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ESTIMATES FOR THE KOEBE CONSTANT AND THE SECOND COEFFICIENT FOR SOME CLASSES OF UNIVALENT FUNCTIONS

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1. Introduction. Let S be the set of all normalized univalent analytic functions $f(z) = z + a_2z^2 + \ldots$ in the open unit disk U. Then $f(U)$ contains the disk $\{|w| < \frac{1}{4}\}$. Here $\frac{1}{4}$ is the best possible constant and is referred to as the Koebe constant for S. On the other extreme, $f(U)$ cannot contain the disk $\{|w| < 1\}$ unless f is the identity mapping.

In order to interpolate between the class S and the identity mapping, one may introduce the families $S(d)$, $\frac{1}{4} \leqq d \leqq 1$, of functions $f \in S$ such that $f(U)$ contains the disk $\{|w| < d\}$. Then $S(d_1) \supset S(d_2)$ for $d_1 < d_2$, $S(\frac{1}{4}) = S$, and $S(1)$ contains only the identity mapping. It is obvious that d is the "Koebe constant" for $S(d)$. The relation between d and the second coefficient a_2 has been studied by E. Netanyahu [**5**, **6**].

In this article we shall introduce new families of univalent functions that interpolate in a natural way between S and the identity mapping. We shall give estimates both for the "Koebe constants" for these families and for the second coefficients a_2 of functions in these families.

2. Definitions. For $0 < \rho \leq \infty$ let S^ρ consist of those functions $f \in S$ such that the inverse function f^{-1} has a univalent analytic continuation to $\{|w| < \rho\}$. Then $S^{\rho_1} \supset S^{\rho_2}$ for $\rho_1 < \rho_2$, $S^\rho = S$ for $0 < \rho \leqq \frac{1}{4}$, and S^∞ contains only the identity mapping. It is obvious that $S(d) \subset S^d$, and by means of examples in Sections 3 and 4 we shall see that this containment is proper for $d \neq \frac{1}{4}$. Therefore, it is an interesting question to determine the radius d_ρ of the largest disk $\{|w| < d_\rho\}$ that is contained in $f(U)$ for every $f \in S^\rho$. We call d_ρ the *Koebe constant for* S^ρ.

We also introduce a family $\mathscr{S}^\rho$ that is closely related to S^ρ. For $0 < \rho < \infty$ let $\mathscr{S}^\rho$ consist of those functions $f \in S$ such that $f = \psi \circ \phi^{-1}$ where ϕ and ψ are analytic and univalent in U, normalized so that $\phi(0) = \psi(0) = 0$, $\phi'(0) = \psi'(0)$, and

$$\{|w| < 1\} \subset \phi(U), \{|w| < \rho\} \subset \psi(U).$$

It is clear that $\mathscr{S}^\rho$ is a subset of S^ρ: $\mathscr{S}^\rho \subset S^\rho$. In Section 8 we shall show that it is a proper subset for $\frac{1}{4} < \rho < \infty$. In addition, if we define

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$S^\infty = \bigcap_{0<\rho<\infty} S^\rho$, then $S^{\rho_1} \supset S^{\rho_2}$ for $\rho_1 < \rho_2$, $S^\rho = S$ for $0 < \rho \leqq \frac{1}{4}$, and S^∞ contains only the identity mapping. By analogy we define ∂_ρ to be the *Koebe constant for* S^ρ. By choosing ϕ as the identity mapping we see that $S(d) \subset \mathscr{S}^d$. However, in Section 4 we note that $\partial_d < d$, so that this containment is also proper, except for $d = \frac{1}{4}$.

The classes S^1 and $\mathscr{S}^1$ were introduced by M. Lewin [**4**]. They have been studied in [**4**], [**5**], [**3**], and [**7**]. The functions in S^1 are called *bi-univalent.*

3. Examples.

The following examples will be useful.

Example 1. For $\frac{1}{4} < d < 1$, let $w = k(z, d)$ be the normalized ($k(z, d) = z + \ldots$) mapping of the unit disk onto the complement of an arc of the circle $|w| = d$, with midpoint $-d$, and a slit along the negative axis from $-d$ to $-\infty$. As extreme cases, let $w = k(z, 1)$ be the identity mapping, and let $k(z, \frac{1}{4})$ be the Koebe function that maps U onto the complement of just the slit along the negative axis from $-\frac{1}{4}$ to $-\infty$.

Of course, $k(z, \frac{1}{4}) = z/(1 - z)^2$, and for $\frac{1}{4} < d \leqq 1$ the function $k(z, d)$ is defined implicitly by

$$k(z, d) = \frac{dt(1 - \epsilon t)}{\epsilon - t} \quad \text{and} \quad \frac{dt}{(1 + t)^2} = \frac{\epsilon z}{(1 + z)^2}$$

where $4d = (1 + \epsilon)^2$. We choose the branch of $t = t(z)$ that takes $0 < z < 1$ to the positive real axis.

The function $k(z, d)$ belongs to $S(d)$ and, hence, to S^d and $\mathscr{S}^d$. It is extremal in $S(d)$ for the following two results, which we shall need later.

LEMMA 1. ([**1**]). *For $f \in S(d)$ and $|z| < 1$, we have*

$$-k(-|z|, d) \leqq |f(z)| \leqq k(|z|, d).$$

LEMMA 2. ([**6**]). *For $f(z) = z + a_2 z^2 + \ldots$ in $S(d)$, we have*

$$|a_2| \leqq \frac{2}{d}(1 - \sqrt{d})(3\sqrt{d} - 1).$$

Equality occurs if and only if $f(z) = e^{-i\alpha} k(e^{i\alpha} z, d)$ for some real α.

In [**5, 6**] Netanyahu actually considered the class $S(d) \backslash \bigcup_{\hat{d} > d} S(\hat{d})$ in terms of our notation. However, since the bound in Lemma 2 is a decreasing function of d, it is valid for the full class $S(d)$.

Example 2. The functions $f(z) = -(1/d)k(-k^{-1}(dz, d), d')$ belong to $\mathscr{S}^{d'/d}$. (Here k^{-1} is with respect to the first argument.) They will be useful in the explicit determination of ∂_ρ in Section 4.

Example 3. For $0 < \theta \leqq \pi$, the function

$$f(z) = \frac{z - \frac{1}{2}(1 + e^{-i\theta})z^2}{(1 - z)^2}$$

is close-to-convex and maps the unit disk onto the entire plane except for a straight line slit. Since $f(\pm i) = -\frac{1}{2} \pm \frac{1}{4}i(1 + e^{-i\theta})$, the slit passes through the point $-\frac{1}{2}$. In addition, $f'(e^{i\theta}) = 0$ so that

$$f(e^{i\theta}) = -\tfrac{1}{4} + \frac{i}{4}\cot\frac{\theta}{2}$$

is the tip of the slit. If δ_f denotes the distance of the slit from the origin, then $\delta_f = \frac{1}{2}\cos(\theta/2)$ for $0 < \theta \leqq \pi/2$ and $\delta_f = \frac{1}{4}\csc(\theta/2)$ for $\pi/2 \leqq \theta \leqq \pi$.

For $0 < \theta < \pi/2$ the Schwarz reflection principle allows us to continue f^{-1} analytically and univalently across the slit. Two points restrict the continuation. One is, of course, the tip of the slit. The other is the reflection of the origin at $e^{i\theta/2}\cos\theta/2$, which leads to a pole of f^{-1}. The latter is closer to the origin than the tip of the slit if $0 < \theta < \pi/6$. The three different situations are summarized as follows: For $0 < \theta \leqq \pi/6$ the function f belongs to S^ρ where

$$\rho = 2\delta_f = \cos(\theta/2).$$

For $\pi/6 \leqq \theta \leqq \pi/2$ the function f belongs to S^ρ where

$$\rho = |f(e^{i\theta})| = \tfrac{1}{4}\csc(\theta/2) \text{ and } \delta_f = \tfrac{1}{2}\cos(\theta/2).$$

For $\pi/2 \leqq \theta \leqq \pi$ the function f belongs to S^ρ where

$$\rho = \delta_f = |f(e^{i\theta})| = \tfrac{1}{4}\csc(\theta/2).$$

Example 4. The Möbius transformation $f(z) = \rho z/(\rho - z)$ belongs to S^ρ, and

$$\delta_f = \min_t |f(e^{it})| = \frac{\rho}{\rho + 1}.$$

Examples 3 and 4 provide a family of functions that vary continuously from the Koebe function $k(z, \frac{1}{4})$ to the identity mapping. They also provide the following upper estimates for d_ρ.

PROPOSITION. *The following are upper bounds for the Koebe constants d_ρ for the families S^ρ:*

$$d_\rho \leqq \begin{cases} \rho & \text{for } \frac{1}{4} \leqq \rho \leqq \frac{\sqrt{2}}{4} \\ \sqrt{16\rho^2 - 1}/8\rho & \text{for } \frac{\sqrt{2}}{4} \leqq \rho \leqq \frac{1}{4}\csc\frac{\pi}{12} \approx .966 \\ \frac{1}{2}\rho & \text{for } \frac{1}{4}\csc\frac{\pi}{12} \leqq \rho \leqq 1 \\ \rho/(\rho + 1) & \text{for } \rho \geqq 1. \end{cases}$$

We shall improve these bounds at the end of the next section.

4. The Koebe constant in $\mathscr{S}^\rho$. The following is a preliminary result.

LEMMA 3. *Let f belong to $\mathscr{S}^\rho$. Then for $\frac{1}{4} \leqq \rho \leqq 4$*

$$(a)\ \min_{1/4 \leqq \rho x, x \leqq 1} -\frac{1}{x} k(-k^{-1}(|z|x, x), \rho x) \leqq |f(z)| \leqq \max_{1/4 \leqq \rho x, x \leqq 1} \frac{1}{x} k(-k^{-1}(-|z|x, x), \rho x)$$

and for $\rho > 4$

$$(b)\ \min_{1/4 \leqq \rho x \leqq 1} -\frac{1}{x} k(-k^{-1}(|z|x, \tfrac{1}{4}), \rho x) \leqq |f(z)| \leqq \max_{1/4 \leqq \rho x \leqq 1} \frac{1}{x} k(-k^{-1}(-|z|x, \tfrac{1}{4}), \rho x).$$

These estimates are sharp.

Proof. If $f \in \mathscr{S}^\rho$, then $f = \psi \circ \phi^{-1}$ where $\phi(0) = \psi(0) = 0$, $a \equiv \phi'(0) = \psi'(0)$, and $U \subset \phi(U)$, $\{|w| < \rho\} \subset \psi(U)$. Thus $(1/a)\psi \in S(\rho x)$ and $(1/a)\phi \in S(x)$ for $x = 1/|a|$, and so by Lemma 1 we have

$$|f(\phi(\zeta))| = |\psi(\zeta)| \leqq \frac{1}{x} k(|\zeta|, \rho x).$$

Here $f(\phi(\zeta))$ for $|\phi(\zeta)| \geqq 1$ is defined by analytic continuation.

Let now $z = \phi(\zeta)$ so that

$$|f(z)| \leqq \frac{1}{x} k(|\zeta|, \rho x) \quad \text{for } |\zeta| < 1.$$

By applying Lemma 1 to ϕ, we have

$$-\frac{1}{x} k(-|\zeta|, x) \leqq |\phi(\zeta)| = |z|.$$

It follows by monotonicity that $|\zeta| \leqq -k^{-1}(-|z|x, x)$ and

$$|f(z)| \leqq \frac{1}{x} k(-k^{-1}(-|z|x, x), \rho x).$$

This estimate is sharp since the functions of Example 2 (§3) belong to the class. The upper estimate in (a) follows from considering all admissible choices of x. The lower estimate in (a) is proved similarly.

If $\rho > 4$, then ψ exists only for $x < \frac{1}{4}$. The necessary adjustments are reflected in (b).

We shall now determine the Koebe constants for the families $\mathscr{S}^\rho$ explicitly.

THEOREM 1. *The Koebe constants for the families* $\mathcal{S}^{\rho}$ *are*

$$\partial_\rho = \rho\frac{2(1+\sqrt{\rho})^2 - (2\sqrt{\rho}-1)\sqrt{(1+\sqrt{\rho})(4+\sqrt{\rho})}}{2(1+\sqrt{\rho})^2 + (2\sqrt{\rho}-1)\sqrt{(1+\sqrt{\rho})(4+\sqrt{\rho})}}$$

or $\frac{1}{4} \leqq \rho \leqq 4$

and

$$\partial_\rho = \rho\frac{\sqrt{4+\rho}-\sqrt{\rho}}{\sqrt{4+\rho}+\sqrt{\rho}} \qquad \text{for } \rho > 4.$$

Proof. Case 1: $\frac{1}{4} \leqq \rho \leqq 4$. It follows from Lemma 3 that

$$\partial_\rho = \min_{1/4 \leqq \rho x, x \leqq 1} -\frac{1}{x}k(-k^{-1}(x,x), \rho x).$$

If $\zeta = k^{-1}(x, x)$, then

$$x = k(\zeta, x) = xt\frac{1-\epsilon t}{\epsilon - t} \quad \text{where}$$

$$\frac{xt}{(1+t)^2} = \frac{\epsilon\zeta}{(1+\zeta)^2}$$

and $4x = (1+\epsilon)^2$. Since $\epsilon t^2 - 2t + \epsilon = 0$, we have $t = (1-\sqrt{1-\epsilon^2})/\epsilon$. Therefore

$$\frac{\epsilon\zeta}{(1+\zeta)^2} = \frac{xt}{(1+t)^2} = \frac{x\epsilon}{2(1+\epsilon)} = \tfrac{1}{4}\epsilon\sqrt{x}$$

and $\zeta = (2-\sqrt{x}-2\sqrt{1-\sqrt{x}})/\sqrt{x}$. We are interested in $-(1/x)k(-\zeta, \rho x)$, and so we compute

$$\frac{-\zeta}{(1-\zeta)^2} = \frac{-\sqrt{x}}{4(1-\sqrt{x})}.$$

If

$$\frac{\rho x\tau}{(1+\tau)^2} = \frac{-\eta\zeta}{(1-\zeta)^2}$$

where $4\rho x = (1+\eta)^2$, then

$$-\eta\tau = 2\rho\sqrt{x}(1-\sqrt{x}) + \eta - 2\rho\sqrt{x}(1-\sqrt{x})\sqrt{1+\frac{\eta}{\rho\sqrt{x}(1-\sqrt{x})}}$$

and

$$-\eta/\tau = 2\rho\sqrt{x}(1-\sqrt{x}) + \eta + 2\rho\sqrt{x}(1-\sqrt{x})\sqrt{1+\frac{\eta}{\rho\sqrt{x}(1-\sqrt{x})}}.$$

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Therefore

$$-\frac{1}{x}k(-\zeta, \rho x) = \rho\frac{1-\eta\tau}{1-\eta/\tau} = \rho\frac{1-\sqrt{\rho}h(y)}{1+\sqrt{\rho}h(y)}$$

where

$$h(y) = \frac{1-y}{1+\sqrt{\rho}(1-y)}\sqrt{1+\frac{2\sqrt{\rho}y-1}{\rho y(1-y)}}$$

and $y = \sqrt{x}$ is restricted by $\frac{1}{2} \leqq y$, $\sqrt{\rho}y \leqq 1$. For $\frac{1}{4} \leqq \rho \leqq 4$ the maximum of $h(y)$ over the given interval occurs at $y = (1+\sqrt{\rho})/(3\sqrt{\rho})$. The indicated value for

$$\partial_\rho = \rho\frac{1-\sqrt{\rho}\max h(y)}{1+\sqrt{\rho}\max h(y)}$$

follows by substitution.

Case 2: $\rho > 4$. In this case, it follows from Lemma 3 that

$$\partial_\rho = \min_{1/4\leqq\rho x\leqq 1} -\frac{1}{x}k(-k^{-1}(x, \tfrac{1}{4}), \rho x).$$

Thus we are interested in $-(1/x)k(-\zeta, \rho x)$ where

$$x = k(\zeta, \tfrac{1}{4}) = \frac{\zeta}{(1-\zeta)^2}.$$

Now

$$-\frac{1}{x}k(-\zeta, \rho x) = \rho\frac{1-\eta\tau}{1-\eta/\tau}$$

where

$$\frac{\rho x\tau}{(1+\tau)^2} = \frac{-\eta\zeta}{(1-\zeta)^2} = -\eta x \quad\text{and}\quad 4\rho x = (1+\eta)^2.$$

Since

$$-\eta\tau = \eta + \tfrac{1}{2}\rho - \tfrac{1}{2}\sqrt{\rho^2+4\rho\eta}$$

and

$$-\eta/\tau = \eta + \tfrac{1}{2}\rho + \tfrac{1}{2}\sqrt{\rho^2+4\rho\eta},$$

we may write

$$-\frac{1}{x}k(-\zeta, \rho x) = \rho\frac{1-\sqrt{\rho}H(\eta)}{1+\sqrt{\rho}H(\eta)}$$

where

$$H(\eta) = \frac{\sqrt{\rho+4\eta}}{2+2\eta+\rho}$$

and η is in the interval from 0 to 1. The maximum of $H(\eta)$ occurs for $\eta = 1$, and the indicated value for

$$\partial_\rho = \rho \frac{1 - \sqrt{\rho}\max H(\eta)}{1 + \sqrt{\rho}\max H(\eta)}$$

follows by substitution.

Remarks. Since $\mathscr{S}^\rho \subset S^\rho$, it follows that $d_\rho \leqq \partial_\rho$. Therefore the values of ∂_ρ from Theorem 1 provide upper bounds for d_ρ. They improve the ones at the conclusion of Section 3. In the next section we shall use the fact that $d_\rho = \partial_\rho = \frac{1}{4}$ for $0 < \rho \leqq \frac{1}{4}$ and $d_\rho \leqq \partial_\rho < \min\{\rho, 1\}$ for $\frac{1}{4} < \rho < \infty$.

5. The Koebe constant in S^ρ. Let f belong to S^ρ, $0 < \rho < \infty$. Then a continuation of the function $g(w) = f^{-1}(\rho w)/\rho$ belongs to $S^{1/\rho}$. Therefore by Lemma 1 we have

$$|g(w)| \leqq k(|w|, d_{1/\rho}) \quad \text{in } |w| < 1,$$

where as before, $d_{1/\rho}$ denotes the Koebe constant for the class $S^{1/\rho}$. That is, for all z with $|f(z)| < \rho$ we have

$$|z| = \rho \left| g\left(\frac{f(z)}{\rho}\right) \right| \leqq \rho k\left(\frac{|f(z)|}{\rho}, d_{1/\rho}\right).$$

Since f is arbitrary and $d_\rho \leqq \rho$, we may conclude by letting $|z| \to 1$ that

$$(1) \quad 1 \leqq \rho k\left(\frac{d_\rho}{\rho}, d_{1/\rho}\right).$$

There is no restriction on ρ, and so we may replace ρ by $1/\rho$ in (1) to obtain the dual inequality

$$(2) \quad 1 \leqq \frac{1}{\rho} k(\rho d_{1/\rho}, d_\rho).$$

Motivated by the inequalities (1) and (2), we shall consider the system of equations

$$(3) \quad \begin{aligned} 1 &= \rho k\left(\frac{x}{\rho}, \tilde{y}\right) & \tilde{y} &= \max\{y, \tfrac{1}{4}\} \\ 1 &= \frac{1}{\rho} k(\rho y, \tilde{x}) & \tilde{x} &= \max\{x, \tfrac{1}{4}\} \end{aligned}$$

for fixed ρ, $0 < \rho < \infty$. The following theorem shows that an iterative solution of this system leads to a lower bound for d_ρ.

THEOREM 2. *For fixed* ρ, $0 < \rho < \infty$, *define* $x_0 = y_0 = 0$ *and*

$$(4) \quad \begin{aligned} \tilde{y}_n &= \max\{y_n, \tfrac{1}{4}\}, & x_{n+1} &= \rho k^{-1}\left(\frac{1}{\rho}, \tilde{y}_n\right) \\ \tilde{x}_n &= \max\{x_n, \tfrac{1}{4}\}, & y_{n+1} &= \frac{1}{\rho} k^{-1}(\rho, \tilde{x}_n) \end{aligned}$$

for $n \geqq 0$. *Then the sequences* $\{x_n\}$, $\{y_n\}$, $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$ *are nondecreasing, and their respective limits* x, y, $\tilde{x}$, $\tilde{y}$ *are a solution of the system* (3). *The Koebe constant* d_ρ *for the family* S^ρ *satisfies*

(5) $\quad \tilde{x} \leqq d_\rho \quad$ *and* $\quad \tilde{y} \leqq d_{1/\rho}$.

In particular, for $\rho \geqq 4$

$$d_\rho = \rho \frac{\sqrt{4+\rho} - \sqrt{\rho}}{\sqrt{4+\rho} + \sqrt{\rho}}.$$

Proof. Since $k(r, d) > r$ for $0 < r < 1$, it follows that x_n, y_n, $\tilde{x}_n$, $\tilde{y}_n$ are bounded by 1 and that the sequences are well defined. In addition, assuming that $\tilde{x}_n \leqq d_\rho$ and $\tilde{y}_n \leqq d_{1/\rho}$, by (1), (2), and (4) we have

$$\rho k\left(\frac{x_{n+1}}{\rho}, \tilde{y}_n\right) = 1 \leqq \rho k\left(\frac{d_\rho}{\rho}, d_{1/\rho}\right) \leqq \rho k\left(\frac{d_\rho}{\rho}, \tilde{y}_n\right)$$

and

$$\frac{1}{\rho} k(\rho y_{n+1}, \tilde{x}_n) = 1 \leqq \frac{1}{\rho} k(\rho d_{1/\rho}, d_\rho) \leqq \frac{1}{\rho} k(\rho d_{1/\rho}, \tilde{x}_n)$$

since $k(r, d)$ is decreasing in its second variable. From this it follows that $x_{n+1} \leqq d_\rho$ and $y_{n+1} \leqq d_{1/\rho}$ since $k(r, d)$ is increasing in its first variable. Thus, by induction, we have $x_n \leqq \tilde{x}_n \leqq d_\rho$ and $y_n \leqq \tilde{y}_n \leqq d_{1/\rho}$ for all n.

Assume now that $x_{n-1} \leqq x_n$ and $y_{n-1} \leqq y_n$. Then $\tilde{x}_{n-1} \leqq \tilde{x}_n$ and $\tilde{y}_{n-1} \leqq \tilde{y}_n$. Again, since $k(r, d)$ is decreasing in its second variable and increasing in its first variable, the relations

$$\rho k\left(\frac{x_n}{\rho}, \tilde{y}_n\right) \leqq \rho k\left(\frac{x_n}{\rho}, \tilde{y}_{n-1}\right) = 1 = \rho k\left(\frac{x_{n+1}}{\rho}, \tilde{y}_n\right)$$

and

$$\frac{1}{\rho} k(\rho y_n, \tilde{x}_n) \leqq \frac{1}{\rho} k(\rho y_n, \tilde{x}_{n-1}) = 1 = \frac{1}{\rho} k(\rho y_{n+1}, \tilde{x}_n)$$

imply that $x_n \leqq x_{n+1}$ and $y_n \leqq y_{n+1}$. Therefore, by induction, the sequences are all nondecreasing. Since $k(r, d)$ is continuous, the limits provide a solution of the system (3), and they remain lower bounds for d_ρ and $d_{1/\rho}$, respectively.

If $\rho \geqq 4$, then $d_{1/\rho} = \frac{1}{4}$ and so $\tilde{y} = \frac{1}{4}$. Consequently, the first equation in (3) implies

$$1 = \rho k\left(\frac{x}{\rho}, \frac{1}{4}\right) = \frac{x}{(1 - x/\rho)^2}$$

and $x = \rho(\sqrt{4+\rho} - \sqrt{\rho})/(\sqrt{4+\rho} + \sqrt{\rho})$. Therefore d_ρ has the indicated lower bound. One has equality since the function

$$f(z) = \rho k^{-1}\left(\frac{z}{\rho}, \frac{1}{4}\right)$$

belongs to S^ρ for $\rho \geqq 4$.

Remarks. Note that $d_\rho = \partial_\rho$ for $\rho \geqq 4$. The bounds obtained in Theorem 2 are computable. The following table compares ∂_ρ to the lower bounds (5) that we have computed for d_ρ. Also recall that ∂_ρ provides an upper bound for d_ρ for all ρ.

ρ	1/4	1/2	3/4	1	4/3	2	4	10	100
$d_\rho \geqq$	.250	.281	.348	.399	.454	.537	.686	.839	.980
∂_ρ	.250	.332	.389	.433	.481	.553	.686	.839	.980

The lower bound for the Koebe constant d_1 for the bi-univalent class S^1 was computed more closely and found to be .39979.... We have not been able to improve it to .4.

6. The second coefficient for S^ρ. It is possible to give a sharp estimate for the coefficient a_2 in the class $\mathcal{S}^\rho$.

THEOREM 3. *For $f(z) = z + a_2z^2 + \ldots$ in the class $\mathcal{S}^\rho$ we have*

$$|a_2| \leqq \begin{cases} \dfrac{2}{3\rho}(4\sqrt{\rho} - 1 - \rho) & \text{for } \tfrac{1}{4} \leqq \rho \leqq 4 \\ 2/\rho & \text{for } \rho > 4 \end{cases}$$

and the estimates are sharp.

Proof. Case 1: $\frac{1}{4} \leqq \rho \leqq 4$. Since $f(z) = z + a_2z^2 + \ldots$ belongs to $\mathcal{S}^\rho$, we may write $f = \psi \circ \phi^{-1}$ where for $x = 1/|a|$

$$\frac{1}{a}\psi(z) = z + b_2z^2 + \ldots \text{ is in } S(\rho x)$$

and

$$\frac{1}{a}\phi(z) = z + c_2z^2 + \ldots \text{ is in } S(x).$$

Since $a_2 = (b_2 - c_2)/a$, Lemma 2 implies

$$|a_2| \leqq x(|b_2| + |c_2|) \leqq \frac{2}{\rho}(1 - \sqrt{\rho x})(3\sqrt{\rho x} - 1) + 2(1 - \sqrt{x})(3\sqrt{x} - 1).$$

At the same time, x is restricted by $\frac{1}{4} \leqq \rho x$, $x \leqq 1$. As x varies through its admissible values, the maximum occurs when

$$\sqrt{x} = \frac{1}{3} + \frac{1}{3\sqrt{\rho}},$$

and the first bound is obtained by substituting this value. This estimate is sharp for the corresponding function of the form of Example 2 (§3).

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Case 2: $\rho > 4$. In this case

$$\frac{1}{\rho} f^{-1}(\rho w) = w - a_2 \rho w^2 + \dots$$

belongs to S, so that $|a_2|\rho \leqq 2$. Equality occurs for

$$f(z) = \rho k^{-1}\left(\frac{z}{\rho}, \frac{1}{4}\right),$$

which belongs to $\mathscr{S}^{\rho}$.

7. Bounds for the second coefficient in S^ρ. We shall derive three different estimates for the second coefficient a_2 of a function $f(z) = z + a_2 z^2 + \dots$ belonging to S^ρ. Although valid for all ρ, they will be valuable for large, small, and intermediate ranges of ρ, respectively.

First bound. If $f \in S^\rho$, then $\rho^{-1}f^{-1}(\rho w) = w - a_2\rho w^2 + \dots$ belongs to S, so that

$$|a_2| \leqq 2/\rho$$

just as in the second part of Theorem 3. Although valid for all ρ, this bound is not of interest for $\rho < 1$. However, it is sharp for all $\rho \geqq 4$ since the function $\rho k^{-1}(z/\rho, \frac{1}{4})$ belongs to S^ρ for $\rho \geqq 4$.

Second bound. For $f \in S^\rho$ we have a bound $d_\rho \geqq \tilde{x}$ from Theorem 2, and so by Lemma 2 we have the estimate

$$|a_2| \leqq g(\rho) \quad \text{where} \quad g(\rho) = \frac{2}{\tilde{x}}(1 - \sqrt{\tilde{x}})(3\sqrt{\tilde{x}} - 1).$$

For $0 < \rho \leqq \frac{1}{4}$ this bound coincides with the sharp estimate $|a_2| \leqq 2$ in $S = S^\rho$.

Third bound. This estimate parallels the work of Lewin [**4**] for the biunivalent class. Using the notation of [**2**], we let

$$l_1 = a_2, \quad l_2 = a_3 - a_2^2, \quad l_3 = a_4 - \frac{5}{2}a_2a_3 - \frac{3}{2}a_2^3,$$

$$l_4 = a_5 - 3a_2a_4 - \frac{3}{2}a_3^2 + \frac{37}{6}a_2^2a_3 - \frac{8}{3}a_2^4$$

be associated with $f(z) = z + a_2z^2 + \dots$ in S^ρ. Then for the function $\rho^{-1}f^{-1}(\rho w)$ the corresponding $\tilde{l}_j$ satisfy $\tilde{l}_j = -\rho^j l_j$, $j = 1, 2, 3, 4$.

With this notation the Grunsky inequalities for $f \in S$ and $N = 2$ become (see [**2**] or [**4**, p. 67])

$$|l_2x_1^2 + 2(l_3 + \tfrac{1}{2}l_1l_2)x_1x_2 + (l_4 + l_1l_3 + \tfrac{1}{3}l_1^2l_2)x_2^2| \leqq |x_1|^2 + \tfrac{1}{2}|x_2|^2$$

for all $x_1, x_2 \in \mathbf{C}$. The corresponding inequalities for $\rho^{-1}f^{-1}(\rho w)$ are

$$|-\rho^2 l_2 y_1^2 + 2\rho^3(-l_3 + \tfrac{1}{2}l_1 l_2)y_1 y_2 + \rho^4(-l_4 + l_1 l_3 - \tfrac{1}{3}l_1^2 l_2)y_2^2| \leqq |y_1|^2 + \tfrac{1}{2}|y_2|^2$$

for all $y_1, y_2 \in \mathbf{C}$. We choose $x_1 = l_1$, $x_2 = \beta > 0$, $y_1 = -l_1/\rho$, $y_2 = \beta/\rho^2 > 0$ and add both inequalities to obtain

$$2\beta(2 + \beta)|l_1 l_3| \leqq \left(1 + \frac{1}{\rho^2}\right)|l_1|^2 + \frac{1}{2}\left(1 + \frac{1}{\rho^4}\right)\beta^2.$$

The optimal choice of β is

$$\beta = \frac{\rho^2|l_1|^2(\rho^2 + 1)(c + 1)}{\rho^4 + 1}$$

where

$$c = \sqrt{1 + \frac{2(\rho^4 + 1)}{\rho^2|l_1|^2(\rho^2 + 1)}},$$

and it leads to the inequality

$$(6) \quad |l_1 l_3| \leqq \frac{\rho^4 + 1}{2\rho^4(c + 1)}.$$

In a similar fashion the Grunsky inequalities for $\sqrt{f(z^2)} \in S$ with $N = 3$ and $x_2 = 0$ become (see [**4**, p. 67])

$$\left|\frac{1}{2}l_1 x_1^2 + \left(l_2 + \frac{1}{4}l_1^2\right)x_1 x_3 + \left(\frac{1}{2}l_3 + \frac{1}{4}l_1 l_2 + \frac{1}{24}l_1^3\right)x_3^3\right| \leqq |x_1|^2 + \frac{1}{3}|x_3|^2$$

for all $x_1, x_3 \in \mathbf{C}$. The corresponding inequalities for $\rho^{-1}f^{-1}(\rho w)$ are

$$\left|-\frac{1}{2}\rho l_1 y_1^2 + \rho^2\left(-l_2 + \frac{1}{4}l_1\right)y_1 y_3 + \rho^3\left(-\frac{1}{2}l_3 + \frac{1}{4}l_1 l_2 - \frac{1}{24}l_1^3\right)y_3^2\right| \leqq |y_1|^2 + \frac{1}{3}|y_3|^2$$

for all $y_1, y_3 \in \mathbf{C}$. Now choose $x_1 = l_1$, $x_3 = \beta > 0$, $y_1 = -il_1/\sqrt{\rho}$, $y_3 = i\beta/(\rho\sqrt{\rho})$ and add both inequalities to obtain

$$\left|\left(1 + \frac{1}{2}\beta + \frac{1}{12}\beta^2\right)l_1^3 + \beta^2 l_3\right| \leqq \left(1 + \frac{1}{\rho}\right)|l_1|^2 + \frac{1}{3}\left(1 + \frac{1}{\rho^3}\right)\beta^2.$$

Multiplying by $|l_1|$ and using the bound in (6), we find

$$\left(1 + \frac{1}{2}\beta + \frac{1}{12}\beta^2\right)|l_1|^4 \leqq \beta^2\frac{\rho^4 + 1}{2\rho^4(c + 1)} + \frac{\rho + 1}{\rho}|l_1|^3 + \frac{\rho^3 + 1}{3\rho^3}\beta^2|l_1|,$$

and so

$$\left(\frac{1}{12}|l_1|^4 - \frac{\rho^4+1}{2\rho^4(c+1)} - \frac{\rho^3+1}{3\rho^3}|l_1|\right)\beta^2 + \frac{1}{2}|l_1|^4\beta + |l_1|^4 - \frac{\rho+1}{\rho}|l_1|^3 \leqq 0$$

for all positive β. By letting $\beta \to \infty$ we learn that the coefficient of β^2 is nonpositive, and evidently the coefficient of β is nonnegative. Therefore, the maximum of this quadratic expression in β occurs for a positive β. Since this maximum value is nonpositive, the discriminant of the expression is also nonpositive. This discriminant condition is

$$\frac{1}{4}|l_1|^8 - 4\left(\frac{1}{12}|l_1|^4 - \frac{\rho^4+1}{2\rho^4(c+1)} - \frac{\rho^3+1}{3\rho^3}|l_1|\right) \times \left(|l_1|^4 - \frac{\rho+1}{\rho}|l_1|^3\right) \leqq 0.$$

There, if we let $h(\rho)$ be the smallest positive zero of the function

$$\frac{1}{4}x^5 - 4\left(\frac{1}{12}x^4 - \frac{\rho^4+1}{2\rho^4[c(x)+1]} - \frac{\rho^3+1}{3\rho^3}x\right)\left(x - \frac{\rho+1}{\rho}\right)$$

where

$$c(x) = \sqrt{1 + \frac{2(\rho^4+1)}{\rho^2(\rho^2+1)x^2}},$$

then $|a_2| = |l_1| \leqq h(\rho)$.

The following theorem simply summarizes the bounds that have been obtained.

THEOREM 4. *If $f(z) = z + a_2z^2 + \ldots$ belongs to S^ρ, then*

$$|a_2| \leqq \min\left\{\frac{1}{\rho}, g(\rho), h(\rho)\right\}.$$

In the following table we have computed this estimate of $|a_2|$ in the class S^ρ for various values of ρ.

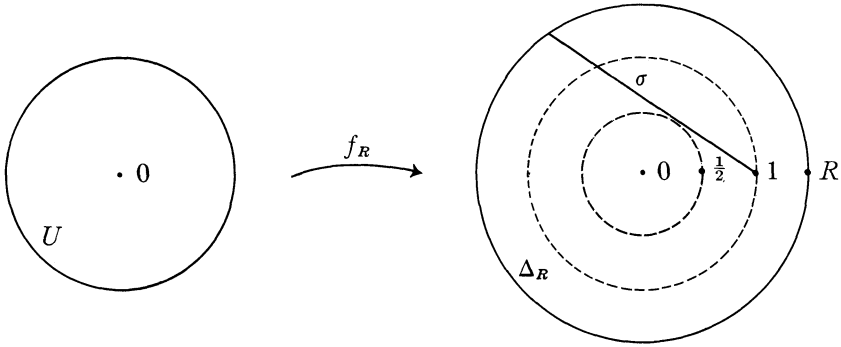
ρ	1/4	1/2	3/4	1	4/3	2	4	10	100
$\lvert a_2\rvert \leqq$	2.000	1.973	1.782	1.509	1.337	1.000	.500	.200	.020

The special case $\rho = 1$ is Lewin's result [**4**] in the biunivalent case.

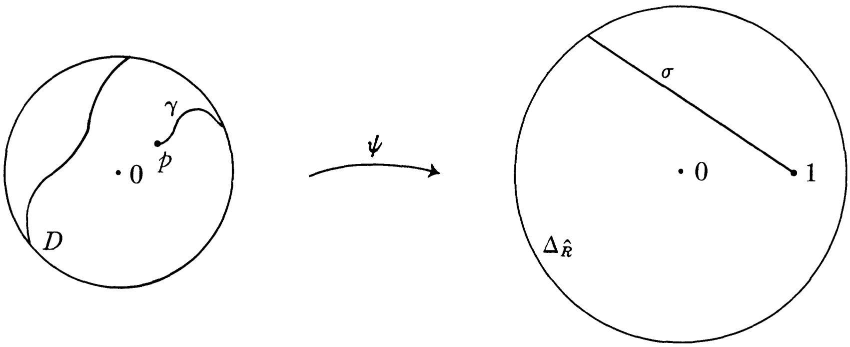
8. A bi-univalent function that is not in $\mathcal{S}^1$. The class $\mathcal{S}^\rho$ is always a subset of S^ρ. However, for $0 < \rho \leqq \frac{1}{4}$ both classes are S, and for $\rho = \infty$

both contain only the identity mapping. The question arises whether the classes are the same for $\frac{1}{4} < \rho < \infty$.

By constructing a function in $S^1 \backslash \mathscr{S}^1$ we shall show that the classes S^1 and $\mathscr{S}^1$ are not the same. This settles a question raised in [**4**]. Similar constructions can be made for other values of ρ.



For $R > 1$, let Δ_R denote the disk $|w| < R$ minus a straight line slit σ that osculates the upper half of the circle $|w| = \frac{1}{2}$ and ends at the point $w = 1$. Let f_R be the Riemann mapping function, $f_R(0) = 0, f_R'(0) > 0$, of the unit disk U onto Δ_R. As $R \to \infty$ the functions f_R converge to a rotation and dilation of a function of the form of Example 3 in §3. In particular, $\lim_{R\to\infty} f_R'(0) > 1$. As $R \to 1$ the functions f_R converge to a function that is bounded by 1 and is different from the identity. Therefore, $\lim_{R\to 1} f_R'(0) < 1$. Thus we may choose a value $\hat{R}$ so that $f_{\hat{R}}'(0) = 1$.



The branch of the inverse function $f_{\hat{R}}^{-1}$ that is defined for $|w| < \frac{1}{2}$ extends by the Schwarz reflection principle to a univalent analytic function in $|w| < 1$. Therefore $f_{\hat{R}} \in S^1$.

We shall show that $f_{\hat{R}}$ does not belong to $\mathscr{S}^1$. Indeed, suppose that

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$f_{\hat{R}} = \psi \circ \phi^{-1}$ where ϕ and ψ are analytic and univalent in U, $\phi(0) = \psi(0) = 0$, $\phi'(0) = \psi'(0)$, and $\phi(U) \supset U$, $\psi(U) \supset U$. Then $D = \phi^{-1}(U)$ is a subset of U, and ψ maps D onto $\Delta_{\hat{R}}$.

Since ψ extends to U and $\psi(U) \supset U$, part of the boundary of D is an analytic slit γ that ψ maps to the segment of σ that is in $|w| < 1$. Since $\psi(U)$ is simply connected, it contains the point $w = 1$, and we denote the corresponding endpoint of γ by p. In the other direction, the function ϕ maps D onto U. On the one hand, each point of $\gamma \backslash \{p\}$ corresponds to two points of ∂U. On the other hand, ϕ has a unique continuation to U, and hence to γ. This presents a contradiction.

Consequently, the function $f_{\hat{R}}$ belongs to S^1, but not to $\mathscr{S}^1$.

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