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LETTERS TO THE EDITOR

IMMIGRATION IN THE TWO-SEX POPULATION PROCESS

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Abstract

Known results for the two-sex birth–death population process are extended to the case where immigration of females is allowed. It is shown that the methods used follow those for the standard birth, death and immigration process.

BIRTH–DEATH PROCESS

In an initial paper issued in 1983, Tapaswi and Roychoudhury reconsidered the two-sex population model originally discussed by Goodman (1953). They were at first able to obtain only partial results, but in a second paper early in 1984 they derived the expression for the joint probability generating function (p.g.f.) of the process in terms of the solutions of a confluent hypergeometric equation. They then proceeded to explore the marginal p.g.f.'s of the process in detail. At approximately the same time, Gani and Tin (1984) who had read Tapaswi and Roychoudhury's first paper (1983), obtained a similar solution, concentrating rather on the structure and integral equation for the p.g.f. of the process.

The purpose of this letter is to indicate that the existing results can readily be extended to the case of the two-sex population process with immigration. Consider a linear birth–death process of females $X(t)$ and males $Y(t)$, $t \geqq 0$; only females can give birth to females at the rate λp and to males at the rate λq. The female death rate is μ while that for males is μ'; immigration is at the rate ν for females only.

It is easily shown that the forward Kolmogorov equations for the probabilities

$$P_{xy}(t) = P\{X(t) = x,\ Y(t) = y \mid X(0) = 1,\ Y(0) = 0\}$$

of this process are

$$\begin{aligned}\frac{d}{dt} P_{xy} &= \{\lambda p(x-1) + \nu\} P_{x-1,y} + \lambda q x P_{x,y-1} - \{(\lambda + \mu)x + \mu' y + \nu\} P_{xy} \\ &\quad + \mu(x+1) P_{x+1,y} + \mu'(y+1) P_{x,y+1} \qquad (x, y = 0, 1, 2, \cdots)\end{aligned}$$

where $P_{xy}(t) = 0$ for $x, y < 0$, and $P_{10}(0) = 1$. The p.g.f. $\psi(u, v, t) = \sum_{x,y=0}^{\infty} P_{xy}(t) u^x v^y$ is

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then readily seen to satisfy the partial differential equation

$$(1) \qquad \frac{\partial \psi}{\partial t} = (\lambda p u^2 + \lambda q u v - (\lambda + \mu) u + \mu) \frac{\partial \psi}{\partial u} + \mu'(1 - v) \frac{\partial \psi}{\partial v} + \nu(u - 1)\psi.$$

As expected, this differs only in its last term $\nu(u-1)\psi$ from the equation for the process without immigration.

From (1), one obtains directly that the means $M_x(t) = E(X(t))$, $M_y(t) = E(Y(t))$ of the process satisfy the differential equations

$$(2) \qquad \frac{dM_x}{dt} = (\lambda p - \mu) M_x + \nu, \qquad \frac{dM_y}{dt} = \lambda q M_x - \mu' M_y.$$

For the case $\lambda p \neq \mu$, their solutions are

$$M_x(t) = \frac{1}{\lambda p - \mu} \{\exp((\lambda p - \mu)t)(\lambda p + \nu - \mu) - \nu\}$$

$$M_y(t) = \frac{\lambda q \nu (\exp((\lambda p - \mu)t) - \exp(-\mu' t))}{(\lambda p - \mu)(\lambda p - \mu + \mu')} + \frac{\lambda q \nu (\exp(-\mu' t) - 1)}{\mu'(\lambda p - \mu)} + \frac{\lambda q}{\lambda p - \mu + \mu'} (\exp((\lambda p - \mu)t) - \exp(-\mu' t)).$$

Higher moments can be similarly derived, as can also the analogous results when $\lambda p = \mu$.

Using Bartlett's ((1978), pp. 82–84) approach to processes with immigration, it is readily seen that if $\phi(u, v, t)$ is the solution of (1) when $\nu = 0$ then

$$(3) \qquad \psi(u, v, t) = \phi(u, v, t) \exp\left\{\int_0^t \nu[\phi(u, v, t - \tau) - 1]\, d\tau\right\}.$$

It can easily be shown that $\psi(u, v, t)$ in (3) satisfies the equation (1).

Now if $z_1(\theta)$, $z_2(\theta)$ are the two independent solutions of the second order differential equation, arising from the auxiliary equations of (1),

$$(4) \qquad \theta^2 \frac{d^2 z}{d\theta^2} + \theta \frac{(\mu' - \mu - \lambda p - \lambda q \theta)}{\mu'} \frac{dz}{d\theta} + \frac{\lambda \mu p}{\mu'^2} z = 0$$

and we write for the case $\lambda p \neq \mu$,

$$(w_1 - \lambda p w_2) z_1(\theta) + \mu' \theta w_2 z_1(\theta) = \theta^{\lambda p / \mu'} H_1(w_1, \mu' w_2, \theta),$$
$$(w_1 - \lambda p w_2) z_2(\theta) + \mu' \theta w_2 z_2(\theta) = \theta^{\mu / \mu'} H_2(w_1, \mu' w_2, \theta),$$

Gani and Tin (1984) have shown that

$$\phi(u, v, t)$$

$$(5) \qquad = \frac{1}{\lambda p} \left\{ \frac{\begin{array}{l} H_1(\lambda p u - \lambda p, \mu', (1 - v)) H_2(\mu, \mu', (1 - v) \exp(-\mu' t)) \\ \quad - H_1(\lambda p, \mu', (1 - v) \exp(-\mu' t)) H_2(\lambda p u - \mu, \mu', (1 - v)) \exp(-(\lambda p - \mu)t) \end{array}}{\begin{array}{l} H_1(\lambda p u - \lambda p, \mu', (1 - v)) H_2(1, 0, (1 - v) \exp(-\mu' t)) \\ \quad - H_1(1, 0, (1 - v) \exp(-\mu' t)) H_2(\lambda p u - \mu, \mu', (1 - v)) \exp(-(\lambda p - \mu)t) \end{array}} \right\}.$$

Tapaswi and Roychoudhury (1984) obtained an equivalent form for $\phi(u, v, t)$ after reducing (4) to a confluent hypergeometric equation.

Substituting (5) into (3), one obtains, after some simplification, the solution

$$\psi(u, v, t) =$$

$$(6) \qquad \phi(u, v, t)\left\{\frac{\begin{array}{r}H_1(\lambda pu-\lambda p, \mu', (1-v))H_2(1, 0, (1-v)) \\ -H_1(1, 0, (1-v))H_2(\lambda pu-\lambda p, \mu', (1-v))\end{array}}{\begin{array}{r}H_1(\lambda pu-\lambda p, \mu', (1-v))H_2(1, 0, (1-v)\exp(-\mu' t))\exp((\lambda p-\mu)t) \\ -H_1(1, 0, (1-v)\exp(-\mu' t))H_2(\lambda pu-\mu, \mu', (1-v))\end{array}}\right\}^{\nu/\lambda p}$$

to Equation (1). An equivalent result can also be derived for the case where $\lambda p = \mu$. Note that if v is set equal to 1 in (6), the p.g.f. reduces to that of the ordinary birth, death and immigration model. While the equation (6) is too complicated to lend itself to practical usage, it is interesting to know that standard methods will yield the p.g.f. for the two-sex population process with immigration explicitly.

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