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A characterization of flat metrics on tori by ergodicity

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Abstract. The Riemannian flat metrics on tori T^2 are characterized by a weakly ergodic property of the geodesic flows.

0. *Introduction*

Let M be a complete Riemannian manifold with finite volume and SM the unit tangent bundle. Let $g^t : SM \to SM$ be the geodesic flow and $\pi : SM \to M$ the projection. For any $v \in SM$ if $\gamma_v(t) = \pi(g^t v)$ for any $t \in (-\infty, \infty)$, then $\gamma_v : (-\infty, \infty) \to M$ is a geodesic. For any integrable function f on SM we define a function $f^* : SM \to \mathbb{R} \cup \{-\infty, \infty\}$ by

$$f^*(v) = \liminf_{T\to\infty} \frac{1}{T} \int_0^T f(g^t v)\, dt,$$

for any $v \in SM$. By the Birkhoff ergodic theorem we know that lim inf = lim sup = lim for almost all $v \in SM$ on the right-hand side, and for those v we write $f^{**}(v)$ instead of $f^*(v)$.

The geodesic flows of non-positively curved compact Riemannian manifolds with rank 1 are ergodic (see [**1**]). And any Riemannian metric on tori with non-positive curvature is flat (see [**4**], [**12**]). The torus often plays a distinguished role from other topological types. Although the geodesic flows of flat tori are not ergodic, they satisfy the following condition.

(0.1) If f is an integrable function on M, then $(f \circ \pi)^*$ is constant for almost all $v \in SM$.

If the geodesic flows are ergodic, then they satisfy the condition (0.1). The purpose of the present note is to prove:

THEOREM. *Let T^2 be a torus and A the set of all $v \in ST^2$ such that $\gamma_v(t_0)$ has no conjugate points $\gamma_v(t)$, $t > t_0$, along γ_v for some t_0. Suppose* $\mathrm{vol}(A) > 0$ *and the geodesic flow of T^2 satisfies the condition* (0.1). *Then T^2 is flat.*

In § 1 we give the preliminaries and prove our theorem in § 2. In § 3 we discuss the assumption $\mathrm{vol}(A) > 0$.

1. *Preliminaries*

(1.1) *Riccati equation.* Let M be a complete Riemannian manifold with dimension 2 and $\gamma : [0, \infty) \to M$ a geodesic. Assume that $\gamma(0)$ has no conjugate points $\gamma(t)$,

$t>0$, along γ. By introducing an orthonormal frame

$$\{E_0(t)=\dot\gamma(t), E_1(t)\}$$

along γ we can write every Jacobi field Z orthogonal to $\dot\gamma$ as $Z=zE_1$ with

$$z''(t)+K(t)z(t)=0,$$

where $K(t)$ is the Gauss curvature of M at $\gamma(t)$ for all $t\geq 0$. It follows from our assumption that $z(t)\neq 0$ for any $t>0$ if $z(0)=0$, $z'(0)>0$. Thus, if $s(t)=z'(t)/z(t)$ for all $t>0$, $s(t)$ satisfies the Riccati equation

$$s'(t)+s(t)^2+K(t)=0, \tag{1.1}$$

for any $t>0$.

LEMMA 1.1 (see [7], [11]). *If $K(t)>-k^2$ for some $k>0$ and for all $t\geq 0$, then $s(t)$ satisfies*

$$-k\leq s(t)\leq k\coth kt,$$

for all $t>0$. In particular, $s(t)/t\to 0$ as $t\to\infty$.

(1.2) *A weakly ergodic property.* Let SM be the unit tangent bundle of M ($\dim M=n$) and $g^t: SM\to SM$ the geodesic flow of M. If $\mu=\eta\wedge\sigma$, where η and σ are the volume forms of M and S^{n-1} (resp.), then μ is a volume form of SM and is preserved by the geodesic flow g^t.

LEMMA 1.2. *Let M be a complete Riemannian manifold with finite volume. Assume that the geodesic flow g^t satisfies the condition* (0.1). *Then, there is a set B in SM such that* $\operatorname{vol}(B)=\operatorname{vol}(SM)$ *and* $(\chi_U\circ\pi)^*(v)>0$ *for any open set U in M and any $v\in B$, where $\chi_U: M\to\mathbb{R}$ is the characteristic function of U.*

Proof. Let $\{U_n;\ n=1,2,3,\dots\}$ be an open base of the topology of M. From the condition (0.1), we have, for each n,

$$(\chi_{U_n}\circ\pi)^{**}(v)=c_n,$$

for almost all $v\in SM$. And, because of the Birkhoff ergodic theorem, c_n is not zero. For each n let B_n be the set of all vectors $v\in SM$ such that $(\chi_{U_n}\circ\pi)^{**}(v)=c_n$. Put $B=\bigcap_{n=1}^\infty B_n$. Then B is the set in SM we required.

2. *Proof of theorem*

Let $K: T\to\mathbb{R}$ be the Gauss curvature. By the Gauss–Bonnet formula and the condition (0.1), we have

$$(K\circ\pi)^{**}(v)=0,$$

for almost all $v\in ST^2$. Hence, we can find a vector $v\in SM$ such that

(1) $(\chi_U\circ\pi)^*(v)>0$ for any open set U in T^2;

(2) $(K\circ\pi)^{**}(v)=0$;

(3) if $\gamma:[0,\infty)\to T^2$ is the geodesic with $\dot\gamma(0)=v$, then $\gamma(0)$ has no conjugate points along γ.

From this we want to prove that $K(p)=0$ for any $p\in T^2$. Assume that

$$s'(t)+s(t)^2+K(t)=0 \tag{2.1}$$

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is the Riccati equation along γ constructed as in 1.1. Then, by lemma 1.1 and condition (2), we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T s(t)^2\,dt=0.$$

Let p be a point in T^2 and $B_p(l)$ the convex ball with centre p and radius l. We first prove:

ASSERTION. *There exists a sequence* $\{t_n\}\subset[0,\infty)$ *such that*

(1)* $t_n\to\infty$ *as* $n\to\infty$;

(2)* *if* $s_n:[0,l]\to\mathbb{R}$ *is given by* $s_n(t)=s(t_n+t)$, *then* $\int_0^l s_n(t)^2\,dt\to 0$ *as* $n\to\infty$, *and* $s_n(t)\to 0$ *for almost all* $t\in[0,l]$ *as* $n\to\infty$,

(3)* *if* $\gamma_n:[0,l]\to T^2$ *is given by* $\gamma_n(t)=\gamma(t_n+t)$, *then* γ_n *converges to a geodesic* $\gamma_0:[0,l]\to T^2$ *through* $p=\gamma(l/2)$ *as* $n\to\infty$.

Proof of assertion. Let $k\geq 4$ be an integer. Since $B_p(l/k)$ is convex, $\gamma^{-1}(B_p(l/k))$ is the union of intervals whose lengths are less than or equal to $2l/k$, say (a'_1, b'_1), $(a'_2, b'_2),\ldots,(a'_i, b'_i),\ldots$; $a'_1<b'_1<a'_2<b'_2<\cdots<a'_i<b'_i<\cdots\to\infty$. Put

$$a_i=(a'_i+b'_i)/2-l/2,\quad b_i=(a'_i+b'_i)/2+l/2,$$

for each $i=1,2,\ldots$. Then, $\gamma[a_i,b_i]\subset B_p(l)$ and $\gamma(a_i)$, $\gamma(b_i)\notin B_p(l/k)$. Suppose

$$\liminf_{i\to\infty}\int_{a_i}^{b_i} s(t)^2\,dt>\alpha>0.$$

For any $T>0$ with $\gamma(T)\notin B_p(l)$, we have

$$\begin{aligned}\frac{1}{T}\int_0^T s(t)^2\,dt&\geq\frac{1}{T}\left[\sum_{i=1}^{n(T)}\int_{a_i}^{b_i} s(t)^2\,dt\right]\\ &\geq\frac{1}{T}\left[\sum_{i=1}^{m}\int_{a_i}^{b_i} s(t)^2\,dt\right]+\frac{\alpha}{lT}\sum_{i=m+1}^{n(T)}(b_i-a_i)\\ &\geq\frac{\alpha}{lT}\sum_{i=m+1}^{n(T)}(b'_i-a'_i)\\ &=\frac{\alpha}{lT}\int_0^T\chi_{B_p(l/k)}(\gamma(t))\,dt-\frac{\alpha}{lT}\sum_{i=1}^{m}(b'_i-a'_i),\end{aligned}$$

where $n(T)$ and m are chosen so that $b_{n(T)}<T<a_{n(T)+1}$ and $\inf_{i\geq m}\int_{a_i}^{b_i} s(t)^2\,dt>\alpha$. This implies that

$$0=\lim_{T\to\infty}\frac{1}{T}\int_0^T s(t)^2\,dt\geq(\alpha/l)(\chi_{B_p(l/k)}\circ\pi)^*(v),$$

contradicting (1). Thus we can find an integer $i(k)$ such that

$$\gamma((a_{i(k)}+b_{i(k)})/2)\in B_p(l/k)\quad\text{and}\quad\int_{a_{i(k)}}^{b_{i(k)}} s(t)^2\,dt\leq\frac{1}{k}.$$

If $t_k=a_{i(k)}$ for all $k\geq 4$, the sequence $\{t_k\}$ satisfies the condition (1)* and the first part of (2)*. For the second part of (2)* and (3)* we have only to choose a suitable subsequence $\{t_n\}$ of $\{t_k\}$ if necessary.

We return to the proof of the theorem. Rewriting (2.1) in terms of (2)*, we get for each n

$$s_n'(t)+s_n(t)^2+K_n(t)=0 \tag{2.2}$$

for any $t\in[0, l]$, where $K_n(t)=K(t_n+t)$. Suppose $K(p)\neq 0$, say $K(p)>0$. Then, there exist $a<b\in[0, l]$ such that $K_0(t)=K(\gamma_0(t))>0$ for any $t\in[a, b]$ and $s_n(a)$, $s_n(b)\to 0$ as $n\to\infty$. On the other hand, by integrating (2.2) on the interval $[a, b]$ and taking n to infinity, we have

$$\int_a^b K_0(t)\, dt=0,$$

a contradiction. This completes the proof of the theorem.

3. *Geodesic rays without conjugate points*

In the present section we discuss the assumption $\mathrm{vol}\,(A)>0$.

The case of tori of revolution. Let $T^2=S^1\times S^1$ be the torus and let $f: S^1\to\mathbb{R}$ be a positive function with minimum at y_0. We consider a Riemannian metric, $ds^2=f(y)^2\,dx^2+dy^2$, $(x, y)\in S^1\times S^1$, on T^2. It is proved in [3], [5], [8] and [10] that all points p at which f assumes its minimum are poles, i.e. p has no conjugate points along any geodesic ray emanating from p. We know that $\mathrm{vol}\,(A)>0$ from the following argument. The Clairaut theorem implies that $f(\alpha(t))\cos\theta_\alpha(t)$ is constant for any geodesic $\alpha:(-\infty, \infty)\to T^2$, where $\theta_\alpha(t)$ is the angle of $\dot\alpha(t)$ and the curve $y=\text{const}$. Hence, if a geodesic $\alpha:(-\infty, \infty)\to T^2$ satisfies that

$$f(\alpha(0))\cos\theta_\alpha(0)<\min f, \tag{3.1}$$

then $\alpha[0, \infty)$ intersects the curve $y=y_0$, and, therefore, contains poles. This proves that $\dot\alpha(0)\in A$ for each such geodesic. Since the set of those vectors which satisfy (3.1) has a positive measure, it follows that $\mathrm{vol}\,(A)>0$.

The case of generic tori. The author does not know whether the assumption of our theorem, $\mathrm{vol}\,(A)>0$, holds in general. We can only have, as a straightforward modification of a result stated by V. Bangert, (see [2]):

PROPOSITION 3.1. *Let T^2 be the torus with Riemannian metric and M be the universal covering of T^2. Then, there exist uncountably many minimizing geodesics emanating from each point p of M. In particular, $S_pM\cap A$ has the cardinal number of the continuum.*

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