point (θ, t).

Case 2. For flow (1.10b)

Similarly, due to Lemma 2.7, the upper bound for κ is achieved when Φ_2 attains its maximum at $t = 0$. If $\Phi_2(\theta, t)$ attains its maximum at point (θ_3, t), where $t > 0$, then it has

$$
(\Phi_2)_\theta = 0 \quad \text{and} \quad (\Phi_2)_{\theta\theta} \leq 0. \tag{2.14}
$$

Set $v = \frac{\kappa}{\tilde{\kappa}}\tilde{h}$. Due to (2.2b), the evolution equation for v is

$$
v_t = \frac{\tilde{k}}{\tilde{h}}v^2 v_{\theta\theta} + v^2\left(\frac{\tilde{k}}{\tilde{h}}v - \frac{\Lambda_2}{\tilde{h}}\right). \tag{2.15}
$$

Then, the first and the second derivatives for Φ_2 with respect to θ are

$$
(\Phi_2)_\theta = \frac{v_\theta}{h-m_0} - \frac{vh_\theta}{(h-m_0)^2}
$$

and

$$
\begin{aligned}
(\Phi_2)_{\theta\theta} &= \frac{v_{\theta\theta}}{h-m_0} - \frac{2h_\theta v_\theta}{(h-m_0)^2} + \frac{2h_\theta^2 v}{(h-m_0)^3} - \frac{vh_{\theta\theta}}{(h-m_0)^2} \\
&= \frac{v_{\theta\theta}}{h-m_0} - \frac{2h_\theta}{h-m_0}(\Phi_2)_\theta - \frac{vh_{\theta\theta}}{(h-m_0)^2}.
\end{aligned}
$$

Combining with (2.14), it deduces that

$$
v_{\theta\theta} \leq \frac{h_{\theta\theta}}{h-m_0}v. \tag{2.16}
$$

Thus, at point (θ_3, t), we can get

$$
0 \leq (\Phi_2)_t = \frac{v_t}{h-m_0} - \frac{vh_t}{(h-m_0)^2}
$$

$$
\begin{aligned}
\leq & \frac{\tilde{\kappa} h_{\theta\theta}}{(h-m_0)^2 \tilde{h}} v^3 + \frac{\tilde{\kappa}}{(h-m_0)\tilde{h}} v^3 - \frac{\Lambda_2}{(h-m_0)\tilde{h}} v^2 - \frac{(\Lambda_2 \tilde{h} - v) v}{(h-m_0)^2} \\
\leq & - \frac{\Lambda_2 \tilde{h}}{h-m_0} \Phi_2 + 2\Phi_2^2 - \frac{m_0 \tilde{k}(h-m_0)}{\tilde{h}} \Phi_2^3,
\end{aligned}
$$

derived from (2.15), (2.16) and (1.6). If $\Phi_2 \gg 1$, then

$$
0 \leq (\Phi_2)_t \leq \left(2 + \frac{2\Lambda_2 \tilde{h}_{\max}}{C_2} - \frac{m_0 \tilde{\kappa}(h-m_0)}{\tilde{h}} \Phi_2 \right) \Phi_2^2.
$$

This implies that

$$
\Phi_2 \leq \left(1 + \frac{\Lambda_2 \tilde{h}_{\max}}{C_2} \right) \frac{\tilde{h}_{\max}}{m_0 \tilde{\kappa}_{\min} C_2} := C_6.
$$

Hence, by Lemma 2.7, we have

$$
\begin{aligned}
\kappa(\theta, t) = \frac{\Phi_2(\theta, t)(h(\theta, t) - m_0)\tilde{\kappa}(\theta)}{\tilde{h}(\theta)} & \leq \frac{\Phi_2(\theta_2, t)(h(\theta, t) - m_0)\tilde{\kappa}(\theta)}{\tilde{h}(\theta)} \\
& \leq \frac{C_6(C_1 - m_0)\tilde{\kappa}_{\max}}{\tilde{h}_{\min}} := C_7,
\end{aligned}
$$

which completes the desired result. □

Proposition 2.9. *The curvature $\kappa(\cdot, t)$ of the evolving curve has a uniformly lower bound as long as the flow (1.10a) or (1.10b) exists.*

Proof. **Case 1.** For flow (1.10a)

Consider the auxiliary function

$$
\Psi_1 = \log \frac{1}{\kappa} - M_1 \log h + \frac{1}{2}\left(h^2 + h_\theta^2\right),
$$

where $M_1 = \max\limits_{S^1 \times (0,T)} (h^2 + h_\theta^2) + 1$. Assume that Ψ_1 achieves its maximum at (θ_0, t_0), then, at this point,

$$
\begin{aligned}
0 = (\Psi_1)_\theta &= \kappa \left(\frac{1}{\kappa}\right)_\theta - M_1 \frac{h_\theta}{h} + (h h_\theta + h_\theta h_{\theta\theta}) \\
&= -\frac{\kappa_\theta}{\kappa} - M_1 \frac{h_\theta}{h} + \frac{h_\theta}{\kappa},
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
0 \geq (\Psi_1)_{\theta\theta} &= -\left(\frac{\kappa_\theta}{\kappa}\right)_\theta - M_1 \frac{h_{\theta\theta} h - h_\theta^2}{h^2} + \frac{h_{\theta\theta}\kappa - h_\theta \kappa_\theta}{\kappa^2} \\
&= -\frac{\kappa_{\theta\theta}\kappa - \kappa_\theta^2}{\kappa^2} - M_1 \frac{h_{\theta\theta} h - h_\theta^2}{h^2} + \frac{h_{\theta\theta}\kappa - h_\theta \kappa_\theta}{\kappa^2},
\end{aligned} \tag{2.18}
$$

and

$$
\begin{aligned}
0 \leq (\Psi_1)_t &= \kappa \left(\frac{1}{\kappa}\right)_t - M_1 \frac{h_t}{h} + (h h_t + h_\theta h_{t\theta}) \\
&= \kappa(h_t + h_{t\theta\theta}) - M_1 \frac{h_t}{h} + (h h_t + h_\theta h_{t\theta}).
\end{aligned}
$$

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Set $f=\log\left(\frac{h}{\tilde\kappa}\right)$. From (2.1a), it follows that

$$\log(h\Lambda_1-h_t)=f+\log\kappa.$$

Differentiating the above equation gives

$$\frac{h_\theta\Lambda_1-h_{t\theta}}{h\Lambda_1-h_t}=f_\theta+\frac{\kappa_\theta}{\kappa}$$

and

$$\frac{h_{\theta\theta}\Lambda_1-h_{t\theta\theta}}{h\Lambda_1-h_t}-\frac{(h_\theta\Lambda_1-h_{t\theta})^2}{(h\Lambda_1-h_t)^2}=f_{\theta\theta}+\frac{\kappa_{\theta\theta}\kappa-\kappa_\theta^2}{\kappa^2}.$$

Then, by (2.17) and (2.18),

$$\begin{aligned}
\frac{(\Psi_1)_t}{h\Lambda_1-h_t}=&\kappa\frac{h_t+h_{t\theta\theta}}{h\Lambda_1-h_t}-\frac{M_1}{h}\frac{h_t}{h\Lambda_1-h_t}+\frac{hh_t}{h\Lambda_1-h_t}+\frac{h_\theta h_{t\theta}}{h\Lambda_1-h_t}\\
=&\kappa\left(-\frac{h_{\theta\theta}\Lambda_1-h_{t\theta\theta}}{h\Lambda_1-h_t}+\frac{h_{\theta\theta}\Lambda_1+h\Lambda_1-h\Lambda_1+h_t}{h\Lambda_1-h_t}\right)-\frac{M_1}{h}\frac{h\Lambda_1-h\Lambda_1+h_t}{h\Lambda_1-h_t}\\
&+\frac{h(h\Lambda_1-h\Lambda_1+h_t)}{h\Lambda_1-h_t}+\frac{h_\theta h_{t\theta}}{h\Lambda_1-h_t}\\
=&\kappa\left(-\frac{(h_\theta\Lambda_1-h_{t\theta})^2}{(h\Lambda_1-h_t)^2}-\frac{\kappa_{\theta\theta}\kappa-\kappa_\theta^2}{\kappa^2}-f_{\theta\theta}\right)+\frac{(1+h^2-M_1)\Lambda_1}{h\Lambda_1-h_t}-\kappa+\frac{M_1}{h}-h+\frac{h_\theta h_{t\theta}}{h\Lambda_1-h_t}\\
\leq&\kappa\left(M_1\frac{h_{\theta\theta}h-h_\theta^2}{h^2}-\frac{h_{\theta\theta}\kappa-h_\theta\kappa_\theta}{\kappa^2}-f_{\theta\theta}\right)+\frac{(1+h^2-M_1)\Lambda_1}{h\Lambda_1-h_t}-\kappa+\frac{M_1}{h}-h+\frac{h_\theta h_{t\theta}}{h\Lambda_1-h_t}\\
=&-\kappa M_1\frac{h_\theta^2}{h^2}-\kappa f_{\theta\theta}-h_\theta f_\theta-\frac{1}{\kappa}+\frac{(1+h^2+h_\theta^2-M_1)\Lambda_1}{h\Lambda_1-h_t}-(1+M_1)\kappa+\frac{2M_1}{h}.
\end{aligned}$$

By $f=\log\left(\frac{h}{\tilde\kappa}\right)$, one has

$$f_\theta=-\frac{\tilde\kappa_\theta}{\tilde\kappa}+\frac{h_\theta}{h}$$

and

$$f_{\theta\theta}=\frac{\tilde\kappa_\theta^2-\tilde\kappa_{\theta\theta}\tilde\kappa}{\tilde\kappa^2}+\frac{h_{\theta\theta}h-h_\theta^2}{h^2}=\frac{\tilde\kappa_\theta^2-\tilde\kappa_{\theta\theta}\tilde\kappa}{\tilde\kappa^2}-\frac{h_\theta^2}{h^2}+\frac{1}{\kappa h}-1,$$

which together with the smoothness of $\tilde\kappa$, Lemma 2.7 and Lemma 2.8 imply that $|f_\theta|\leq C_8$, $|\tilde\kappa_{\theta\theta}|\leq C_9$ and

$$\begin{aligned}
-\kappa f_{\theta\theta}&=-\frac{\kappa\tilde\kappa_\theta^2}{\tilde\kappa^2}+\frac{\tilde\kappa_{\theta\theta}\kappa}{\tilde\kappa}+\frac{\kappa h_\theta^2}{h^2}-\frac{1}{h}+\kappa\\
&\leq\frac{\tilde\kappa_{\theta\theta}^2+\kappa^2}{2\tilde\kappa}+\frac{\kappa h_\theta^2}{h^2}-\frac{1}{h}+\kappa\\
&\leq\frac{C_7^2+C_9^2}{2\tilde\kappa_{\max}}+\frac{C_5\tilde C_1^2}{C_2^2}+C_5-\frac{1}{h},
\end{aligned}$$

where $\tilde C_1$ is the lower bound for h_θ. Then, we have

$$\begin{aligned}
0\leq\frac{(\Psi_1)_t}{h\Lambda_1-h_t}&\leq\frac{C_7^2+C_9^2}{2\tilde\kappa_{\max}}+\frac{h_\theta^2+f_\theta^2}{2}-\frac{C_{10}}{\kappa}+\frac{C_5\tilde C_1^2}{C_2^2}+C_5+\frac{2M_1-1}{h}\\
&\leq\frac{C_7^2+C_9^2}{2}+\frac{\tilde C_1^2+C_8^2}{2}+\frac{2M_1-1}{C_2}+\frac{C_5\tilde C_1^2}{C_2^2}+C_5-\frac{C_{10}}{\kappa},
\end{aligned}$$

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which implies that $\kappa(\theta,t)$ has a positive lower bound independent of t.

Case 2. For flow (1.10b)

For the flow (1.10b), there is the support function associated with a fixed curve denoted by $\tilde{\gamma}$ in the evolution equation. However, applying the technique from Case 1 to obtain a lower bound for κ is challenging. So we need find another approach to achieve our goal.

Set $w = v_\theta$, then

$$\begin{aligned} w_t &= (v_\theta)_t = (v_t)_\theta \\ &= \frac{\tilde{\kappa}}{\tilde{h}} v^2 w_{\theta\theta} + \left(\frac{\tilde{\kappa}}{\tilde{h}} v^2\right)_\theta w_\theta + \frac{\tilde{\kappa}}{\tilde{h}} v^2 w + \left(\frac{\tilde{\kappa}}{\tilde{h}} v^2\right)_\theta \left(v - \frac{\Lambda_2}{\tilde{\kappa}}\right) + \frac{\tilde{\kappa}_\theta}{\tilde{\kappa}\tilde{h}} \Lambda_2 v^2. \end{aligned} \tag{2.19}$$

We first show the upper bound for w, and consider the quantity

$$\Psi_2 = w + \alpha v$$

for $\alpha > 0$. The first and second derivatives for Ψ_2 with respect to θ are

$$\begin{aligned} (\Psi_2)_\theta &= w_\theta + \alpha w, \\ (\Psi_2)_{\theta\theta} &= w_{\theta\theta} + \alpha w_\theta. \end{aligned}$$

Combining with (2.19), we can get

$$\begin{aligned} (\Psi_2)_t =& w_t + \alpha v_t \\ =& \frac{\tilde{\kappa}}{\tilde{h}} v^2 w_{\theta\theta} + \left(\frac{\tilde{\kappa}}{\tilde{h}} v^2\right)_\theta w_\theta + \frac{\tilde{\kappa}}{\tilde{h}} v^2 w + \left(\frac{\tilde{\kappa}}{\tilde{h}} v^2\right)_\theta \left(v - \frac{\Lambda_2}{\tilde{\kappa}}\right) + \frac{\tilde{\kappa}_\theta}{\tilde{\kappa}\tilde{h}} \Lambda_2 v^2 \\ &+ \alpha \frac{\tilde{\kappa}}{\tilde{h}} v^2 \left(w_\theta + v - \frac{\Lambda_2}{\tilde{\kappa}}\right) \\ =& \frac{\tilde{\kappa}}{\tilde{h}} v^2 w_{\theta\theta} + \left[\left(\frac{\tilde{\kappa}}{\tilde{h}} v^2\right)_\theta + \alpha \frac{\tilde{\kappa}}{\tilde{h}} v^2\right] w_\theta + \frac{\tilde{\kappa}}{\tilde{h}} v^2 w + \left[\left(\frac{\tilde{\kappa}}{\tilde{h}} v^2\right)_\theta + \alpha \frac{\tilde{\kappa}}{\tilde{h}} v^2\right] \left(v - \frac{\Lambda_2}{\tilde{\kappa}}\right) + \frac{\tilde{\kappa}_\theta}{\tilde{\kappa}\tilde{h}} \Lambda_2 v^2 \\ =& \frac{\tilde{\kappa}}{\tilde{h}} v^2 (\Psi_2)_{\theta\theta} + \left(\frac{\tilde{\kappa}}{\tilde{h}} v^2\right)_\theta (\Psi_2)_\theta + F(\theta,t) \end{aligned}$$

and

$$\begin{aligned} F(\theta,t) =& \left[\left(\frac{\tilde{\kappa}}{\tilde{h}}\right)_\theta v^2 + \frac{2\tilde{\kappa}}{\tilde{h}} v w\right](-\alpha w) + \frac{\tilde{\kappa}}{\tilde{h}} v^2 w + \left[\left(\frac{\tilde{\kappa}}{\tilde{h}}\right)_\theta v^2 + \frac{2\tilde{\kappa}}{\tilde{h}} v w + \alpha \frac{\tilde{\kappa}}{\tilde{h}} v^2\right] \left(v - \frac{\Lambda_2}{\tilde{\kappa}}\right) + \frac{\tilde{\kappa}_\theta}{\tilde{\kappa}\tilde{h}} \Lambda_2 v^2 \\ =& v \left[-\frac{2\alpha\tilde{\kappa}}{\tilde{h}} w^2 + \left(-\alpha \left(\frac{\tilde{\kappa}}{\tilde{h}}\right)_\theta v + \frac{3\tilde{\kappa}}{\tilde{h}} v - \frac{2\Lambda_2}{\tilde{h}}\right) w + \left(\frac{\tilde{\kappa}}{\tilde{h}}\right)_\theta v^2 + \alpha \frac{\tilde{\kappa}}{\tilde{h}} v^2 - \frac{\alpha\Lambda_2}{\tilde{h}} v + \frac{\tilde{h}_\theta}{\tilde{h}^2} \Lambda_2 v\right]. \end{aligned}$$

Because v and Λ_2 are bounded, $F < 0$ holds when w is large enough or equivalently Ψ_2 is large enough. From the maximum principle, we can get the upper bound for Ψ_2 and thus for w. Similarly, we have the upper bound for $-w$ by setting $\Psi_2 = -w + \alpha v$ with $\alpha > 0$. Thus, $|w| \le C_{11}$ as long as the flow (1.10b) exists.

Next, we can obtain the lower bound for κ by the upper bound for w. For any $\theta_3, \theta_4 \in S^1$ and $t > 0$, we have

$$\left|\log \frac{v(\theta_4,t)}{v(\theta_3,t)}\right| = \left|\int_{\theta_3}^{\theta_4} \frac{w(\theta,t)}{v(\theta,t)} d\theta\right| \le C_{11} \int_{\theta_3}^{\theta_4} \frac{\tilde{\kappa}(\theta)}{\kappa(\theta,t)\tilde{h}(\theta)} d\theta$$

$$= C_{11}\int_{\theta_3}^{\theta_4}\frac{\tilde{h}(\theta)}{\kappa(\theta,t)}\,\frac{\tilde{\kappa}(\theta)}{\tilde{h}(\theta)^2}d\theta$$
$$\le C_{11}\tilde{C}\mathscr{L}(t) \le C_{12},$$

where $\tilde{C} = \max\{\frac{\tilde{\kappa}(\theta)}{\tilde{h}(\theta)^2} \mid \theta \in S^1\}$ and $C_{12} = C_{11}\tilde{C}\mathscr{L}(0)$. This implies

$$\frac{v_{\max}(t)}{v_{\min}(t)} \le e^{C_{12}}.$$

Notice that

$$\frac{2\pi}{v_{\max}(t)} \le \int_0^{2\pi}\frac{1}{v(\theta,t)}d\theta \le \tilde{C}\mathscr{L}(0).$$

This deduces that

$$v(\theta,t) \ge v_{\min}(t) \ge v_{\max}(t)e^{-C_{12}} \ge \frac{2\pi e^{-C_{12}}}{\tilde{C}\mathscr{L}(0)}.$$

Recall that $v = \frac{\kappa}{\tilde{\kappa}}\tilde{h}$, we can get $\kappa(\theta,t) \ge \frac{2\pi e^{-C_{12}}}{\tilde{C}\tilde{C}'\mathscr{L}(0)}$, where $\tilde{C}' = \max\{\frac{\tilde{h}(\theta)}{\tilde{\kappa}(\theta)} \mid \theta \in S^1\}$. □

Then, we have the long-term existence for the flows (1.10a) and (1.10b).

Theorem 2.10. *The flows* (1.10a) *and* (1.10b) *exist in the time interval* $[0,\infty)$.

Proof. By Propositions 2.3 and 2.9, we have known that curvature κ has uniformly positive lower and upper bounds. Since $\Lambda_1(t)$ and $\Lambda_2(t)$ are bounded, (2.1a) or (2.2a) is uniformly parabolic. From standard regularity theory of uniformly parabolic equations (cf. Krylov [37]), it yields that

$$|h^{(i)}| \le M_i, \tag{2.20}$$

where $h^{(i)}$ is the i-th derivative of h, and M_i is a positive constant only depending on i and initial curve γ_0, and the i-th derivative of curvature κ, $\kappa^{(i)}$, has uniformly bound, that is,

$$|\kappa^{(i)}| \le N_i \tag{2.21}$$

holds for a positive constant N_i only depending on i and initial curve γ_0. This deduces the long-term existence and uniqueness of the smooth solution to the flow (1.10a) or (1.10b). □

To obtain the final shape and the convergence of the flow (1.10a) or (1.10b), we need to show the boundedness of $\left|\frac{d^2\mathscr{L}}{dt^2}\right|$.

Lemma 2.11. *For the flow* (1.10a) *or* (1.10b)*, there exists a positive constant* C_{13} *such that*

$$\left|\frac{d^2\mathscr{L}}{dt^2}\right| \le C_{13}. \tag{2.22}$$

Proof. For flow (1.10a), by Lemma 2.4, we have

$$\frac{d^2\mathscr{L}}{dt^2} = -\left(\int_0^{2\pi}\frac{\kappa h}{\tilde{\kappa}^2}d\theta - \Lambda_1(t)\int_0^{2\pi}\frac{h}{\tilde{\kappa}}d\theta\right)_t.$$

This together with (2.1b), (1.7) and (2.1a) can yield

$$\frac{d^2\mathscr{L}}{dt^2} = -\int_0^{2\pi}\frac{\kappa_t h}{\tilde{\kappa}^2}d\theta - \int_0^{2\pi}\frac{\kappa h_t}{\tilde{\kappa}^2}d\theta + \frac{d\Lambda_1(t)}{dt}\int_0^{2\pi}\frac{h}{\tilde{\kappa}}d\theta$$

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$$= -\int_0^{2\pi} \frac{\kappa^2 h}{\tilde{\kappa}^2}\left(\frac{\kappa h}{\tilde{\kappa}}\right)_{\theta\theta} d\theta - \int_0^{2\pi} \frac{\kappa^3 h^2}{\tilde{\kappa}^3} d\theta + \int_0^{2\pi} \frac{\kappa^2 h}{\tilde{\kappa}^3} d\theta + \frac{\mathscr{L}}{\mathscr{A}}\frac{d\mathscr{L}}{dt}$$
$$= -\int_0^{2\pi} \frac{\kappa^2 h}{\tilde{\kappa}^2}\left(\frac{\kappa h}{\tilde{\kappa}}\right)_{\theta\theta} d\theta - \int_0^{2\pi} \frac{\kappa^3 h^2}{\tilde{\kappa}^3} d\theta + \int_0^{2\pi} \frac{\kappa^2 h}{\tilde{\kappa}^3} d\theta - \frac{\mathscr{L}}{\mathscr{A}}\int_0^{2\pi} \frac{\kappa h}{\tilde{\kappa}^2} d\theta + \frac{\mathscr{L}^3}{\mathscr{A}^2}.$$

From (2.20) and (2.21), it yields that $\left|\frac{d^2\mathscr{L}}{dt^2}\right|$ has a uniform bound.

Similarly, for flow (1.10b), we can get

$$\frac{d^2\mathscr{L}}{dt^2} = -\left(\int_0^{2\pi} \frac{\kappa\tilde{h}}{\tilde{\kappa}^2} d\theta - \Lambda_2(t)\int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} d\theta\right)_t$$
$$= -\int_0^{2\pi} \frac{\kappa_t\tilde{h}}{\tilde{\kappa}^2} d\theta + \frac{d\Lambda_2(t)}{dt}\int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} d\theta$$
$$= -\int_0^{2\pi} \frac{\kappa^2\tilde{h}}{\tilde{\kappa}^2}\left(\frac{\kappa\tilde{h}}{\tilde{\kappa}}\right)_{\theta\theta} d\theta - \int_0^{2\pi} \frac{\kappa^3\tilde{h}^2}{\tilde{\kappa}^3} d\theta + \frac{\alpha}{\mathscr{L}}\int_0^{2\pi} \frac{\kappa^2\tilde{h}}{\tilde{\kappa}^3} d\theta + \frac{\alpha^2}{\mathscr{L}^2}\int_0^{2\pi} \frac{\kappa h}{\tilde{\kappa}^2} d\theta - \frac{\alpha^3}{\mathscr{L}^3}.$$

By (2.20) and (2.21), the desired result is concluded. □

Proposition 2.12. *Under the flow (1.10a) or (1.10b), we have*

$$\lim_{t\to\infty} \frac{\kappa(\theta,t)}{\tilde{\kappa}(\theta)} = \sqrt{\frac{\alpha}{\mathscr{A}}}. \tag{2.23}$$

Proof. It follows from Theorem 2.10 that the flow (1.10a) or (1.10b) exists on $[0,\infty)$. Let $\kappa(\theta,t_i)$ be a convergent subsequence of the curvature, where $t_i \to \infty$ as $i \to \infty$. Denote by κ_∞ the limit of $\kappa(\theta,t_i)$. Recall that $\frac{d\mathscr{L}}{dt}$ is non-positive. We have

$$\int_0^\infty \frac{d\mathscr{L}}{dt} dt \geq -\mathscr{L}(0).$$

Then, by (2.11) and Lemma 2.4, it yields that

$$\lim_{t\to\infty} \frac{d\mathscr{L}}{dt} = 0.$$

This deduces that $\frac{\kappa}{\tilde{\kappa}}$ is equal to a constant m, and so is $\frac{\tilde{h}}{h}$ as time t goes to infinity from the equality case for the Cauchy-Schwarz inequality. In other words, $\frac{\kappa_\infty}{\tilde{\kappa}} = m$ and $\frac{\tilde{h}}{h_\infty} = m$. Since

$$\mathscr{A} = \int_0^{2\pi} \frac{h_\infty}{\kappa_\infty} d\theta = \frac{1}{m^2}\int_0^{2\pi} \frac{\tilde{h}_\infty}{\tilde{\kappa}_\infty} d\theta = \frac{\alpha}{m^2},$$

one has $m = \sqrt{\frac{\alpha}{\mathscr{A}}}$, which is the limit (2.23) for subsequence $\{\kappa(\theta,t_i)\}$. Since every convergent subsequence of $\{\kappa(\cdot,t)\}$ tends to the same limit, the curvature κ itself converges. □

From (2.20) and (2.21), we have the C^∞ convergence of the flow (1.10a) or (1.10b).

Corollary 2.13. *For the flow (1.10a) or (1.10b), the convergence of the derivatives*

$$\lim_{t\to\infty} \frac{\partial^i\kappa(\theta,t)}{\partial\theta^i} = \sqrt{\frac{\alpha}{\mathscr{A}}}\frac{\partial^i\tilde{\kappa}(\theta)}{\partial\theta^i}$$

holds for $i = 1, 2, 3, \cdots$.

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Proof of Theorem 1.1. Theorem 2.2 asserts the short time existence of the flow (1.10a) or (1.10b). Lemma 2.5, Theorem 2.6 and Theorem 2.10 demonstrate that any origin symmetric convex curve evolving according to (1.10a) or (1.10b) retains its symmetry, ensuring its long-term existence. Lemma 2.7 and Proposition 2.12 deduce that the evolving curve $\gamma(\cdot, t)$ cannot escape to infinity and converges asymptotically to a fixed curve γ_∞ which is congruent to $\sqrt{\frac{\alpha}{\mathscr{A}}}\tilde{\gamma}$. Finally, Corollary 2.13 establishes the C^∞ convergence of flow (1.10a) or (1.10b) as time t approaches infinity. □

Remark 2.14. *Motivated by the insightful work in Li-Wang [41], the support function of the evolving curve has a uniform bound as long as flow* (1.10b) *exists, even if the initial curve γ_0 and $\tilde{\gamma}$ are non-symmetric.*

Let γ be the evolving curve of length l and enclosed area A. The Bonnesen inequality (cf.[24]) states that

$$lr - A - \pi r^2 \geq 0 \tag{2.24}$$

holds for $r_{in} \leq r \leq r_{out}$, where r_{in} and r_{out} are the inradius and outradius of the domain enclosed by γ, respectively. Then, we have

$$0 < \frac{l - \sqrt{l^2 - 4\pi A}}{2\pi} \leq r_{in} \leq r_{out} \leq \frac{l + \sqrt{l^2 - 4\pi A}}{2\pi}.$$

Due to Lemma 2.4, it has

$$l + \sqrt{l^2 - 4\pi A} \leq 2l \leq \frac{2}{\tilde{h}_{\min}}\mathscr{L}(t) \leq \frac{2}{\tilde{h}_{\min}}\mathscr{L}(0)$$

and

$$\frac{l - \sqrt{l^2 - 4\pi A}}{2\pi} = \frac{\mathscr{A}(t)}{l + \sqrt{l^2 - 4\pi A}} \geq \frac{\tilde{h}_{\min}\mathscr{A}(0)}{2\mathscr{L}(0)}.$$

Therefore, $2r_0 \leq r_{in} \leq r_{out} \leq 2R_0$, where $2r_0 = \frac{\tilde{h}_{\min}\mathscr{A}(0)}{2\mathscr{L}(0)}$ and $2R_0 = \frac{\mathscr{L}(0)}{\pi\tilde{h}_{\min}}$.

For any fixed time $T' \in [0, T)$, denote by $S(T')$ the circle of radius $r(T') = 2r_0$ inscribed in $\gamma(T')$, with centered $P \in R^2$. For $t \geq T'$, let $S(T')$ shrink according to flow $\frac{\partial S}{\partial t} = \Lambda\kappa N$, where $\Lambda = \max\limits_{\theta \in S^1} \frac{\tilde{h}(\theta)}{\tilde{\kappa}(\theta)}$. Since each $S(t)$ is a circle as long as it exists, the radius of $S(t)$ is

$$r(t) = \sqrt{r^2(T') - 2\Lambda(t - T')} \quad \textit{for} \quad t \in \left[T', T' + \frac{r^2(T')}{2\Lambda}\right).$$

Comparing the velocity of the flow $\frac{\partial S}{\partial t} = \Lambda\kappa N$ with that of the flow (1.10b)*, it shows that the circle $S(t)$ is always enclosed by $\gamma(t)$ for all $t \in [T', \min\{T' + \frac{r^2(T')}{2\Lambda}, T\})$. Hence, the support function of the evolving curve (choose P as the origin) satisfies $h(\theta, t) \geq r(t)$ for $(\theta, t) \in [0, 2\pi] \times [T', \min\{T' + \frac{r^2(T')}{2\Lambda}, T\})$. Set $\Delta t = \frac{r^2(T')}{4\Lambda}$, we have*

$$h(\theta, t) \geq r(t) \geq \sqrt{2}r_0$$

for $(\theta, t) \in [0, 2\pi] \times [T', \min\{T' + \Delta t, T\})$.

Recalling that the Minkowski length $\mathscr{L}(t)$ is decreasing along the flow (1.10b)*, we obtain the upper bound for the support function, that is, $h(\theta, t) \leq \tilde{\kappa}_{\max}\mathscr{L}(t) \leq \tilde{\kappa}_{\max}\mathscr{L}(0)$ holds for $(\theta, t) \in [0, 2\pi] \times [T', \min\{T' + \Delta t, T\})$. The desired result is obtained.*

The discussion above is primarily based on Li-Wang [41, pp. 359-361]. To study the gradient flow of an anisoperimetric ratio for convex plane curves without symmetry assumptions, they first established upper and lower bounds for the evolving curve using Bonnesen's inequality.

Note that the results appearing in Proposition 2.3 and Proposition 2.9 are independent of the symmetry. Furthermore, the C^∞ convergence for flow (1.10b) is achieved when the initial curve γ_0 and $\tilde{\gamma}$ are without symmetry condition. Although an anisotropic model analogous to (1.10b) when $\tilde{\gamma}$ is symmetric was discussed in [48], this work introduces a different method that yields a better convergence result.

3 Entropy inequalities

In this section, we intend to discuss two entropy inequalities, which are the log-Minkowski inequality and the curvature entropy inequality. To do this, we first introduce two monotonic entropy functions under flows (1.10a) and (1.10b).

Lemma 3.1. *For the flow (1.10a), the entropy function*

$$\mathscr{E}_1(t) = \int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \log \frac{h}{\tilde{h}} d\theta$$

is decreasing as time t goes to infinity.

Proof. By (2.1a), it has

$$\begin{aligned} \frac{d\mathscr{E}_1(t)}{dt} &= \int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \frac{h_t}{h} d\theta \\ &= -\int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \left(\frac{\kappa}{\tilde{\kappa}} - \Lambda_1(t) \right) d\theta \\ &= -\int_0^{2\pi} \frac{\kappa \tilde{h}}{\tilde{\kappa}^2} d\theta + \frac{\mathscr{L}\alpha}{\mathscr{A}}. \end{aligned}$$

It follows from the Wulff Gage inequality [24, Thm.0.7] that $\frac{d\mathscr{E}_1(t)}{dt} \leq 0$ with the equality holds if and only if γ and $\tilde{\gamma}$ are homothetic. □

Lemma 3.2. *For the flow (1.10b), the entropy function*

$$\mathscr{E}_2(t) = \int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \log \frac{\kappa}{\tilde{\kappa}} d\theta$$

is decreasing as time t goes to infinity.

Proof. From (2.15) and $v = \frac{\kappa}{\tilde{\kappa}} \tilde{h}$, it yields

$$\begin{aligned} \frac{d\mathscr{E}_2(t)}{dt} &= \int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \frac{v_t}{v} d\theta \\ &= \int_0^{2\pi} v \left(v_{\theta\theta} + v - \frac{\Lambda_2(t)}{\tilde{\kappa}} \right) d\theta \end{aligned}$$

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$$= \int_0^{2\pi} v(v_{\theta\theta} + v) d\theta - \frac{\alpha}{\mathscr{L}} \int \frac{\kappa \tilde{h}}{\tilde{\kappa}^2} d\theta.$$

To ensure the monotonicity for $\mathscr{E}_2(t)$, we state an inequality introduced firstly by Andrews in [1, p.322 (1.6)], which says that for two curves with corresponding support functions s_1 and s_2,

$$\int_0^{2\pi} s_1 \tau[s_1] d\theta \int_0^{2\pi} s_2 \tau[s_2] d\theta \leq \left(\int_0^{2\pi} s_1 \tau[s_2] d\theta \right)^2, \tag{3.1}$$

where $\tau[s_i] = \frac{\partial^2 s_i}{\partial \theta^2} + s_i$, $i = 1, 2$. Notice that the above inequality still holds if one of the functions (say s_2) has τ strictly positive, and the other is an arbitrary smooth function. Choose $s_1 = v$ and $s_2 = h$, we have

$$\int_0^{2\pi} v(v_{\theta\theta} + v) d\theta \int_0^{2\pi} h(h_{\theta\theta} + h) d\theta \leq \left(\int_0^{2\pi} v(h_{\theta\theta} + h) d\theta \right)^2,$$

which together with (1.7) and (1.8) leads to

$$\int_0^{2\pi} v(v_{\theta\theta} + v) d\theta \leq \frac{\alpha^2}{\mathscr{A}}.$$

Combining with the Wulff Gage inequality [24, Thm.0.7], it has

$$\frac{d\mathscr{E}_2(t)}{dt} \leq \frac{\alpha^2}{\mathscr{A}} - \frac{\alpha}{\mathscr{L}} \frac{\mathscr{L}\alpha}{\mathscr{A}} = 0,$$

which implies the desired result.

The quantity $\frac{d\mathscr{E}_2(t)}{dt}$ is equal to 0 if and only if the equalities hold in both the Andrews inequality (3.1) and the Wulff Gage inequality. From [1, p.322], we have known that the equality holds in (3.1) only when $s_1(z) = ps_2(z) + \langle z, q \rangle$ for some constant p and some point $q \in R^2$. And the equality holds in the Wulff Gage inequality if and only if γ and $\tilde{\gamma}$ are homothetic. Thus, $\frac{d\mathscr{E}_2(t)}{dt} = 0$ if and only if γ and $\tilde{\gamma}$ are homothetic. □

Proof of Theorem 1.2. Denote by K and L the planar convex bodies with boundary γ and $\tilde{\gamma}$. To obtain the log-Minkowski inequality (1.11), we need only to show the following inequality

$$\int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \log \frac{h}{\tilde{h}} d\theta \geq \frac{\alpha}{2} \log \frac{\mathscr{A}}{\alpha}. \tag{3.2}$$

From Lemma 3.1, we have known that $\mathscr{E}_1(t)$ is decreasing under flow (1.10a). Since the flow (1.10a) exists in $[0, \infty)$,

$$\int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \log \frac{h}{\tilde{h}} d\theta = \mathscr{E}_1(t) \geq \mathscr{E}_1(\infty) = \int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \log \frac{h_\infty}{\tilde{h}} d\theta.$$

From the fact that $\frac{\tilde{h}}{h_\infty} = \sqrt{\frac{\alpha}{\mathscr{A}}}$ and (1.8), it yields the inequality (3.2). The equality holds in (3.2) if and only if $\mathscr{E}_1(t)$ is a constant, which implies that γ and $\tilde{\gamma}$ are homothetic. That is, the equality in the log-Minkowski inequality holds if and only if K and L are homothetic. □

Proof of Theorem 1.4. Denote by K and L the planar convex bodies with boundary γ and $\tilde{\gamma}$. To show the curvature entropy inequality (1.12), we need only to show the following inequality

$$\int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \log \frac{\kappa}{\tilde{\kappa}} d\theta \geq \frac{\alpha}{2} \log \frac{\alpha}{\mathscr{A}}. \tag{3.3}$$

Because $\mathscr{E}_2(t)$ is decreasing under the flow (1.10b) and this flow exists in $[0,\infty)$, we get

$$\int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \log \frac{\kappa}{\tilde{\kappa}} d\theta = \mathscr{E}_2(t) \geq \mathscr{E}_2(\infty) = \int_0^{2\pi} \frac{\tilde{h}}{\tilde{\kappa}} \log \frac{\kappa_\infty}{\tilde{\kappa}} d\theta.$$

Again, by $\frac{\kappa_\infty}{\tilde{\kappa}} = \sqrt{\frac{\alpha}{\mathscr{A}}}$ and (1.8), we have the inequality (3.3). The equality holds in (3.3) if and only if $\mathscr{E}_2(t)$ is a constant, which implies that γ and $\tilde{\gamma}$ are homothetic. This leads to equality in the curvature entropy inequality if and only if K and L are homothetic. □

Remark 3.3. *The curvature entropy inequality* (1.12) *still holds without symmetry condition but with a dilation position condition for two planar convex bodies.*

By Remark 2.14, the symmetry requirements for γ and $\tilde{\gamma}$ can be omitted in flow (1.10b)*. Note that the Wulff Gage inequality holds without symmetry condition but with a dilation position condition for two planar convex bodies (see [65]). Since the entropy $\mathcal{E}_2(t)$ is origin-independent and decreases as $t \to \infty$ without symmetry requirements, we conclude that inequality* (3.3) *remains valid without symmetry. Consequently, the curvature entropy inequality* (1.12) *holds under a dilation position condition.*

In conclusion of the article, we also wish to provide the proof of the Brunn-Minkowski inequality in the planar case using flows (1.10a) and (1.10b). In fact, let K and L be the planar convex bodies with boundaries γ and $\tilde{\gamma}$. Then, by the Steiner formula [24, Pro.1.4], we have

$$V(K+L) = \frac{1}{2}(\mathscr{A} + 2\mathscr{L} + \alpha).$$

Consider the function

$$F(t) = V(K+L)^{\frac{1}{2}} - V(K)^{\frac{1}{2}} - V(L)^{\frac{1}{2}}.$$

It follows from Lemma 2.4 that

$$\frac{dF}{dt} = \frac{1}{2} V(K+L)^{-\frac{1}{2}} \frac{d\mathscr{L}}{dt} \leq 0$$

with equality holds if and only if γ and $\tilde{\gamma}$ are homothetic. By Proposition 2.12, it yields that

$$F(t) \geq F(\infty) = (1+m)V(L)^{\frac{1}{2}} - mV(L)^{\frac{1}{2}} - V(L)^{\frac{1}{2}} = 0,$$

which implies the Brunn-Minkowski inequality, that is,

$$V(K+L)^{\frac{1}{2}} \geq V(K)^{\frac{1}{2}} + V(L)^{\frac{1}{2}}.$$

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