GAUSS SUMS FOR $U(2n, q^2)$

by DAE SAN KIM†

For my mother, Byung Keum Lim

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Abstract. For a lifted nontrivial additive character λ' and a multiplicative character λ of the finite field with q^2 elements, the "Gauss" sums $\sum \lambda'(\operatorname{tr} g)$ over $g \in SU(2n, q^2)$ and $\sum \lambda(\det g)\lambda'(\operatorname{tr} g)$ over $g \in U(2n, q^2)$ are considered. We show that the first sum is a polynomial in q with coefficients involving averages of "bihyperkloosterman sums" and that the second one is a polynomial in q with coefficients involving powers of the usual twisted Kloosterman sums. As a consequence, we can determine certain "generalized Kloosterman sums over nonsingular Hermitian matrices", which were previously determined by J. H. Hodges only in the case that one of the two arguments is zero.

1. Introduction. Let λ be a nontrivial additive character of the finite field $\mathbf{F}_q$, λ' the lifting of λ to $\mathbf{F}_{q^2}$ (cf. (2.3)), χ a multiplicative character of $\mathbf{F}_{q^2}$. Then we consider the exponential sum

$$\sum_{g \in SU(2n, q^2)} \lambda'(\operatorname{tr} g), \tag{1.1}$$

where $SU(2n, q^2)$ is a special unitary group over $\mathbf{F}_{q^2}$ (cf. (2.8)) and $\operatorname{tr} g$ is the trace of g. Also, we consider

$$\sum_{g \in U(2n, q^2)} \chi(\det g)\lambda'(\operatorname{tr} g), \tag{1.2}$$

where $U(2n, q^2)$ is a unitary group over $\mathbf{F}_{q^2}$ (cf. (2.4) and (2.6)) and $\det g$ is the determinant of g.

The purpose of this paper is to find explicit expressions for the sums (1.1) and (1.2). We will show that (1.1) is a polynomial in q with coefficients involving the averages (over $\mathbf{F}_q^\times$) of certain bihyperkloosterman sums over $\mathbf{F}_{q^2}$ (cf. (4.18) and (4.20)). On the other hand, (1.2) is a polynomial in q with coefficients involving powers of twisted Kloosterman sums.

In [**2**], Hodges expressed certain exponential sums in terms of what we call the "generalized Kloosterman sum over nonsingular Hermitian matrices" $K_{Herm,t}(A, B)$, where A, B are $t \times t$ Hermitian matrices over $\mathbf{F}_{q^2}$ (*cf.* (7.1)). Some of its general properties were investigated in [**2**], and, for A or B zero, it was evaluated in [**1**]. However, they have never been explicitly computed for both A and B nonzero. From a corollary to the main theorem in [2] and Theorem 6.1, we will able to find an explicit expression for $K_{Herm,2n}(a^2C^{-1}, C)$, where C is a nonsingular Hermitian matrix over $\mathbf{F}_{q^2}$ of size $2n$ and $0 \neq a \in \mathbf{F}_q$.

Similar sums for other classical groups over a finite field have been considered ([**3**]–[**7**]) and the results for these sums will appear in various places.

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THEOREM A. *The sum* $\sum_{g\in SU(2n,q^2)}\lambda'(\operatorname{tr} g)$ *in* (1.1) *equals*

$$q^{2n^2-n-2}\left\{\sum_{r=0}^{[n/2]} q^{r(2r+1)}\begin{bmatrix} n \\ 2r\end{bmatrix}_{q^2}\prod_{j=1}^{2r}(q^j+(-1)^j)\right.$$
$$\times\sum_{l=1}^{[n-2r+2/2]} q^{2l}\sum_{\alpha\in\mathbf{F}_q^\times} BK_{n-2r+1-2l}(\lambda';1;(-1)^{l-1}\alpha;1;(-1)^{l-1}\alpha^{-1}:q^2)$$
$$\times\sum\prod_{\nu=1}^{l-1}(q^{2j_\nu-4\nu}-1)$$
$$+\sum_{r=0}^{[n-1/2]} q^{(r+1)(2r+1)}\begin{bmatrix} n \\ 2r+1\end{bmatrix}_{q^2}\prod_{j=1}^{2r+1}(q^j+(-1)^j)$$
$$\times\sum_{l=1}^{[n-2r+1/2]} q^{2l}\sum_{\alpha\in\mathbf{F}_q^\times} BK_{n-2r-2l}(\lambda';1;(-1)^{l-1}\alpha\theta;1;(-1)^{l-1}\alpha^{-1}\theta^{-1}:q^2)$$
$$\left.\times\sum\prod_{\nu=1}^{l-1}(q^{2j_\nu-4\nu}-1)\right\},$$

where $BK_t(\lambda';-;-;-;-:q^2)$ *is the bihyperkloosterman sum over* $\mathbf{F}_{q^2}$ *defined in* (4.20), $\theta\in\mathbf{F}_{q^2}$ *is a fixed nonzero element with* $\operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}\theta=0$, *and the first and second unspecified sums are respectively over all integers* $j_1,\cdots,j_{l-1}$ *satisfying* $2l-1\le j_{l-1}\le j_{l-2}\le\cdots\le j_1\le n-2r+1$ *and over the same set of integers satisfying* $2l-1\le j_{l-1}\le j_{l-2}\le\cdots\le j_1\le n-2r$.

THEOREM B. *The sum* $\sum_{g\in U(2n,q^2)}\chi(\det g)\lambda'(\operatorname{tr} g)$ *in* (1.2) *equals*

$$q^{2n^2-n-2}\sum_{r=0}^{n}\chi(-1)^r q^{\frac{1}{2}r(r+1)}\begin{bmatrix} n \\ r\end{bmatrix}_{q^2}\prod_{j=1}^{r}(q^j+(-1)^j)$$
$$\times\sum_{l=1}^{[n-r+2/2]} q^{2l}K(\lambda',\chi^{q-1};1,1:q^2)^{n-r+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{2j_\nu-4\nu}-1),$$

where $K(\lambda',\chi^{q-1};1,1:q^2)$ *is the usual twisted Kloosterman sum defined in* (4.5), *and the innermost sum is over all integers* $j_1,\cdots,j_{l-1}$ *satisfying* $2l-1\le j_{l-1}\le j_{l-2}\le\cdots\le j_1\le n-r+1$.

THEOREM C. *Let* $0\ne a\in\mathbf{F}_q$. *Then, for any nonsingular Hermitian matrix over* $\mathbf{F}_{q^2}$ *of size* $2n$, *the following Kloosterman sum over nonsingular Hermitian matrices* (*cf.* (7.1)) *is independent of* C, *and*

$$K_{Herm,2n}(a^2C^{-1},C)=\sum_{g\in U(2n,q^2)}\lambda'(\operatorname{tr} g),$$

so that it equals the expression in Theorem B above with χ *trivial*, $\lambda'=\lambda'_a$ (*cf.* (2.2) *and* (2.3)).

The above Theorem A, B and C are respectively stated as Theorem 5.2, Theorem 6.1 and Theorem 7.1.

2. Preliminaries. In this section, we will fix some notations that will be used throughout this paper, describe some basic groups and mention the q-binomial theorem.

Let $\mathbf{F}_q$ and $\mathbf{F}_{q^2}$ denote respectively the finite field with q elements, $q=p^d$(p any prime, d a positive integer), and the quadratic extension of $\mathbf{F}_q$, and let $\tau: \mathbf{F}_{q^2} \to \mathbf{F}_{q^2}$ be the Frobenius automorphism of $\mathbf{F}_{q^2}$ given by

$$\alpha^\tau = \alpha^q. \tag{2.1}$$

Note that, for $\alpha \in \mathbf{F}_{q^2}$,

$$\mathrm{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}\alpha = \alpha + \alpha^\tau,\ N_{\mathbf{F}_{q^2}/\mathbf{F}_q}\alpha = \alpha\alpha^\tau.$$

Let λ be an additive character of $\mathbf{F}_q$. Then $\lambda = \lambda_a$ for a unique $a \in \mathbf{F}_q$, where, for $\alpha \in \mathbf{F}_q$,

$$\lambda_a(\alpha) = \exp\left\{\frac{2\pi i}{p}(a\alpha + (a\alpha)^p + \ldots + (a\alpha)^{p^{d-1}})\right\}. \tag{2.2}$$

It is nontrivial if $a \neq 0$. For such a λ, λ' denotes the additive character λ lifted to $\mathbf{F}_{q^2}$. Thus

$$\lambda' = \lambda \circ \mathrm{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}. \tag{2.3}$$

Note that λ' is nontrivial if λ is. Likewise, for a multiplicative character ψ of $\mathbf{F}_q$, the lifting of that to $\mathbf{F}_{q^2}$ is denoted by ψ'. So $\psi' = \psi \circ N_{\mathbf{F}_{q^2}/\mathbf{F}_q}$.

Here $\mathrm{tr}\, A$ and $\det A$ denote respectively the trace and determinant of A for a square matrix A, $*B = {}^t(\beta_{ij}^\tau)$ for any matrix $B = (\beta_{ij})$ over $\mathbf{F}_{q^2}$ (cf. (2.1)), where the "t" indicates the transpose. We will say that B is Hermitian if $*B = B$.

$GL(n, q)$ is the group of all nonsingular $n \times n$ matrices with entries in $\mathbf{F}_q$. Then

$$U(2n, q^2) = \{g \in GL(2n, q^2) \mid *gJg = J\}, \tag{2.4}$$

where

$$J = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}. \tag{2.5}$$

We write $g \in U(2n, q^2)$ as

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C, D are of size n. Then (2.4) is given by

$$\begin{aligned} &U(2n, q^2) \\ &= \left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in GL(2n, q^2) \mid *AC + *CA = 0, *AD + *CB = 1_n, *BD + *DB = 0\right\} \\ &= \left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in GL(2n, q^2) \mid A*B + B*A = 0, A*D + B*C = 1_n, C*D + D*C = 0\right\}. \end{aligned} \tag{2.6}$$

$P(2n, q^2)$ is the maximal parabolic subgroup of $U(2n, q^2)$ defined by

$$P(2n, q^2) = \left\{\begin{bmatrix} A & 0 \\ 0 & *A^{-1} \end{bmatrix}\begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \,\middle|\, \begin{matrix} A \in GL(n, q^2),\ B \text{ is of} \\ \text{n×n over } \mathbf{F}_{q^2} \text{ with } *B + B = 0 \end{matrix}\right\}. \tag{2.7}$$

$$SU(2n, q^2) = \{g \in U(2n, q^2) \mid \det g = 1\}, \tag{2.8}$$

which is a subgroup of index $q + 1$ in $U(2n, q^2)$.

For integers n, r with $0 \le r \le n$, we define the q-binomial coefficients as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{j=0}^{r-1} (q^{n-j} - 1)/(q^{r-j} - 1). \tag{2.9}$$

The order of the group $GL(n, q)$ is denoted by

$$g_n(q) = \prod_{j=0}^{n-1} (q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n} (q^j - 1). \tag{2.10}$$

Then we have:

$$\frac{g_n(q)}{g_{n-r}(q) g_r(q)} = q^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_q, \tag{2.11}$$

for integers n, r with $0 \le r \le n$.

For x an indeterminate, n a nonnegative integer,

$$(x; q)_n = (1 - x)(1 - xq) \dots (1 - xq^{n-1}). \tag{2.12}$$

Then the q-binomial theorem says

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{\binom{r}{2}} x^r = (x; q)_n. \tag{2.13}$$

$[y]$ denotes the greatest integer $\le y$, for a real number y.

3. Bruhat decomposition. In this section, we will discuss the Bruhat decomposition of $U(2n, q^2)$ with respect to the maximal parabolic subgroup $P(2n, q^2)$ of $U(2n, q^2)$ (cf. (2.7)).

This decomposition (in fact, its slight variants (3.8) and Cor. 3.3) will play a decisive role in deriving Theorem 5.2 and Theorem 6.1. The next theorem about the decomposition can be proved by slightly modifying the corresponding proof in [**7**]. So we will not provide a proof for that. We demonstrate, as a simple application of this decomposition, that it yields the well-known formula for the order of the group $U(2n, q^2)$ when combined with the q-binomial theorem.

THEOREM 3.1. (*a*) *There is a one-to-one correspondence*

$$P(2n, q^2) \backslash U(2n, q^2) \to GL(n, q^2) \backslash \Lambda$$

given by

$$P(2n, q^2) \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto GL(n, q^2)[C \quad D],$$

where

$$\Lambda = \{[C \quad D] \mid C, D \ n \times n \text{ matrices over } \mathbf{F}_{q^2}, \ \text{rank}[C \quad D] = n, \ C^*D + D^*C = 0\}.$$

(*b*) *For given* $[C \quad D] \in \Lambda$, *there exists a unique* r $(0 \le r \le n)$, $g \in GL(n, q^2)$, $p \in P(2n, q^2)$ *such that*

$$g[C \quad D]p = \begin{bmatrix} 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix}.$$

(*c*)

$$U(2n, q^2) = \coprod_{r=0}^{n} P\sigma_r P,$$

where $P = P(2n, q^2)$ *and*

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix} \in U(2n, q^2). \tag{3.1}$$

Put

$$\begin{aligned} Q = Q(2n, q^2) &= \{g \in P(2n, q^2) \mid \det g = 1\} \\ &= \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^*A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \,\middle|\, A \in GL(n, q^2), \det A \in \mathbf{F}_q^\times, {}^*B + B = 0 \right\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} Q^- = Q^-(2n, q^2) &= \{g \in P(2n, q^2) \mid \det g = -1\} \\ &= \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^*A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \,\middle|\, A \in GL(n, q^2), \operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}(\det A) = 0, {}^*B + B = 0 \right\}. \end{aligned} \tag{3.3}$$

Then $Q(2n, q^2)$ is a subgroup of index $q + 1$ in $P(2n, q^2)$ and

$$U(2n, q^2) = \coprod_{r=0}^{n} P\sigma_r Q. \tag{3.4}$$

Write, for each r $(0 \le r \le n)$,

$$A_r = A_r(q^2) = \{p \in P(2n, q^2) \mid \sigma_r p \sigma_r^{-1} \in P(2n, q^2)\}, \tag{3.5}$$

$$B_r = B_r(q^2) = \{p \in Q(2n, q^2) \mid \sigma_r p \sigma_r^{-1} \in P(2n, q^2)\}. \tag{3.6}$$

Then B_r is a subgroup of A_r of index $q + 1$.

Expressing $U(2n, q^2)$ as a disjoint union of right cosets of $P(2n, q^2)$, the Bruhat decomposition in (c) of Theorem 3.1 and the decomposition in (3.4) can be rewritten as follows.

Corollary 3.2.

$$U(2n, q^2) = \coprod_{r=0}^{n} P\sigma_r (A_r \backslash P), \tag{3.7}$$

$$U(2n, q^2) = \coprod_{r=0}^{n} P\sigma_r (B_r \backslash Q), \tag{3.8}$$

where $P = P(2n, q^2)$, *and* σ_r, Q, A_r, B_r *are respectively as in* (3.1), (3.2), (3.5), (3.6).

Observing that $\det \sigma_r = (-1)^r$, we get from (3.8) the following decomposition for $SU(2n, q^2)$.

Corollary 3.3.

$$SU(2n, q^2) = \coprod_{\substack{0 \le r \le n \\ r \text{ even}}} Q\sigma_r (B_r \backslash Q) \amalg \left(\coprod_{\substack{0 \le r \le n \\ r \text{ odd}}} Q^- \sigma_r (B_r \backslash Q) \right),$$

where $Q^- = Q^-(2n, q^2)$ *is as in* (3.3).

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Write $p \in P(2n, q^2)$ *as*

$$p = \begin{bmatrix} A & 0 \\ 0 & {}^*A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix}, \tag{3.9}$$

with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad {}^*A^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ -{}^*B_{12} & B_{22} \end{bmatrix}, \tag{3.10}$$
$$B_{11} + {}^*B_{11} = 0, \qquad B_{22} + {}^*B_{22} = 0.$$

Here A_{11}, A_{12}, A_{21}, *and* A_{22} *are respectively of sizes* $r \times r$, $r \times (n-r)$, $(n-r) \times r$, *and* $(n-r) \times (n-r)$, *and similarly for* ${}^*A^{-1}$, B.

Then, by multiplying out, we see that $\sigma_r p \sigma_r^{-1} \in P(2n, q^2)$ *if and only if* $A_{11}B_{11} - A_{12}{}^*B_{12} = 0$, $A_{12} = 0$, $E_{21} = 0$ *if and only if* $A_{12} = 0$, $B_{11} = 0$. *Hence*

$$|A_r(q^2)| = g_r(q^2) g_{n-r}(q^2) q^{n^2} q^{r(2n-3r)}, \tag{3.11}$$

where $g_n(q^2)$ *is as in* (2.10). *Also*,

$$|P(2n, q^2)| = g_n(q^2) q^{n^2}. \tag{3.12}$$

From (2.11), (3.11) and (3.12), we get

$$|A_r(q^2) \backslash P(2n, q^2)| = q^{r^2} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}. \tag{3.13}$$

This will be used later in Section 5 and 6. Also, from (2.10), (3.12) and (3.13),

$$|P(2n, q^2)|^2 \, |A_r(q^2)|^{-1} = q^{2n^2-n} \prod_{j=1}^{n} (q^{2j} - 1) \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} q^{r^2}. \tag{3.14}$$

From (3.7),

$$|U(2n, q^2)| = \sum_{r=0}^{n} |P(2n, q^2)|^2 \, |A_r(q^2)|^{-1}. \tag{3.15}$$

Applying the q-binomial theorem (2.12) with $x = -q$ and with q^2 instead of q, and from (3.14) and (3.15), we get the following theorem. We note here that the result in Theorem 3.4 and Proposition 3.5 were mentioned in [**1**].

THEOREM 3.4.

$$|U(2n, q^2)| = q^{\binom{2n}{2}} \prod_{j=1}^{2n} (q^j - (-1)^j). \tag{3.16}$$

Proof.

$$\begin{aligned} |U(2n, q^2)| &= q^{2n^2-n} \prod_{j=1}^{n} (q^{2j} - 1) \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} q^{r^2} \\ &= q^{2n^2-n} (-q; q^2)_n \prod_{j=1}^{n} (q^{2j} - 1) \\ &= q^{\binom{2n}{2}} \prod_{j=1}^{2n} (q^j - (-1)^j). \end{aligned}$$

PROPOSITION 3.5. *For each positive integer r, let h_r denote the number of all $r\times r$ nonsingular Hermitian matrices over $\mathbf{F}_{q^2}$. Then*

$$h_r = q^{\binom{r}{2}} \prod_{j=1}^{r} (q^j + (-1)^j). \tag{3.17}$$

4. Certain Kloosterman sums. For a nontrivial additive character λ of $\mathbf{F}_q$, ψ a multiplicative character of $\mathbf{F}_q$, $t\times t$ matrices A, B over $\mathbf{F}_q$, we define the "twisted" Kloosterman sum $K_{GL(t,q)}(\lambda,\psi;A,B)$ for $GL(t,q)$ as

$$K_{GL(t,q)}(\lambda,\psi;A,B) := \sum_{g\in GL(t,q)} \psi(\det g)\lambda(\operatorname{tr}(Ag+Bg^{-1})). \tag{4.1}$$

Further, if $A=a1_t$, $B=b1_t(a,b\in\mathbf{F}_q)$ are scalar matrices, then (4.1) will be simply written as

$$K_{GL(t,q)}(\lambda,\psi;a,b) := \sum_{g\in GL(t,q)} \psi(\det g)\lambda(a\operatorname{tr} g + b\operatorname{tr} g^{-1}). \tag{4.2}$$

Also, we define the Kloosterman sum $K_{SL(t,q)}(\lambda;A,b)$ for $SL(t,q)$ as

$$K_{SL(t,q)}(\lambda;A,B) := \sum_{g\in SL(t,q)} \lambda(\operatorname{tr}(Ag+Bg^{-1})). \tag{4.3}$$

Again, if $A=a1_t$, $B=b1_t$ $(a, b\in\mathbf{F}_q)$ are scalar matrices, then (4.3) is written simply as

$$K_{SL(t,q)}(\lambda;a,b) := \sum_{g\in SL(t,q)} \lambda(a\operatorname{tr} g + b\operatorname{tr} g^{-1}). \tag{4.4}$$

For ψ trivial, an explicit expression for (4.2) was obtained in [**3**]. Also, (4.2) becomes trivial unless both a and b are not zero, as we note in the following.

REMARKS. (1) If $a=b=0$, then $K_{GL(t,q)}(\lambda,\psi;0,0)=g_t(q)$ (cf. (2.10)) for ψ trivial and $K_{GL(t,q)}(\lambda,\psi;0,0)=0$ for ψ nontrivial.

(2) If exactly one of a, b is zero, say $a\neq 0$, $b=0$, then it is $\psi(a)^{-t}\sum_{g\in GL(t,q)}\psi(\det g)\lambda(\operatorname{tr} g)$ and equals $\psi(a)^{-t}q^{\binom{t}{2}}G(\psi,\lambda)^t$, where $G(\psi,\lambda)$ is the usual Gauss sum $G(\psi,\lambda)=\sum_{\alpha\in\mathbf{F}_q^\times}\psi(\alpha)\lambda(\alpha)$. For this, see [6].

(3) For $t=1$, (4.2) is the usual twisted Kloosterman sum which is denoted by

$$K(\lambda,\psi;a,b:q) = K_{GL(1,q)}(\lambda,\psi;a,b) = \sum_{\alpha\in\mathbf{F}_q^\times} \psi(\alpha)\lambda(a\alpha+b\alpha^{-1}). \tag{4.5}$$

Assume now that $ab\neq 0$. Then, following an argument analogous to that leading up to (4.16), we get the recursive formula (4.6) in below. For $t\geq 2$, $a,b\in\mathbf{F}_q^\times$,

$$\begin{aligned} K_{GL(t,q)}(\lambda,\psi;a,b) &= q^{t-1}K_{GL(t-1,q)}(\lambda,\psi;a,b)K(\lambda,\psi;a,b) \\ &\quad + \psi(-a^{-1}b)q^{2t-2}(q^{t-1}-1)K_{GL(t-2,q)}(\lambda,\psi;a,b), \end{aligned} \tag{4.6}$$

where $K(\lambda,\psi;a,b)=K(\lambda,\psi;a,b:q)$ is the usual twisted Kloosterman sum in (4.5), and we understand that $K_{GL(0,q)}(\lambda,\psi;a,b)=1$.

As in [3], we can deduce, by induction on t, the following theorem from the recursive formula (4.6).

THEOREM 4.1. *For integers* $t \geq 1$, a, $b \in \mathbf{F}_q^{\times}$, *the twisted Kloosterman sum* $K_{GL(t,q)}(\lambda, \psi; a, b)$ *defined by* (4.2) *is*

$$K_{GL(t,q)}(\lambda, \psi; a, b) = q^{\frac{1}{4}(t-2)(t+1)} \sum_{l=1}^{[t+2/2]} \psi(-a^{-1}b)^{l-1} q^l K(\lambda, \psi; a, b:q)^{t+2-2l}$$
$$\times \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1) \tag{4.7}$$

where $K(\lambda, \psi; a, b:q)$ *is the usual twisted Kloosterman sum in* (4.5) *and the inner sum is over all integers* $j_1, \ldots, j_{l-1}$ *satisfying* $2l-1 \leq j_{l-1} \leq j_{l-2} \leq \ldots \leq j_1 \leq t+1$. *Here we agree that the inner sum in* (4.7) *is* 1 *for* $l = 1$.

REMARK. The inner sum in (4.7) is equivalently given by

$$\sum \prod_{\nu=1}^{l-1} (q^{j_\nu} - 1),$$

where the sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l-3 \leq j_1 \leq t-1$, $2l-5 \leq j_2 \leq j_1 - 2, \ldots, 1 \leq j_{l-1} \leq j_{l-2} - 2$ (with the understanding $j_0 = t+1$ for $l = 2$).

Next, we want to determine the Kloosterman sum $K_{SL(t,q)}(\lambda; a, b)$ for $SL(t, q)$ in (4.4). If a or b is zero, then it was treated in [6].

REMARKS. (a) If $a = b = 0$, then $K_{SL(t,q)}(\lambda; 0, 0) = |SL(t, q)| = (1/q^{-1})g_t(q) = q^{\binom{t}{2}} \prod_{j=2}^{t} (q^j - 1)$.

(b) If exactly one of a, b is zero, say $a \neq 0$, $b = 0$, then $K_{SL(t,q)}(\lambda; a, 0) = q^{\binom{t}{2}} K_{t-1}(\lambda; a, \ldots, a; a:q)$, where, for positive integers $r, a_1, \ldots, a_r$, $b \in \mathbf{F}_q^{\times}$, the hyper-kloosterman sum is defined as

$$K_r(\lambda; a_1, \ldots, a_r; b:q) = \sum_{\alpha_1, \ldots, \alpha_r \in \mathbf{F}_q^{\times}} \lambda(a_1\alpha_1 + \ldots + a_r\alpha_r + b\alpha_1^{-1} \ldots \alpha_r^{-1}).$$

For this, see [6].

Assume, from now on, that $ab \neq 0$. In order to determine (4.4), we need to consider a sum which is slightly more general than that. Namely, for each $\alpha \in \mathbf{F}_q^{\times}$ (and with fixed $a, b \in \mathbf{F}_q^{\times}$), we will consider

$$s_t(\alpha) = \sum_{g \in SL(t,q)} \lambda\left(a \operatorname{tr} \begin{bmatrix} 1_{t-1} & 0 \\ 0 & \alpha \end{bmatrix} g + b \operatorname{tr}\left(\begin{bmatrix} 1_{t-1} & 0 \\ 0 & \alpha \end{bmatrix} g\right)^{-1}\right), \tag{4.8}$$

where $t \geq 2$.

For $t = 1$, we agree that

$$s_1(\alpha) = \lambda(a\alpha + b\alpha^{-1}). \tag{4.9}$$

The decomposition in (4.4) of [3] can be modified to give

$$SL(t, q) = Q(t-1, 1; q) \coprod Q(t-1, 1; q)\sigma(C(t, q) \backslash Q(t-1, 1; q)), \tag{4.10}$$

where

$$Q(t-1,1;q)=\left\{\begin{bmatrix} A & B \\ 0 & (\det A)^{-1}\end{bmatrix} \,\middle|\, A\in GL(t-1,q),\ B \text{ is of } (t-1)\times 1 \text{ over } \mathbf{F}_q\right\},$$

$$C(t,q)=\left\{g\in Q(t-1,1;q)\,\middle|\,\sigma g\sigma^{-1}\in Q(t-1,1;q)\right\},$$

and

$$\sigma=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1_{t-2} & 0 \\ -1 & 0 & 0\end{bmatrix}$$

One can check that

$$|C(t,q)\backslash Q(t-1,1;q)|=q(q^{t-1}-1)/(q-1). \tag{4.11}$$

Using the decomposition (4.10), $s_t(\alpha)$ in (4.8) can be written as

$$\sum_{g\in Q(t-1,1;q)}\lambda\left(a\operatorname{tr}\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}g+b\operatorname{tr}\left(\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}g\right)^{-1}\right)+|c(t,q)\backslash Q(t-1,1;q)|$$
$$\times\sum_{g\in Q(t-1,1;q)}\lambda\left(a\operatorname{tr}\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}g\sigma+b\operatorname{tr}\left(\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}g\sigma\right)^{-1}\right). \tag{4.12}$$

Here one should note that, for each $q\in Q(t-1,1;q)$,

$$\sum_{g\in Q(t-1,1;q)}\lambda\left(a\operatorname{tr}\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}g\sigma q+b\operatorname{tr}\left(\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}g\sigma q\right)^{-1}\right)$$
$$=\sum_{g\in Q(t-1,1;q)}\lambda\left(a\operatorname{tr}\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}q'g\sigma+b\operatorname{tr}\left(\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}q'g\sigma\right)^{-1}\right)$$
$$=\sum_{g\in Q(t-1,1;q)}\lambda\left(a\operatorname{tr}\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}g\sigma+b\operatorname{tr}\left(\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}g\sigma\right)^{-1}\right),$$

where $q'=\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}^{-1}q\begin{bmatrix}1_{t-1} & 0\\ 0 & \alpha\end{bmatrix}\in Q(t-1,1;q)$.

Writing $g\in Q(t-1,1;q)$ as $g=\begin{bmatrix}A & B\\ 0 & (\det A)^{-1}\end{bmatrix}$, the first sum in (4.12) equals

$$q^{t-1}\sum_{A\in GL(t-1,q)}\lambda(a\operatorname{tr}A+b\operatorname{tr}A^{-1}+a\alpha(\det A)^{-1}$$
$$+b\alpha^{-1}(\det A))=q^{t-1}\sum_{\delta\in\mathbf{F}_q^\times}\lambda(a\alpha\delta^{-1}+b\alpha^{-1}\delta)$$
$$\times\sum_{g\in SL(t-1,q)}\lambda\left(a\operatorname{tr}\begin{bmatrix}1_{t-2} & 0\\ 0 & \delta\end{bmatrix}g+b\operatorname{tr}\left(\begin{bmatrix}1_{t-2} & 0\\ 0 & \delta\end{bmatrix}g\right)^{-1}\right)$$
$$=q^{t-1}\sum_{\delta\in\mathbf{F}_q^\times}\lambda(a\alpha\delta^{-1}+b\alpha^{-1}\delta)s_{t-1}(\delta). \tag{4.13}$$

On the other hand, for the second sum in (4.12) we write $g \in Q(t-1,1;q)$ as $g = \begin{bmatrix} A & B \\ 0 & (\det A)^{-1} \end{bmatrix}$ with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where A_{11}, A_{12}, A_{21}, A_{22} are respectively of sizes 1×1, $1 \times (t-2)$, $(t-2) \times 1$, $(t-2) \times (t-2)$, similarly for A^{-1}, and B_1, B_2 are respectively of sizes 1×1, $(t-2) \times 1$.

Now, the second sum in (4.12) is

$$\sum_A \lambda(a \operatorname{tr} A_{22} + b \operatorname{tr} E_{22}) \sum_{B_1} \lambda(-(a + \alpha^{-1} b E_{11} \det A) B_1) \sum_{B_2} \lambda(-\alpha^{-1} b E_{12} B_2 \det A). \quad (4.14)$$

The subsum over B_2 in (4.14) is nonzero if and only if $E_{12} = 0$, in which case it is q^{t-2}. Further, we have $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} E_{11}^{-1} & 0 \\ * & E_{22}^{-1} \end{bmatrix}$ in that case. So the subsum over B_1 is nonzero if and only if $a + \alpha^{-1} b E_{11} \det A = a + \alpha^{-1} b \det A_{22} = 0$ if and only if $\det A_{22} = -ab^{-1}\alpha$, in which case it is q. From these observations, we can conclude that (4.14) equals

$$q^{2t-3}(q-1) \sum \lambda(a \operatorname{tr} A_{22} + b \operatorname{tr} A_{22}^{-1}),$$

where the sum is over all $A_{22} \in GL(t-2, q)$ with $\det A_{22} = -ab^{-1}\alpha$. Thus this can be written as

$$q^{2t-3}(q-1) s_{t-2}(-ab^{-1}\alpha). \quad (4.15)$$

From (4.11), (4.12), (4.13), and (4.15), we see that, for $t \geq 2$,

$$s_t(\alpha) = q^{t-2} \sum_{\delta \in \mathbf{F}_q^\times} \lambda(a\alpha\delta^{-1} + b\alpha^{-1}\delta) s_{t-1}(\delta) + q^{2t-2}(q^{t-1} - 1) s_{t-2}(-ab^{-1}\alpha), \quad (4.16)$$

where, for $\beta \in \mathbf{F}_q^\times$, $s_1(\beta)$ is as in (4.9) and we agree that

$$s_0(\beta) = \begin{cases} 1, & \text{if } \beta = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.17)$$

This convention is natural, since, for $t = 2$, (4.14) is

$$\sum_{A_{11}, B_1} \lambda(-(a + b\alpha^{-1}) B_1) = \begin{cases} q(q-1), & \text{if } -ab^{-1}\alpha = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.17)$$

For positive integers r and $a_1, \ldots, a_r, b, c_1, \ldots, c_r, d \in \mathbf{F}_q^\times$, we define $BK_r(\lambda; a_1, \ldots, a_r; b; c_1, \ldots, c_r; d : q)$, called *bihyperkloosterman sum over* $\mathbf{F}_q$, as

$$BK_r(\lambda; a_1, \ldots, a_r; b; c_1, \ldots, c_r; d : q)$$

$$:= \sum_{\alpha_1, \ldots, \alpha_r \in \mathbf{F}_q^\times} \lambda\left(\sum_{j=1}^r a_j \alpha_j + b \prod_{j=1}^r \alpha_j^{-1} + \sum_{j=1}^r c_j \alpha_j^{-1} + d \prod_{j=1}^r \alpha_j \right). \quad (4.18)$$

Also, we agree that

$$BK_0(\lambda; a_1, \ldots, a_r; b; c_1, \ldots, c_r; d:q) = \lambda(b+d).$$

If $a_1 = \ldots = a_r = a$, $c_1 = \ldots = c_r = c$, then the notation in (4.18) will be abbreviated to

$$BK_r(\lambda; a; b; c; d:q) = \sum_{\alpha_1,\ldots,\alpha_r \in \mathbf{F}_q^\times} \lambda\Big(a\sum_{j=1}^r \alpha_j + b\prod_{j=1}^r \alpha_j^{-1} + c\sum_{j=1}^r \alpha_j^{-1} + d\prod_{j=1}^r \alpha_j\Big). \quad (4.20)$$

Now, it is elementary to see that

$$\sum_{\delta \in \mathbf{F}_q^\times} \lambda(a_{r+1}\alpha\delta^{-1} + c_{r+1}\alpha^{-1}\delta) BK_r(\lambda; a_1, \ldots, a_r; b\delta; c_1, \ldots, c_r; d\delta^{-1})$$
$$= BK_{r+1}(\lambda; a_1, \ldots, a_{r+1}; b\alpha; c_1, \ldots, c_{r+1}; d\alpha^{-1}). \quad (4.21)$$

From the recursive relation in (4.16) and using a suitable form of (4.21), the following theorem can be proved, by induction on t, in exactly the same manner as in the proof of Theorem 4.3 of [**3**].

THEOREM 4.2. *For integers* $t \geq 1$, *and* $\alpha, a, b \in \mathbf{F}_q^\times$, the sum

$$s_t(\alpha) = \sum_{g \in SL(t,q)} \lambda\left(a \operatorname{tr}\begin{bmatrix} 1_{t-1} & 0 \\ 0 & \alpha \end{bmatrix} g + b \operatorname{tr}\left(\begin{bmatrix} 1_{t-1} & 0 \\ 0 & \alpha \end{bmatrix} g\right)^{-1}\right)$$

defined in (4.8) *and* (4.9) *is*

$$s_t(\alpha) = q^{\frac{1}{2}(t-2)(t+1)} \sum_{l=1}^{[t+2/2]} q^l BK_{t+1-2l}(\lambda; a; a(-ab^{-1})^{l-1}\alpha; b; b(-a^{-1}b)^{l-1}\alpha^{-1}:q)$$
$$\times \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1), \quad (4.22)$$

where $BK_t(\lambda; a; b; c; d:q)$ *is the bihyperkloosterman sum in* (4.20), *and the inner sum runs over all integers* $j_1, \ldots, j_{l-1}$ *satisfying* $2l-1 \leq j_{l-1} \leq j_{l-2} \leq \ldots \leq j_1 \leq t+1$ *with the convention that the inner sum in* (4.22) *is* 1 *for* $l = 1$.

Here we understand that

$$BK_{t+1-2[t+2/2]}(\lambda; a; a(-ab^{-1})^{[t+2/2]-1}\alpha; b; b(-a^{-1}b)^{[t+2/2]-1}\alpha^{-1}:q)$$
$$= \begin{cases} \lambda(a(-ab^{-1})^{k-1}\alpha + b(-a^{-1}b)^{k-1}\alpha^{-1}), \textit{ for } t = 2k-1 \textit{ odd (cf. } (4.19)), \\ s_0((-ab^{-1})^k\alpha), \text{ for } t = 2k \textit{ even (cf. } (4.17)). \end{cases}$$

Setting $\alpha = 1$ in (4.22), we get the following.

COROLLARY 4.3. *For integers* $t \geq 1$, a, $b \in \mathbf{F}_q^\times$, *the Kloosterman sum* $K_{SL(t,q)}(\lambda; a, b)$ *for* $SL(t,q)$, *defined in* (4.4), *is given by*

$$K_{SL(t,q)}(\lambda; a, b) = q^{\frac{1}{2}(t-2)(t+1)} \sum_{l=1}^{[t+2/2]} q^l BK_{t+1-2l}(\lambda; a; a(-ab^{-1})^{l-1}; \quad b; \quad b(-a^{-1}b)^{l-1}:q)$$
$$\times \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1),$$

where the inner sum runs over all integers $j_1, \ldots, j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le t+1$. *For bihyperkloosterman sums, see* (4.18), (4.20) *and* (4.23).

5. $SU(2n, q^2)$ case. In this section, we will consider the sum in (1.1)

$$\sum_{g \in SU(2n,q^2)} \lambda'(\operatorname{tr} g),$$

where λ' is the nontrivial additive character λ of $\mathbf{F}_q$ lifted to $\mathbf{F}_{q^2}$. As we will see, it is a polynomial in q with coefficients involving certain averages (over $\mathbf{F}_q^\times$) of bihyperkloosterman sums (cf. (4.20)).

Using the decomposition in Corollary 3.3, the sum in (1.1) can be written as

$$\sum_{\substack{0 \le r \le n \\ r \text{ even}}} |B_r \backslash Q| \sum_{g \in Q} \lambda'(\operatorname{tr} g\sigma_r) + \sum_{\substack{0 \le r \le n \\ r \text{ odd}}} |B_r \backslash Q| \sum_{g \in Q^-} \lambda'(\operatorname{tr} g\sigma_r), \tag{5.1}$$

where $B_r = B_r(q^2)$, $Q = Q(2n, q^2)$, $Q^- = Q^-(2n, q^2)$, σ_r are respectively as in (3.6), (3.2), (3.3), (3.1). For the second sum in (5.1), one should note that $hQ^- = Q^-$ for each $h \in Q$.

Write $g \in Q$ as in (3.9) with $A, {}^*A^{-1}, B$ as in (3.10). Recall here that $\det A \in \mathbf{F}_q^\times$. Then $g\sigma_r$ is $\begin{bmatrix} M & * \\ * & N \end{bmatrix}$ with

$$M = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ -{}^*B_{12} & 1_{n-r} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & E_{12} \\ 0 & E_{22} \end{bmatrix}. \tag{5.2}$$

Thus, for any $r (0 \le r \le n)$,

$$\sum_{g \in Q} \lambda'(\operatorname{tr} g\sigma_r) = \sum \lambda'(\operatorname{tr} A_{11}B_{11} - \operatorname{tr} A_{12}{}^*B_{12} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22}), \tag{5.3}$$

where the sum is over all $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in GL(n, q^2)$ with $\det A \in \mathbf{F}_q^\times$ and over all matrices $B = \begin{bmatrix} B_{11} & B_{12} \\ -{}^*B_{12} & B_{22} \end{bmatrix}$ over $\mathbf{F}_{q^2}$ subject to conditions $B_{11} + {}^*B_{11} = 0$, $B_{22} + {}^*B_{22} = 0$.

For each fixed A, the subum over B in (5.3) is

$$q^{(n-r)^2} \sum_{B_{11}} \lambda'(\operatorname{tr} A_{11}B_{11}) \sum_{B_{12}} \lambda'(-\operatorname{tr} A_{12}{}^*B_{12}), \tag{5.4}$$

since the summand is independent of B_{22}. The sum over B_{12} in (5.4) is nonzero if and only if $A_{12} = 0$, in which case it is $q^{2r(n-r)}$. On the other hand, $\sum_{B_{11}} \lambda'(\operatorname{tr} A_{11}B_{11}) \ne 0$ if and only if A_{11} is Hermitian, in which case it equals q^{r^2}. To see this, we first need the following lemma.

LEMMA 5.1. *Let* λ' *be the nontrivial additive character* λ *of* $\mathbf{F}_q$ *lifted to* $\mathbf{F}_{q^2}$. *Let* $a \in \mathbf{F}_{q^2}$. *Then*

$$\sum \lambda'(ab) = \begin{cases} q, & \text{if } a \in \mathbf{F}_q, \\ 0, & \text{otherwise,} \end{cases} \tag{5.5}$$

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where the sum is over all elements $b \in \mathbf{F}_{q^2}$ *with* $\operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q} b = 0$.

Proof. Choose a nonzero element $\xi \in \mathbf{F}_{q^2}$ with $\operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q} \xi = 0$. Then the sum in (5.5) is

$$\sum_{\alpha \in \mathbf{F}_q} \lambda'(\alpha a \xi) = \sum_{\alpha \in \mathbf{F}_q} \lambda(\alpha \operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q} a\xi),$$

which is nonzero if and only if $\operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q} a\xi = 0$ if and only if $a \in \mathbf{F}_q$. Moreover, in that case it equals q.

Now, let $A_{11} = (\alpha_{ij})$, $B_{11} = (\beta_{ij})$. The condition $B_{11} + {}^*B_{11} = 0$ is equivalent to

$$\operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q} \beta_{ii} = 0, \text{ for } 1 \leq i \leq r,$$
$$\beta_{ij} + \beta_{ji}^{\tau} = 0, \text{ for } 1 \leq i < j \leq r.$$

Then it is not hard to see that

$$\lambda'(\operatorname{tr} A_{11} B_{11}) = \lambda'\Big(\sum_{i=1}^{r} \alpha_{ii}\beta_{ii}\Big)\lambda'\Big(\sum_{1 \leq i < j \leq r} (\alpha_{ji} - \alpha_{ij}^{\tau})\beta_{ij}\Big). \tag{5.6}$$

Hence, in view of Lemma 5.1 and (5.6), the sum over B_{11} in (5.4) is nonzero if and only if $\alpha_{ii} \in \mathbf{F}_q$ for $1 \leq i \leq r$ and $\alpha_{ji} = \alpha_{ij}^{\tau}$ for $1 \leq i < j \leq r$ if and only if A_{11} is Hermitian, in which case it is q^{r^2}.

So far we have shown that the sum in (5.4) is nonzero if and only if $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} nonsingular Hermitian. In addition, in that case it equals

$$q^{(n-r)^2 + 2r(n-r) + r^2} = q^{n^2}.$$

For such an $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$, $\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^{-1} & * \\ 0 & A_{22}^{-1} \end{bmatrix}$. Moreover, since A_{11} is Hermitian, $\det A_{11} \in \mathbf{F}_q^{\times}$ and hence $\det A \in \mathbf{F}_q^{\times}$ is equivalent to $\det A_{22} \in \mathbf{F}_q^{\times}$.

The sum in (5.3) is

$$q^{n^2} \sum_{A_{11}, A_{21}} \sum_{A_{22}} \lambda'(\operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}) = q^{n^2 + 2r(n-r)} h_r \sum \lambda'(\operatorname{tr} g + \operatorname{tr} g^{-1}),$$

where h_r is the number of $r \times r$ nonsingular Hermitian matrices over $\mathbf{F}_{q^2}$ and the sum is over all $g \in GL(n-r, q^2)$ with $\det g \in \mathbf{F}_q^{\times}$. Here we agree that $h_r = 1$ for $r = 0$.

Noting that $|A_r(q^2) \backslash P(2n, q^2)| = |B_r(q^2) \backslash Q(2n, q^2)|$ and from (3.13), (3.17), (5.7), the first sum in (5.1) equals

$$q^{n^2} \sum_{\substack{0 \leq r \leq n \\ r \text{ even}}} q^{\frac{1}{2}r(4n-r-1)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \times \sum \lambda'(\operatorname{tr} g + \operatorname{tr} g^{-1}), \tag{5.8}$$

where g is over all elements in $GL(n-r, q^2)$ with $\det g \in \mathbf{F}_q^{\times}$.

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On the other hand, glancing through the above argument, we see that, for any $r(0 \le r \le n)$,

$$\sum_{g \in Q^-} \lambda'(\operatorname{tr} g\sigma_r)$$

is the same as (5.7) except that the sum is now over all $g \in GL(n-r, q^2)$ with $\operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}(\det g) = 0$. Thus the second sum in (5.1) is

$$q^{n^2} \sum_{\substack{0 \le r \le n \\ r \text{ odd}}} q^{\frac{1}{2}r(4n-r-1)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \times \sum \lambda'(\operatorname{tr} g + \operatorname{tr} g^{-1}),$$

where the sum is over all $g \in GL(n-r, q^2)$ with $\operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}(\det g) = 0$.

Replacing r in (5.8) and (5.9) respectively by $2r$ and $2r+1$, and from Theorem 4.2, we get the following main result of this section.

THEOREM 5.2. *Let λ' be the nontrivial additive character λ of $\mathbf{F}_q$ lifted to $\mathbf{F}_{q^2}$. Then the Gauss sum over $SU(2n, q^2)$*

$$\sum_{g \in SU(2n, q^2)} \lambda'(\operatorname{tr} g)$$

is given by

$$\begin{aligned} q^{2n^2-n-2} \Bigg\{ & \sum_{r=0}^{[n/2]} q^{r(2r+1)} \begin{bmatrix} n \\ 2r \end{bmatrix} \prod_{j=1}^{2r} (q^j + (-1)^j) \\ & \times \sum_{l=1}^{[n-2r+2/2]} q^{2l} \sum_{\alpha \in \mathbf{F}_q^\times} BK_{n-2r+1-2l}(\lambda'; 1; (-1)^{l-1}\alpha; 1; (-1)^{l-1}\alpha^{-1} : q^2) \\ & \times \sum \prod_{v=1}^{l-1} (q^{2j_v - 4v} - 1) + \sum_{r=0}^{[n-1/2]} q^{(r+1)(2r+1)} \begin{bmatrix} n \\ 2r+1 \end{bmatrix}_{q^2} \prod_{j=1}^{2r+1} (q^j + (-1)^j) \\ & \times \sum_{l=1}^{[n-2r+1/2]} q^{2l} \sum_{\alpha \in \mathbf{F}_q^\times} BK_{n-2r-2l}(\lambda'; 1; (-1)^{l-1}\alpha\theta; 1; (-1)^{l-1}\alpha^{-1}\theta^{-1} : q^2) \\ & \times \sum \prod_{v=1}^{l-1} (q^{2j_v - 4v} - 1) \Bigg\}, \end{aligned}$$

where the first and second unspecified sums are respectively over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r+1$ and over the same set of integers satisfying $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r$. Here $BK_l(\lambda';\ -;\ -;\ -;\ - : q^2)$ is the bihyperkloosterman sum over $\mathbf{F}_{q^2}$ defined in (4.20), *and θ is a fixed nonxero element in $\mathbf{F}_{q^2}$ with $\operatorname{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}\theta = 0$.*

6. $U(2n, q^2)$ case. Let χ be a multiplicative character $\mathbf{F}_{q^2}$, and let λ' be the nontrivial additive character λ of $\mathbf{F}_q$ lifted to $\mathbf{F}_{q^2}$. Then we will consider the sum in (1.2)

$$\sum_{g \in U(2n, q^2)} \chi(\det g)\lambda'(\operatorname{tr} g)$$

and find an explicit expression for this.

Using the decomposition in (3.8), the sum in (1.2) can be written as

$$\sum_{r=0}^{n} |B_r \backslash Q| \chi(-1)^r \sum_{g \in P} \chi(\det g) \lambda'(\operatorname{tr} g\sigma_r). \tag{6.1}$$

Write $g \in P$ as in (3.9) with A, ${}^*A^{-1}$, B as in (3.10). Note here that, in contrast to the $SU(2n, q^2)$ case, we don't have any restriction on A. The inner sum in (6.1) is

$$\sum_{A} \chi\left(\frac{\det A}{(\det A)^{\tau}}\right) \lambda'(\operatorname{tr} A_{22} + \operatorname{tr} E_{22}) \sum_{B} \lambda'(\operatorname{tr} A_{11} B_{22} - \operatorname{tr} A_{12}{}^* B_{12}). \tag{6.2}$$

As we saw in Section 5, for each fixed A the sum over B in (6.2) is nonzero if and only if $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} nonsingular Hermitian. In addition, in that case it is q^{n^2}. Also, for such an A, $\dfrac{\det A}{(\det A)^{\tau}} = \dfrac{\det A_{22}}{(\det A_{22})^{\tau}}$, since $\det A_{11} = (\det A_{11})^{\tau}$. Thus (6.2) equals

$$\begin{aligned} q^{n^2} &\sum_{A_{11}, A_{21}} \sum_{A_{22}} \chi\left(\frac{\det A_{22}}{(\det A_{22})^{\tau}}\right) \lambda'(\operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}) \\ &= q^{n^2 + 2r(n-r)} h_r \sum_{g \in GL(n-r, q^2)} \chi\left(\frac{(\det g)^{\tau}}{\det g}\right) \lambda'(\operatorname{tr} g + \operatorname{tr} g^{-1}) \\ &= q^{n^2 + 2r(n-r)} h_r \sum_{g \in GL(n-r, q^2)} \chi^{q-1}(\det g) \lambda'(\operatorname{tr} g + \operatorname{tr} g^{-1}), \end{aligned} \tag{6.3}$$

where h_r denotes the number of $r \times r$ nonsingular Hermitian matrices over $\mathbf{F}_{q^2}$ for a positive integer r and $h_0 = 1$.

So, from (6.1) and (6.3), the sum in (1.2) equals

$$q^{n^2} \sum_{r=0}^{n} |B_r \backslash Q| \, \chi(-1)^r q^{2r(n-r)} h_r K_{GL(n-r, q^2)}(\lambda', \chi^{q-1}; 1, 1) \tag{6.4}$$

in the notation of the twisted Kloosterman sum defined by (4.2).

Finally, we obtain the following main theorem of this section from (3.13), (3.17), (4.7), and (6.4). Recall here that, as we noted in Section 5, $|B_r \backslash Q| = |A_r \backslash P|$.

THEOREM 6.1. *Let χ be a multiplicative character of $\mathbf{F}_{q^2}$, and let λ' be the nontrivial additive character λ of $\mathbf{F}_q$ lifted to $\mathbf{F}_{q^2}$. Then the Gauss sum over $U(2n, q^2)$*

$$\sum_{g \in U(2n, q^2)} \chi(\det g) \lambda'(\operatorname{tr} g)$$

is given by

$$\begin{aligned} q^{2n^2 - n - 2} &\sum_{r=0}^{n} \chi(-1)^r q^{\frac{1}{2}r(r+1)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\ &\times \sum_{l=1}^{[n-r+2/2]} q^{2l} K(\lambda', \chi^{q-1}; 1, 1 : q^2)^{n-r+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1), \end{aligned} \tag{6.5}$$

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where the innermost sum is over all integers $j_1, \ldots, j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-r+1$, *and* $K(\lambda', \chi^{q-1}; 1, 1:q^2)$ *is the usual twisted Kloosterman sum defined in* (4.5).

Separating terms with even r and those with odd r in (6.5), we get the following equivalent form of the above theorem.

Corollary 6.2. *Let* χ, λ' *be as in the above theorem. Then the Gauss sum over* $U(2n, q^2)$

$$\sum_{g \in U(2n,q^2)} \chi(\det g)\lambda'(\operatorname{tr} g)$$

is also given by

$$\begin{aligned} q^{2n^2-n-2}\Bigg\{&\sum_{r=0}^{[n/2]} q^{r(2r+1)}\begin{bmatrix} n \\ 2r \end{bmatrix}_{q^2} \prod_{j=1}^{2r}(q^j+(-1)^j) \\ &\times \sum_{l=1}^{[n-2r+2/2]} q^{2l}K(\lambda', \chi^{q-1}; 1, 1:q^2)^{n-2r+2-2l} \sum \prod_{v=1}^{l-1}(q^{2j_v-4v}-1) \\ &+\chi(-1)\sum_{r=0}^{[n-1/2]} q^{(r+1)(2r+1)}\begin{bmatrix} n \\ 2r+1 \end{bmatrix}_{q^2} \prod_{j=1}^{2r+1}(q^j+(-1)^j) \\ &\times \sum_{l=1}^{[n-2r+1/2]} q^{2l}K(\lambda', \chi^{q-1}; 1; 1:q^2)^{n-2r+1-2l} \sum \prod_{v=1}^{l-1}(q^{2j_v-4v}-1)\Bigg\}, \end{aligned} \tag{6.6}$$

where the first and second unspecified sums are respectively over all integers $j_1, \ldots, j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r+1$ *and over the same set of integers satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r$, *and* $K(\lambda', \chi^{q-1}; 1, 1; q^2)$ *is as in* (4.5).

Remark. Note that $N_{\mathbf{F}_{q^2}/\mathbf{F}_q}(\det g) = 1$ for $g \in U(2n, q^2)$. So if χ is a multiplicative character of $\mathbf{F}_{q^2}$ lifted from that of $\mathbf{F}_q$ then $\chi(\det q) = 1$ for $g \in U(2n, q^2)$.

7. Application to Hodges' Kloosterman sum.

In [**2**], the generalized Kloosterman sum over nonsingular Hermitian matrices is defined as, for $t \times t$ Hermitian matrices A,B over $\mathbf{F}_{q^2}$,

$$K_{Herm,t}(A, B) = \sum_g \lambda_1(\operatorname{tr}(Ag + Bg^{-1})), \tag{7.1}$$

where g runs over the set of all nonsingular Hermitian matrices over $\mathbf{F}_{q^2}$ of size t. Here λ_1 is as in (2.2), and one should note that, for Hermitian matrices C, D over $\mathbf{F}_{q^2}$ of size t, $(\operatorname{tr} CD)^\tau = \operatorname{tr}^*(CD) = \operatorname{tr} {}^*D^*C = \operatorname{tr} DC = \operatorname{tr} CD$ and hence $\operatorname{tr} CD \in \mathbf{F}_q$.

Now, in Theorem 6 of [**2**] we take $m = t = 2n$, $A = B = J$ in (2.5), $X = a1_{2n}$ with $0 \ne a \in \mathbf{F}_q$. Then we get the following identity

$$\sum_{g \in U(2n,q^2)} \lambda_a'(\operatorname{tr} g) = K_{Herm,2n}(a^2J^{-1}, J). \tag{7.2}$$

This is summarized in the following result.

THEOREM 7.1. *For $0 \neq a \in \mathbf{F}_q$, we have the identity*:

$$\sum_{g \in U(2n,q^2)} \lambda'_a(\operatorname{tr} g) = K_{Herm,2n}(a^2J^{-1}, J)$$
$$= K_{Herm,2n}(a^2C^{-1}, C), \qquad (7.3)$$

where λ_a is as in (2.2) *and C is any nonsingular Hermitian matrix over $\mathbf{F}_{q^2}$ of size* $2n$.

REMARK. (1) Here we don't have to assume that $q = p^d$ is a power of an odd prime. In fact, the whole discussion in [2] is valid even for $p = 2$ if the "conjugate" of α in $\mathbf{F}_{q^2}$ means α^{τ}.

(2) The second identity in (7.3) is clear from the definition of Kloosterman sum in (7.1).

From (7.3) and Theorem 6.1, we have the following theorem.

THEOREM 7.2. *Let $0 \neq a \in \mathbf{F}_q$, and let C be any nonsingular Hermitian matrix over $\mathbf{F}_{q^2}$ of size $2n$. Then the following generalized Kloosterman sum over nonsingular Hermitian matrices is the same for any such a C, and*

$$K_{Herm,2n}(a^2C^{-1}, C) = q^{2n^2-n-2} \sum_{r=0}^{n} q^{\frac{1}{2}r(r+1)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j)$$
$$\times \sum_{l=1}^{[n-r+2/2]} q^{2l} K(\lambda'_a; 1, 1 : q^2)^{n-r+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1),$$

where the innermost sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l - 1 \leq j_{l-1} \leq j_{l-2} \leq \ldots \leq j_1 \leq n - r + 1$, and $K(\lambda'_a; 1, 1; q^2)$ is the usual Kloosterman sum given by

$$K(\lambda'_a; 1, 1; q^2) = \sum_{\alpha \in \mathbf{F}_{q^2}^{\times}} \lambda'_a(\alpha + \alpha^{-1}).$$

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DEPARTMENT OF MATHEMATICS
SEOUL WOMEN'S UNIVERSITY
SEOUL 139-774
KOREA

Present address:
DEPARTMENT OF MATHEMATICS
SOGANG UNIVERSITY
SEOUL 121-742
KOREA