DUAL INTEGRAL EQUATIONS WITH A TRIGONOMETRIC KERNEL*

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In this paper, we solve the following dual integral equations

\[
\int_0^\infty \left[ 1 - \frac{2\xi\delta(1 + \xi\delta) + 1 - e^{-2\xi\delta}}{2\xi\delta + \sinh 2\xi\delta} \right] \xi A(\xi) \cos \xi x \, d\xi = f(x), \quad 0 < x < a, \\
\int_0^\infty A(\xi) \cos \xi x \, d\xi = 0, \quad x > a,
\]

where \( \delta \) is a real positive constant and \( f(x) \) is a continuous and integrable function of \( x \) in \([0, a]\). The dual integral equations (1) and (2) arise in a crack problem of elasticity.

Let us rewrite the above integral equations in the form:

\[
\int_0^\infty \xi \psi(\xi)(1 - \xi^2\delta^2 \cosech^2 \xi\delta) \cos \xi x \, d\xi = f(x), \quad 0 < x < a, \\
\int_0^\infty \psi(\xi)(\coth \xi\delta + \xi\delta \cosech^2 \xi\delta) \cos \xi x \, d\xi = 0, \quad x > a,
\]

where

\[
\psi(\xi) = (\coth \xi\delta + \xi\delta \cosech^2 \xi\delta)^{-1} A(\xi).
\]

Equations (3) and (4) may be further put in the form

\[
\int_0^\infty \psi(\xi) \frac{d}{d\delta} \left( -\frac{1}{\delta} + \xi \coth \xi\delta \right) \cos \xi x \, d\xi = \frac{f(x)}{\delta^2}, \quad 0 < x < a, \\
\int_0^\infty \psi(\xi) \frac{d}{d\delta} \left( \frac{1}{\delta} \coth \xi\delta \right) \cos \xi x \, d\xi = 0, \quad x > a.
\]

Integrating equations (6) and (7) with respect to \( \delta \), we obtain

\[
\int_0^\infty \xi \psi(\xi) \left( -\frac{1}{\delta} + \xi \coth \xi\delta \right) \cos \xi x \, d\xi = -\frac{f(x)}{\delta} + g(x), \quad 0 < x < a, \\
\int_0^\infty \psi(\xi) \coth \xi\delta \cos \xi x \, d\xi = 0, \quad x < a,
\]

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where the limits of integration have been taken from $\delta$ to $\infty$ for integrating equation (7) and $g(x)$ is an arbitrary function of $x$.

Integrating equation (8) with respect to $x$ between 0 to $x$, we obtain

$$
\int_0^a \psi(\xi) \left( -\frac{1}{\delta} + \xi \coth \xi \delta \right) \sin \xi x \, d\xi = -\frac{F(x)}{\delta} + G(x), \quad 0 < x < a,
$$

(10)

where

$$
F(x) = \int_0^x f(x) \, dx, \quad G(x) = \int_0^x g(x) \, dx.
$$

(11)

If we assume the representation

$$
\psi(\xi) = \frac{2}{\pi} \xi^{-1} \tanh \xi \delta \int_0^a \phi(t) \sin \xi t \, dt,
$$

(12)

the integral equation (9) is identically satisfied. Now rewriting equation (10) in the form

$$
-\frac{1}{\delta} \int_0^a \psi(\xi) \sin \xi x \, d\xi - \frac{\partial}{\partial x} \int_0^a \psi(\xi) \coth \xi \delta \cos \xi x \, d\xi
$$

$$
= -\frac{1}{\delta} F(x) + G(x), \quad 0 < x < a,
$$

(13)

and then substituting for $\psi(\xi)$ from (12), we find that $\phi$ is the solution of the integral equation

$$
-\frac{1}{\pi \delta} \int_0^a \phi(t) \log \left| \frac{\sinh cx + \sinh ct}{\sinh cx - \sinh ct} \right| \, dt - \phi(x)
$$

$$
= -\frac{1}{\delta} F(x) + G(x), \quad 0 < x < a,
$$

(14)

where $c = \pi/2\delta$ and we have used the following integral

$$
\int_0^\infty \xi^{-1} \tanh \xi \delta \sin \xi x \sin \xi t \, d\xi = \frac{1}{2} \log \left| \frac{\sinh cx + \sinh ct}{\sinh cx - \sinh ct} \right|, \quad \delta > 0,
$$

(15)

for obtaining integral equation (14). Letting $\delta \to \infty$ in equation (14), we find that $G(x) = -\phi(x)$

(16)

and equation (14) simplifies to

$$
\int_0^a \phi(t) \log \left| \frac{\sinh cx + \sinh ct}{\sinh cx - \sinh ct} \right| \, dt = \pi F(x), \quad 0 < x < a.
$$

(17)

With the help of (1) or (2), the solution of the above integral equation is obtained in the following form:

$$
\phi(t) = -\frac{2c}{\pi} \frac{\cosh ct}{(\sinh^2 cx - \sinh^2 ct)^{1/2}}
$$

$$
\times \left[ \sinh ct \int_0^a \frac{(\sinh^2 cx - \sinh^2 ct)^{1/2}}{\sinh^2 cx - \sinh^2 ct} F'(x) \, dx - \frac{F(0)}{\sinh ct} \right], \quad 0 < t < a,
$$

(18)
where prime denotes the derivative with respect to the argument. If \( f(x) \) is a constant, say,

\[
f(x) = p_0,
\]

we find from (18) and (19) that

\[
\phi(t) = -\frac{2cp_0}{\pi} \frac{\sinh ct \cosh ct}{(\sinh^2 ca - \sinh^2 ct)^{1/2}} \int_0^a \frac{(\sinh^2 ca - \sinh^2 cx)^{1/2}}{\sin^2 cx - \sinh^2 ct} \, dx, \quad 0 < t < a. \tag{20}
\]

If we let \( \delta \to \infty \) (or \( c \to 0 \)) in equation (20), we find that

\[
\phi(t) = p_0 t (a^2 - t^2)^{-1/2} \tag{21}
\]

and hence from (5), (12) and (21), we have

\[
A(\xi) = \psi(\xi) = a p_0 \xi^{-1} J_1(a \xi), \tag{22}
\]

which is the solution (see Sneddon (3), pp. 103–104) of the dual integral equations

\[
\int_0^a \xi A(\xi) \cos \xi x \, d\xi = p_0, \quad 0 < x < a, \tag{23}
\]

\[
\int_0^\infty A(\xi) \cos \xi x \, d\xi = 0, \quad x > a. \tag{24}
\]

The integral equations (1) and (2) reduce to (23) and (24) for \( f(x) = p_0 \) and \( \delta \to \infty \).

By evaluating the integral in equation (20), we find that \( \phi(t) \) may be put in the following form.

\[
\phi(t) = \frac{p_0}{\pi} \frac{\sinh^2 ct}{\cosh ca(\sinh^2 ca - \sinh^2 ct)^{1/2}} \times \left[ F(\pi/2 \tanh ca) - \Pi(\pi/2, \frac{\sinh^2 ca}{\sinh^2 ca - \sinh^2 ct}, \tan ca) \right], \quad 0 < t < a, \tag{25}
\]

where \( F \) and \( \Pi \), respectively, denote elliptic integrals of the first and third kind. Now \( A(\xi) \) may be obtained from equations (5), (12) and (25).

REFERENCES

