BULL. AUSTRAL. MATH. SOC. 46A03, 46A20, 49K35
VOL. 66 (2002) [43-56]

A NEW MINIMAX THEOREM AND A PERTURBED JAMES'S THEOREM

M. RUIZ GALÁN AND S. SIMONS

The main result of this paper is a sufficient condition for the minimax relation to hold for the canonical bilinear form on $X \times Y$, where X is a nonempty convex subset of a real locally convex space and Y is a nonempty convex subset of its dual. Using the known "converse minimax theorem", this result leads easily to a nonlinear generalisation of James's ("sup") theorem. We give a brief discussion of the connections with the "sup-limsup theorem" and, in the appendix to the paper, we give a simple, direct proof (using Goldstine's theorem) of the converse minimax theorem referred to above, valid for the special case of a normed space.

0. INTRODUCTION

This paper is in two main parts. The first three sections give various result about bounded functions on an abstract set, while the last section is more functional-analytic in character.

The main functional-analytic result is Theorem 14, which contains a sufficient condition for the minimax relation to hold for the canonical bilinear form on $X \times Y$, where X is a nonempty convex subset of a real locally convex space, E, and Y is a nonempty convex subset of its dual, E^*. (The notation necessary for an understanding of Theorem 14 appears at the beginning of Section 4.) Using the known "converse minimax theorem", Theorem 14 leads easily to Theorem 16, a nonlinear generalisation of James's ("sup") theorem.

Theorem 14 depends on Theorem 13, which gives a sufficient condition for there to exist a set of functions, all of which fail to attain their maximum value on X. The statement that $\widetilde{g} \in \operatorname{liminfsup}_i g_i$ means that $\liminf_i g_i \leqslant \widetilde{g} \leqslant \limsup_i g_i$ on X — we describe such a function $\widetilde{g}$ as an "undetermined function". In the situation of Theorem 14, condition (14.1) ensures that one of these undetermined functions can be chosen to be the restriction to X of an element of E^*, which is exactly what is needed for the proof of Theorem 14. All proofs of James's theorem also seem to need this undetermined function.

Received 11th December, 2001

Theorem 13 has two main components: a convexity argument specific to the canonical bilinear form and Theorem 10, which gives a sufficient condition (again, with an undetermined function) for there to exist a set of functions, all of which fail to attain their maximum value on a set X, but now X is simply any nonempty set with no vector space structure.

Theorem 10 relies on a technique used by Pryce in his proof of James's theorem (see Lemma 9) and a strengthened form of an argument used in [**8**], (see Lemmas 4 and 5).

In Section 2, we digress a little to discuss the "sup–limsup theorem" proved in [**8**], and show how its proof contrasts with what we need to obtain our results on undetermined functions.

In the appendix to this paper, we give a simple, direct proof (using Goldstine's theorem) of the converse minimax theorem referred to above, valid for the special case when E is a normed space.

1. Preliminary results

Let $\mathcal{P}$ be the set of all real sequences $\{\lambda_j\}_{j\geqslant 1}$ such that, for all $j\geqslant 1$, $\lambda_j\geqslant 0$ and $\sum_{j=1}^{\infty}\lambda_j = 1$. ("$\mathcal{P}$" stands for "probability".) We first give an elementary property of $\mathcal{P}$.

Lemma 1. *Suppose that $\{\mu_i\}_{i\geqslant 1}\in\mathcal{P}$ and, for all $i\geqslant 1$, $\{\lambda_{ij}\}_{j\geqslant 1}\in\mathcal{P}$. For all $j\geqslant 1$, let $\nu_j := \sum_{i=1}^{\infty}\mu_i\lambda_{ij}\geqslant 0$. Then $\{\nu_j\}_{j\geqslant 1}\in\mathcal{P}$.*

Proof: Since $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu_i\lambda_{ij} = \sum_{i=1}^{\infty}\mu_i\sum_{j=1}^{\infty}\lambda_{ij} = \sum_{i=1}^{\infty}\mu_i = 1$, from the double series theorem, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\mu_i\lambda_{ij} = 1$, that is to say, $\sum_{j=1}^{\infty}\nu_j = 1$. ∎

Now let H be a real sequentially complete Hausdorff locally convex space and $\{a_j\}_{j\geqslant 1}$ be a bounded sequence in H. If $\{\lambda_j\}_{j\geqslant 1}\in\mathcal{P}$ then $\sum_{j=1}^{\infty}\lambda_j a_j$ is defined in H to be $\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_j a_j$ (which exists by sequential completeness) and, whenever p is a continuous seminorm on H, $p\left(\sum_{j=1}^{\infty}\lambda_j a_j\right)\leqslant\sum_{j=1}^{\infty}\lambda_j p(a_j)$. We write $\mathrm{co}_\sigma\{a_j\colon j\geqslant 1\}$ for the set

$$\left\{\sum_{j=1}^{\infty}\lambda_j a_j\colon\ \{\lambda_j\}_{j\geqslant 1}\in\mathcal{P}\right\}.$$

The operator co_σ has a simple but important stability property, which is contained in the following lemma:

Lemma 2. *Suppose that $\{a_j\}_{j\geqslant 1}$ is a bounded sequence in H and, for all $i\geqslant 1$,*

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$b_i \in \mathrm{co}_\sigma\{a_j\colon\ j \geqslant 1\}$. *Then*

$$\mathrm{co}_\sigma\{b_i\colon\ i \geqslant 1\} \subset \mathrm{co}_\sigma\{a_j\colon\ j \geqslant 1\}.$$

PROOF: For all $i \geqslant 1$, let $\{\lambda_{ij}\}_{j\geqslant 1} \in \mathcal{P}$ and $b_i = \sum_{j=1}^{\infty} \lambda_{ij} a_j$. Let c be an arbitrary element of $\mathrm{co}_\sigma\{b_i\colon\ i \geqslant 1\}$. Then there exists $\{\mu_i\}_{i\geqslant 1} \in \mathcal{P}$ such that $c = \sum_{i=1}^{\infty} \mu_i b_i$. Define $\{\nu_j\}_{j\geqslant 1} \in \mathcal{P}$ as in Lemma 1. We shall show that

$$c = \sum_{j=1}^{\infty} \nu_j a_j. \tag{2.1}$$

This establishes that $c \in \mathrm{co}_\sigma\{a_j\colon\ j \geqslant 1\}$, which gives the required result since c was an arbitrary element of $\mathrm{co}_\sigma\{b_i\colon\ i \geqslant 1\}$. Now, for all $n \geqslant 1$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu_i\lambda_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i\lambda_{ij} + \sum_{i=1}^{n}\sum_{j=n+1}^{\infty} \mu_i\lambda_{ij} + \sum_{i=n+1}^{\infty}\sum_{j=1}^{n} \mu_i\lambda_{ij} + \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} \mu_i\lambda_{ij}.$$

Since it follows from the double series theorem that $\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i\lambda_{ij} \to \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu_i\lambda_{ij}$ as $n \to \infty$, we have

$$\sum_{i=1}^{n}\sum_{j=n+1}^{\infty} \mu_i\lambda_{ij} + \sum_{i=n+1}^{\infty}\sum_{j=1}^{n} \mu_i\lambda_{ij} + \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} \mu_i\lambda_{ij} \to 0 \text{ as } n \to \infty. \tag{2.2}$$

Now, for all $n \geqslant 1$,

$$\begin{aligned}\sum_{i=1}^{n} \mu_i b_i - \sum_{j=1}^{n} \nu_j a_j &= \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{n} \lambda_{ij} a_j + \sum_{j=n+1}^{\infty} \lambda_{ij} a_j \right) - \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \mu_i\lambda_{ij} + \sum_{i=n+1}^{\infty} \mu_i\lambda_{ij} \right) a_j \\ &= \sum_{i=1}^{n} \mu_i \sum_{j=n+1}^{\infty} \lambda_{ij} a_j - \sum_{j=1}^{n}\sum_{i=n+1}^{\infty} \mu_i\lambda_{ij} a_j.\end{aligned}$$

Thus $p\left(\sum_{i=1}^{n} \mu_i b_i - \sum_{j=1}^{n} \nu_j a_j\right) \leqslant \sum_{i=1}^{n}\sum_{j=n+1}^{\infty} \mu_i\lambda_{ij} p(a_j) + \sum_{j=1}^{n}\sum_{i=n+1}^{\infty} \mu_i\lambda_{ij} p(a_j)$ whenever p is a continuous seminorm on H, and it follows from (2.2) and the fact that $\{a_j\}_{j\geqslant 1}$ is bounded that $\sum_{i=1}^{n} \mu_i b_i - \sum_{j=1}^{n} \nu_j a_j \to 0$ as $n \to \infty$. This gives (2.1), which completes the proof of Lemma 2. □

Now let $X \neq \emptyset$. We are going to apply Lemma 2 in Lemma 4, Theorem 7 and Theorem 14, with H the Banach space $\ell_\infty(X)$ of bounded real functions on X with the supremum norm. Lemma 2 is more general than we need for this particular application. However, there are spaces in which Lemma 2 applies that are not Banach spaces — for instance, (when X is infinite) the space $\mathbb{R}^X$ of all real functions on X with the topology of pointwise convergence.

If $f \in \ell_\infty(X)$, we write $S_X(f) := \sup_X f$ and $\operatorname{argmax}_X f$ for $\{t \in X\colon f(t) = S_X(f)\}$. In what follows, $\sum_{i=1}^{0} \ldots$ is always interpreted to be 0.

DEFINITION 3: Let $\{a_j\}_{j\geqslant 1}$ be a bounded sequence in $\ell_\infty(X)$. We say that $\{b_i\}_{i\geqslant 1}$ is a *pseudo-subsequence* of $\{a_j\}_{j\geqslant 1}$ if, for all $i \geqslant 1$, $b_i \in \operatorname{co}_\sigma\{a_j\colon j \geqslant i\}$.

LEMMA 4. *Let $\{a_j\}_{j\geqslant 1}$ be a bounded sequence in $\ell_\infty(X)$ and $\rho \in (0,1)$.*

(a) *Let $\eta > 0$. Then there exists a pseudo-subsequence $\{b_i\}_{i\geqslant 1}$ of $\{a_j\}_{j\geqslant 1}$ such that*

$$(4.1) \qquad k \geqslant 0 \implies S_X\Bigl(\sum_{i=1}^{\infty} \rho^i b_i\Bigr) \geqslant S_X\Bigl(\sum_{i=1}^{k} \rho^i b_i\Bigr) + \rho^k\Bigl[S_X\Bigl(\sum_{i=1}^{\infty} \rho^i b_i\Bigr) - \eta\Bigr].$$

(b) *Let $B < \inf S_X\bigl(\operatorname{co}_\sigma\{a_j\colon j \geqslant 1\}\bigr)$. Then there exists a pseudo-subsequence $\{b_i\}_{i\geqslant 1}$ of $\{a_j\}_{j\geqslant 1}$ such that*

$$(4.2) \qquad k \geqslant 0 \implies S_X\Bigl(\sum_{i=1}^{\infty} \rho^i b_i\Bigr) \geqslant S_X\Bigl(\sum_{i=1}^{k} \rho^i b_i\Bigr) + B \sum_{i=k+1}^{\infty} \rho^i.$$

PROOF: (a) Suppose first that $m \geqslant 1$ and $b_1, \ldots, b_m \in \operatorname{co}_\sigma\{a_j\colon j \geqslant 1\}$. Then

$$(4.3) \qquad S_X\Bigl(\sum_{i=1}^{m} \rho^i b_i\Bigr) \geqslant -\Bigl(\sum_{i=1}^{m} \rho^i\Bigr) \sup_{i=1}^{m} \|b_i\| \geqslant -\Bigl(\sum_{i=1}^{m} \rho^i\Bigr) \sup_{j\geqslant 1} \|a_j\| > -\infty.$$

For all $m \geqslant 1$, let $C_m := \operatorname{co}_\sigma\{a_j\colon j \geqslant m\}$. Then we can choose $b_m \in C_m$ inductively so that

$$(4.4) \qquad S_X\Bigl(\sum_{i=1}^{m-1} \rho^i b_i + \rho^m b_m\Bigr) \leqslant \inf_{b\in C_m} S_X\Bigl(\sum_{i=1}^{m-1} \rho^i b_i + \rho^m b\Bigr) + \eta(\rho/2)^m.$$

$\Bigl($We note from (4.3) that $\inf_{b\in C_m} S_X\Bigl(\sum_{i=1}^{m-1} \rho^i b_i + \rho^m b\Bigr) > -\infty$.$\Bigr)$ Define $c := \sum_{i=1}^{\infty} \rho^i b_i$ and, for all $m \geqslant 1$, $c_m := \sum_{i=1}^{m} \rho^i b_i$ (so $c_0 = 0$). Then (4.4) gives

$$(4.5) \qquad \text{for all } m \geqslant 1, \quad S_X(c_m) \leqslant \inf\nolimits_{b\in C_m} S_X(c_{m-1} + \rho^m b) + \eta(\rho/2)^m.$$

(4.1) is obvious when $k = 0$. Now let $k \geqslant 1$ and $1 \leqslant m \leqslant k$. Then, from Lemma 2, $(1-\rho)(c - c_{m-1})/\rho^m = \sum_{i=0}^{\infty} (\rho^i - \rho^{i+1}) b_{i+m} \in C_m$. Thus, from (4.5) and the sublinearity of S_X on $\ell_\infty(X)$,

$$\begin{aligned} S_X(c_m) &\leqslant S_X\bigl(c_{m-1} + (1-\rho)(c - c_{m-1})\bigr) + \eta(\rho/2)^m \\ &= S_X\bigl((1-\rho)c + \rho c_{m-1}\bigr) + \eta(\rho/2)^m \\ &\leqslant (1-\rho)S_X(c) + \rho S_X(c_{m-1}) + \eta(\rho/2)^m. \end{aligned}$$

Dividing this by ρ^m, we obtain

$$(1/\rho^m - 1/\rho^{m-1})S_X(c) \geqslant S_X(c_m)/\rho^m - S_X(c_{m-1})/\rho^{m-1} - \eta/2^m.$$

Adding up these inequalities for $m = 1, 2, \ldots, k$ (and noting that $c_0 = 0$) yields

$$(1/\rho^k - 1)S_X(c) \geqslant S_X(c_k)/\rho^k - \eta,$$

which gives (4.1) on rearrangement.

(b) Let $\eta := \sum_{i=1}^{\infty} \rho^i \left[\inf S_X\big(\mathrm{co}_\sigma\{a_j\colon\ j \geqslant 1\}\big) - B\right] > 0$, and $\{b_i\}_{i\geqslant 1}$ be chosen as in (a) for this value of η. Lemma 2 gives

$$S_X\left(\sum_{i=1}^{\infty} \rho^i b_i\right) - \eta \geqslant \left(\sum_{i=1}^{\infty} \rho^i\right) \inf S_X\big(\mathrm{co}_\sigma\{a_j\colon\ j \geqslant 1\}\big) - \eta = \sum_{i=1}^{\infty} \rho^i B,$$

and (4.2) follows by substituting this into (4.1). ∎

Lemma 5. *Let $\{b_i\}_{i\geqslant 1}$ be a bounded sequence in $\ell_\infty(X)$ satisfying (4.2) for some $\rho \in (0,1)$ and $B \in \mathbb{R}$. Let $M := \sup_{i\geqslant 1} \|b_i\| \in \mathbb{R}$. Then*

$$p \in \mathrm{argmax}_X\left(\sum_{i=1}^{\infty} \rho^i b_i\right) \quad \Longrightarrow \quad \inf_{k\geqslant 1} b_k(p) \geqslant \frac{B - \rho M}{1 - \rho}. \tag{5.1}$$

Proof: Let $p \in \mathrm{argmax}_X\left(\sum_{i=1}^{\infty} \rho^i b_i\right)$. Then, from (4.2),

$$k \geqslant 0 \quad \Longrightarrow \quad \sum_{i=1}^{\infty} \rho^i b_i(p) \geqslant \sum_{i=1}^{k} \rho^i b_i(p) + B \sum_{i=k+1}^{\infty} \rho^i,$$

from which

$$k \geqslant 1 \quad \Longrightarrow \quad \sum_{i=k}^{\infty} \rho^{i-k} b_i(p) \geqslant B \sum_{i=0}^{\infty} \rho^i,$$

and so

$$k \geqslant 1 \quad \Longrightarrow \quad b_k(p) \geqslant B \sum_{i=0}^{\infty} \rho^i - \sum_{i=1}^{\infty} \rho^i M,$$

which gives (5.1). ∎

Lemma 6. *Let $\{a_j\}_{j\geqslant 1}$ be a bounded sequence in $\ell_\infty(X)$ and $\{b_i\}_{i\geqslant 1}$ be a pseudo-subsequence of $\{a_j\}_{j\geqslant 1}$. Then*

$$\liminf_j a_j \leqslant \liminf_i b_i \leqslant \limsup_i b_i \leqslant \limsup_j a_j \quad \text{on } X.$$

This holds, in particular, if $\{b_i\}_{i\geqslant 1}$ *is a subsequence of* $\{a_j\}_{j\geqslant 1}$.

PROOF: Let $i \geqslant 1$. Then there exists $\{\lambda_j\}_{j\geqslant i} \in \mathcal{P}$ such that $b_i = \sum_{j=i}^{\infty} \lambda_j a_j$. It follows that $b_i \leqslant \sup_{j\geqslant i} a_j$ on X, from which

$$\limsup_i b_i \leqslant \limsup_i \sup_{j\geqslant i} a_j = \limsup_j a_j \quad \text{on } X.$$

The proof that $\liminf_j a_j \leqslant \liminf_i b_i$ on X is similar. ∎

2. The sup–limsup theorem

In this short section, we discuss the "sup–limsup theorem", first proved in a Banach space context in [**8**]. (See also Oja, [**4**, Theorem 2.2, p.2807–2808].) Though it is a digression from our main theme, it provides an interesting comparison. Theorem 7 uses the technique of Lemma 5, organised in a slightly different way. It also uses Lemma 4(a) with $\rho = 1/2$. This contrasts with the situation in Corollary 8, in which we take ρ to be small. The precise place where we need this is the statement "since $\inf_{k\geqslant 1} g_k \leqslant \widetilde{g}$ on X" towards the end of the proof of Corollary 8. (It is not true that $\sup_{k\geqslant 1} g_k \leqslant \widetilde{g}$ on X.) Corollary 8 leads to Theorem 10, the result on "undetermined functions", that leads in turn to our main result, Theorem 14.

This is a good place to mention the paper [**2**] by Godefroy, which contains many other applications of similar ideas to Banach spaces, as well as further references.

THEOREM 7. *Let* $\{f_k\}_{k\geqslant 1}$ *be a bounded sequence in* $\ell_\infty(X)$ *and suppose that* P *is a "peak set" for* $\mathrm{co}_\sigma\{f_k\colon k \geqslant 1\}$, *that is, for all* $f \in \mathrm{co}_\sigma\{f_k\colon k \geqslant 1\}$, $P \cap \operatorname{argmax}_X f \neq \emptyset$. *Then*

$$S_P(\limsup_k f_k) = S_X(\limsup_k f_k).$$

PROOF: It is obvious that $S_P(\limsup_k f_k) \leqslant S_X(\limsup_k f_k)$, so we now prove "$\geqslant$". Let $\eta > 0$. We first choose $x \in X$ so that $\limsup_k f_k(x) > S_X(\limsup_k f_k) - \eta$, and then choose a subsequence $\{a_j\}_{j\geqslant 1}$ of $\{f_k\}_{k\geqslant 1}$ such that $\inf_{j\geqslant 1} a_j(x) \geqslant S_X(\limsup_k f_k) - \eta$. It follows that

$$S_X(\inf_{j\geqslant 1} a_j) \geqslant S_X(\limsup_k f_k) - \eta. \tag{7.1}$$

From Lemma 4(a) with $\rho = 1/2$, there exists a pseudo–subsequence $\{b_i\}_{i\geqslant 1}$ of $\{a_j\}_{j\geqslant 1}$ such that,

$$k \geqslant 0 \implies S_X\Bigl(\sum_{i=1}^{\infty} b_i/2^i\Bigr) \geqslant S_X\Bigl(\sum_{i=1}^{k} b_i/2^i\Bigr) + \Bigl(S_X\Bigl(\sum_{i=1}^{\infty} b_i/2^i\Bigr) - \eta\Bigr)/2^k.$$

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Arguing as in Lemma 6, $\sum_{i=1}^{\infty} b_i/2^i \geqslant \inf_{i\geqslant 1} b_i \geqslant \inf_{j\geqslant 1} a_j$ on X so, from (7.1),

$$k \geqslant 0 \quad\Longrightarrow\quad S_X\Big(\sum_{i=1}^{\infty} b_i/2^i\Big) \geqslant S_X\Big(\sum_{i=1}^{k} b_i/2^i\Big) + \Big(S_X\big(\limsup_k f_k\big) - 2\eta\Big)/2^k.$$

By hypothesis and Lemma 2, there exists $p \in P \cap \operatorname{argmax}_X\Big(\sum_{i=1}^{\infty} b_i/2^i\Big)$. Consequently

$$\begin{aligned} k \geqslant 0 \quad&\Longrightarrow\quad \sum_{i=1}^{\infty} b_i(p)/2^i \geqslant \sum_{i=1}^{k} b_i(p)/2^i + \Big(S_X\big(\limsup_k f_k\big) - 2\eta\Big)/2^k \\ &\Longrightarrow\quad \sum_{i=k+1}^{\infty} b_i(p)/2^i \geqslant \Big(S_X\big(\limsup_k f_k\big) - 2\eta\Big)/2^k \\ &\Longrightarrow\quad \sum_{i=1}^{\infty} b_{i+k}(p)/2^i \geqslant S_X\big(\limsup_k f_k\big) - 2\eta \\ &\Longrightarrow\quad \sup_{i>k} b_i(p) \geqslant S_X\big(\limsup_k f_k\big) - 2\eta. \end{aligned}$$

Letting $k \to \infty$, we obtain that $\limsup_i b_i(p) \geqslant S_X\big(\limsup_k f_k\big) - 2\eta$. Two application of Lemma 6 now give $\limsup_k f_k \geqslant \limsup_j a_j \geqslant \limsup_i b_i$ on X, and so $\limsup_j f_j(p) \geqslant S_X\big(\limsup_k f_k\big) - 2\eta$. The result follows since $\eta > 0$ is arbitrary. ∎

3. The technique of the undetermined function

If $\{g_i\}_{i\geqslant 1}$ is a bounded sequence in $\ell_\infty(X)$, we write $\operatorname{liminfsup}_i g_i$ for the set

$$\big\{\widetilde{g} \in \ell_\infty(X)\colon \liminf_i g_i \leqslant \widetilde{g} \leqslant \limsup_i g_i \text{ on } X\big\}.$$

Corollary 8. *Let $\varphi \in \ell_\infty(X)$ with $\varphi \geqslant 0$ on X, $\{h_j\}_{j\geqslant 1}$ be a bounded sequence in $\ell_\infty(X)$, $A > 0$ and suppose*

$$\text{(8.1)} \qquad S_X\big(h_0 - \limsup_j h_j - \varphi\big) = S_X\big(h_0 - \liminf_j h_j - \varphi\big) \geqslant A$$

for all $h_0 \in \operatorname{co}_\sigma\{h_j\colon j \geqslant 1\}$. Then there exist a pseudo-subsequence $\{g_i\}_{i\geqslant 1}$ of $\{h_j\}_{j\geqslant 1}$ and $g_0 \in \operatorname{co}_\sigma\{g_i\colon i \geqslant 1\}$ such that

$$\widetilde{g} \in \operatorname{liminfsup}_i g_i \quad\Longrightarrow\quad \operatorname{argmax}_X(g_0 - \widetilde{g} - \varphi) = \emptyset.$$

Proof: Let $N := \sup_{n\geqslant 1}\|h_n\|$ and $\rho \in (0,1)$ be so small that $(2N + \|\varphi\| + 1)\rho < A$. Let $\underline{h} := \liminf_j h_j \in \ell_\infty(X)$, $\overline{h} := \limsup_j h_j \in \ell_\infty(X)$ and, for all $j \geqslant 1$,

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$a_j := h_j - \underline{h} - \varphi$, so $\sup_{j\geqslant 1} \|a_j\| \leqslant 2N + \|\varphi\| < \infty$. We now apply Lemma 4(b) with $B := (2N + \|\varphi\| + 1)\rho$. It follows that there exists a pseudo-subsequence $\{g_i\}_{i\geqslant 1}$ of $\{h_j\}_{j\geqslant 1}$ such that

$$k \geqslant 0 \quad\Longrightarrow\quad S_X\Big(\sum_{i=1}^{\infty} \rho^i (g_i - \underline{h} - \varphi)\Big) \geqslant S_X\Big(\sum_{i=1}^{k} \rho^i (g_i - \underline{h} - \varphi)\Big) + B \sum_{i=k+1}^{\infty} \rho^i.$$

Let $\widetilde{g} \in \operatorname{liminfsup}_i g_i$. We have from Lemma 6 that $\widetilde{g} \in \operatorname{liminfsup}_j h_j$ and so the equality part of (8.1) gives us that

$$k \geqslant 0 \quad\Longrightarrow\quad S_X\Big(\sum_{i=1}^{\infty} \rho^i (g_i - \widetilde{g} - \varphi)\Big) \geqslant S_X\Big(\sum_{i=1}^{k} \rho^i (g_i - \widetilde{g} - \varphi)\Big) + B \sum_{i=k+1}^{\infty} \rho^i.$$

For all $i \geqslant 1$, let $b_i := g_i - \widetilde{g} - \varphi$. Now $\sup_{i\geqslant 1} \|b_i\| \leqslant 2N + \|\varphi\| < \infty$, and so we can apply Lemma 5. It follows that if $p \in \operatorname{argmax}_X\big(\sum_{i=1}^{\infty} \rho^i (g_i - \widetilde{g} - \varphi)\big)$ then

$$\inf_{k\geqslant 1} (g_k - \widetilde{g} - \varphi)(p) \geqslant \frac{(2N + \|\varphi\| + 1)\rho - \rho(2N + \|\varphi\|)}{1-\rho} = \frac{\rho}{1-\rho} > 0,$$

which is impossible, since $\inf_{k\geqslant 1} g_k \leqslant \widetilde{g}$ on X and $\varphi(p) \geqslant 0$. Consequently,

$$\operatorname{argmax}_X\Big(\sum_{i=1}^{\infty} \rho^i (g_i - \widetilde{g} - \varphi)\Big) = \emptyset,$$

and the required result follows with $g_0 := \sum_{i=1}^{\infty} \rho^i g_i \Big/ \sum_{i=1}^{\infty} \rho^i$. ∎

The proof of Lemma 9 below is based on a technique used by Pryce in his proof ([5]) of James's theorem.

Lemma 9.

(a) *Let $\{a_j\}_{j\geqslant 1}$ be a bounded sequence in $\ell_\infty(X)$ and $\varepsilon > 0$. Then there exists a subsequence $\{b_j\}_{j\geqslant 1}$ of $\{a_j\}_{j\geqslant 1}$ such that*

$$S_X\big(\liminf_j b_j\big) \geqslant S_X\big(\limsup_j a_j\big) - \varepsilon. \tag{9.1}$$

(b) *Let $\{a_j\}_{j\geqslant 1}$ be a bounded sequence in $\ell_\infty(X)$. Then there exists a subsequence $\{b_j\}_{j\geqslant 1}$ of $\{a_j\}_{j\geqslant 1}$ such that*

$$S_X\big(\liminf_j b_j\big) = S_X\big(\limsup_j b_j\big). \tag{9.2}$$

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(c) *Let $\varphi \in \ell_\infty(X)$ and $\varphi \geqslant 0$ on X. Let $\{f_j\}_{j\geqslant 1}$ be a bounded sequence in $\ell_\infty(X)$. Then there exists a subsequence $\{h_j\}_{j\geqslant 1}$ of $\{f_j\}_{j\geqslant 1}$ such that*

$$S_X\big(h_0 - \limsup_j h_j - \varphi\big) = S_X\big(h_0 - \liminf_j h_j - \varphi\big) \tag{9.3}$$

for all $h_0 \in \mathrm{co}_\sigma\{h_j\colon\ j \geqslant 1\}$.

PROOF: (a) We first choose $x \in X$ so that

$$\limsup_j a_j(x) \geqslant S_X\big(\limsup_j a_j\big) - \varepsilon,$$

and then choose a subsequence $\{b_j\}_{j\geqslant 1}$ of $\{a_j\}_{j\geqslant 1}$ so that

$$\lim_j b_j(x) = \limsup_j a_j(x) \text{ — hence } \liminf_j b_j(x) \geqslant S_X\big(\limsup_j a_j\big) - \varepsilon.$$

This gives (9.1), and (a) follows immediately.

(b) For all $j \geqslant 1$, let $b_j^{(0)} := a_j$. From (a), for all $m \geqslant 1$, we can define inductively a subsequence $\{b_j^{(m)}\}_{j\geqslant 1}$ of $\{b_j^{(m-1)}\}_{j\geqslant 1}$ so that

$$S_X\big(\liminf_j b_j^{(m)}\big) \geqslant S_X\big(\limsup_j b_j^{(m-1)}\big) - 1/m. \tag{9.4}$$

From the diagonal argument, there exists a bounded sequence $\{b_j\}_{j\geqslant 1}$ in $\ell_\infty(X)$ such that, for all $m \geqslant 1$, $\{b_j\}_{j\geqslant m}$ is a subsequence of $\{b_j^{(m)}\}_{j\geqslant 1}$, and (9.2) now follows from (9.4) by using Lemma 6 and letting $m \to \infty$.

(c) Since the set $\mathrm{co}_\sigma\{f_j\colon\ j \geqslant 1\}$ is norm–separable, we can choose $\{d_m\colon\ m \geqslant 1\}$ to be norm–dense in $\mathrm{co}_\sigma\{f_j\colon\ j \geqslant 1\}$. For all $j \geqslant 1$, let $h_j^{(0)} := f_j$. Using (b) with $a_j := d_m - h_j^{(m-1)} - \varphi$, for all $m \geqslant 1$, we can find a subsequence $\{h_j^{(m)}\}_{j\geqslant 1}$ of $\{h_j^{(m-1)}\}_{j\geqslant 1}$ inductively so that

$$S_X\big(d_m - \limsup_j h_j^{(m)} - \varphi\big) = S_X\big(d_m - \liminf_j h_j^{(m)} - \varphi\big). \tag{9.5}$$

From the diagonal argument, there exists a bounded sequence $\{h_j\}_{j\geqslant 1}$ in $\ell_\infty(X)$ such that, for all $m \geqslant 1$, $\{h_j\}_{j\geqslant m}$ is a subsequence of $\{h_j^{(m)}\}_{j\geqslant 1}$. Then, from Lemma 6 and (9.5),

$$\text{for all } m \geqslant 1, \quad S_X\big(d_m - \limsup_j h_j - \varphi\big) = S_X\big(d_m - \liminf_j h_j - \varphi\big).$$

(9.3) now follows since $S_X(\cdot - \varphi)$ is norm–continuous and $\{d_m\colon\ m \geqslant 1\}$ is norm–dense in the set $\mathrm{co}_\sigma\{f_j\colon\ j \geqslant 1\}$, and so certainly norm–dense in the (sub)set $\mathrm{co}_\sigma\{h_j\colon j \geqslant 1\}$. ∎

Theorem 10. *Let $\varphi \in \ell_\infty(X)$ and $\varphi \geqslant 0$ on X. Let $\{f_j\}_{j\geqslant 1}$ be a bounded sequence in $\ell_\infty(X)$, $A > 0$ and*

$$S_X\big(f_0 - \limsup_j f_j - \varphi\big) \geqslant A \tag{10.1}$$

for all $f_0 \in \mathrm{co}_\sigma\{f_j\colon j \geqslant 1\}$. Then there exist a pseudo-subsequence $\{g_i\}_{i\geqslant 1}$ of $\{f_j\}_{j\geqslant 1}$ and $g_0 \in \mathrm{co}_\sigma\{g_i\colon i \geqslant 1\}$ such that

$$\mathrm{argmax}_X(g_0 - \widetilde{g} - \varphi) = \emptyset \ \text{ for all } \ \widetilde{g} \in \mathrm{liminfsup}_i\, g_i.$$

Proof: From Lemma 9(c), there exists a subsequence $\{h_j\}_{j\geqslant 1}$ of $\{f_j\}_{j\geqslant 1}$ satisfying (9.3) for all $h_0 \in \mathrm{co}_\sigma\{h_j : j \geqslant 1\}$. Since $\mathrm{co}_\sigma\{h_j\colon j \geqslant 1\} \subset \mathrm{co}_\sigma\{f_j\colon j \geqslant 1\}$, (8.1) follows by combining this with Lemma 6 and (10.1). The result now follows from Corollary 8. ∎

4. A minimax theorem that implies a nonlinear version of James's theorem

For the rest of this paper, we shall suppose that E is a real locally convex space with dual E^* and $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E\times E^*$. We shall suppose also that X is a nonempty convex subset of E and Y is a nonempty convex subset of E^* such that $\langle\cdot,\cdot\rangle$ is bounded on $X\times Y$. We write "δ" as an alias for $\langle\cdot,\cdot\rangle$, so "$\sup_X \inf_Y \delta$" stands for $\sup_{x\in X}\inf_{x^*\in Y}\langle x, x^*\rangle$ and "$\inf_Y \sup_X \delta$" stands for $\inf_{x^*\in Y}\sup_{x\in X}\langle x, x^*\rangle$. Then, as is well known, $\sup_X \inf_Y \delta \leqslant \inf_Y \sup_X \delta$. We write $\mathrm{dgap}(X,Y) := \inf_Y \sup_X \delta - \sup_X \inf_Y \delta$. "dgap" stands for "duality gap". However, we should caution the reader that some authors use the phrase "duality gap" for the interval $\left[\sup_X \inf_Y \delta, \inf_Y \sup_X \delta\right]$. If $\varphi \in \ell_\infty(X)$, we write $\mathrm{osc}_X\,\varphi$ for the "oscillation" of φ on X, defined by $\mathrm{osc}_X\,\varphi := \sup_X \varphi - \inf_X \varphi$.

We shall need a fact about convex functions for our analysis. We could use a minimax theorem for this, but it is somewhat more direct to use the following result (see [**9**, Lemma 2.1, p.15]), which can also be deduced from Fan–Glicksberg–Hoffman, [**1**, Theorem 1, p.618], after some simple transformations.

Lemma 11. *Let C be a nonempty convex subset of a vector space and $f_1, \ldots, f_n$ be concave real functions on C. Then there exist $\lambda_1, \ldots, \lambda_n \geqslant 0$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and*

$$\sup_C[f_1 \wedge \cdots \wedge f_n] = \sup_C[\lambda_1 f_1 + \cdots + \lambda_n f_n].$$

Lemma 12.

(a) *Suppose that $\inf_Y \sup_X \delta > \beta$. Then, for all $x_1^*, \ldots, x_n^* \in Y$, there exists $x \in X$ such that $\langle x, x_1^*\rangle \wedge \cdots \wedge \langle x, x_n^*\rangle > \beta$.*

(b) *Suppose that $\sup_X \inf_Y \delta < \alpha$. Then, for all $x_1, \ldots, x_n \in X$, there exists $x^* \in Y$ such that $\langle x_1, x^*\rangle \vee \cdots \vee \langle x_n, x^*\rangle < \alpha$.*

PROOF: From Lemma 11, there exist $\lambda_1, \dots, \lambda_n \geqslant 0$ such that $\lambda_1 + \dots + \lambda_n = 1$ and

$$\sup_{x \in X}[\langle x, x_1^* \rangle \wedge \dots \wedge \langle x, x_n^* \rangle] = \sup_{x \in X}[\lambda_1 \langle x, x_1^* \rangle + \dots + \lambda_n \langle x, x_n^* \rangle] \geqslant \sup_{x \in X} \langle x, \lambda_1 x_1^* + \dots + \lambda_n x_n^* \rangle > \beta,$$

since $\lambda_1 x_1^* + \dots + \lambda_n x_n^* \in Y$. This completes the proof of (a), and the proof of (b) is similar. ∎

THEOREM 13. *Suppose that $\varphi \in \ell_\infty(X)$ and $\operatorname{osc}_X \varphi < \operatorname{dgap}(X, Y)$. Then there exist a sequence $\{x_j^*\}_{j \geqslant 1}$ in Y, a pseudo-subsequence $\{g_i\}_{i \geqslant 1}$ of $\{x_j^*|_X\}_{j \geqslant 1}$ in $\ell_\infty(X)$ and $g_0 \in \operatorname{co}_\sigma\{g_i \colon i \geqslant 1\}$ such that*

(13.1) $$\operatorname{argmax}_X(g_0 - \widetilde{g} - \varphi) = \emptyset, \quad \text{for all } \widetilde{g} \in \operatorname{liminfsup}_i g_i.$$

PROOF: Since $\operatorname{osc}_X \varphi$ and (13.1) are unaffected by adding a constant to φ, we can and shall suppose that $\inf_X \varphi = 0$, so $\varphi \geqslant 0$ on X and also $\operatorname{osc}_X \varphi = S_X(\varphi)$. Let $M := \sup |\langle X, Y \rangle|$, and choose $\alpha > \sup_X \inf_Y \delta$ and $\beta < \inf_Y \sup_X \delta$ so that $\beta - \alpha > \operatorname{osc}_X \varphi$. Let x_1^* be an arbitrary element of Y. Then, using (a) and (b) of Lemma 12 alternately, we can find $x_1 \in X$, $x_2^* \in Y$, $x_2 \in X$, ... so that

$$\langle x_n, x_1^* \rangle \wedge \dots \wedge \langle x_n, x_n^* \rangle > \beta \quad \text{and} \quad \langle x_1, x_{n+1}^* \rangle \vee \dots \vee \langle x_n, x_{n+1}^* \rangle < \alpha.$$

Write $f_j := x_j^*|_X \in \ell_\infty(X)$. Then $j \leqslant n \Longrightarrow f_j(x_n) > \beta$ and $n < j \Longrightarrow f_j(x_n) < \alpha$. It follows that, for all $n \geqslant 1$, $\limsup_j f_j(x_n) \leqslant \alpha$. On the other hand, if $f_0 \in \operatorname{co}_\sigma\{f_j \colon j \geqslant 1\}$, then there exists a sequence $\{\lambda_j\}_{j \geqslant 1}$ of elements of $[0, 1]$ such that $\sum_{j=1}^{\infty} \lambda_j = 1$ and $f_0 = \sum_{j=1}^{\infty} \lambda_j f_j$. Thus, for all $n \geqslant 1$,

$$\begin{aligned} f_0(x_n) &= \sum_{j=1}^{\infty} \lambda_j f_j(x_n) = \sum_{j=1}^{n} \lambda_j f_j(x_n) + \sum_{j=n+1}^{\infty} \lambda_j f_j(x_n) \\ &\geqslant \sum_{j=1}^{n} \lambda_j \beta - \sum_{j=n+1}^{\infty} \lambda_j M = \sum_{j=1}^{\infty} \lambda_j \beta - \sum_{j=n+1}^{\infty} \lambda_j \beta - \sum_{j=n+1}^{\infty} \lambda_j M \\ &= \beta - \sum_{j=n+1}^{\infty} \lambda_j (\beta + M). \end{aligned}$$

Thus $S_X(f_0 - \limsup_j f_j) \geqslant f_0(x_n) - \limsup_j f_j(x_n) \geqslant \beta - \alpha - \sum_{j=n+1}^{\infty} \lambda_j(\beta + M)$. If we now let $n \to \infty$, we obtain $S_X(f_0 - \limsup_j f_j) \geqslant \beta - \alpha$, and the result follows from Theorem 10 since $S_X(f_0 - \limsup_j f_j - \varphi) \geqslant \beta - \alpha - S_X(\varphi) > \operatorname{osc}_X \varphi - S_X(\varphi) = 0$. ∎

We now come to our main result, a minimax theorem generalising [**7**, Theorem 14, p.715–6].

Theorem 14. *Suppose that*

(14.1) *every sequence in Y has a $w(E^*,X)$–cluster point in E^**

and there exists $\psi \in \ell_\infty(X)$ such that,

(14.2) *for all $x^* \in E^*$,* $\operatorname{argmax}_X(x^*|_X - \psi) \neq \emptyset$.

Then

(14.3) $$\inf_Y \sup_X \delta = \sup_X \inf_Y \delta.$$

Proof: If (14.3) fails then $\operatorname{dgap}(X,Y) > 0$, so we can choose $\rho > 0$ so that

$$\operatorname{dgap}(X,Y) > \rho \operatorname{osc}_X \psi = \operatorname{osc}_X \rho\psi.$$

From Theorem 13, we can find a sequence $\{x_j^*\}_{j\geqslant 1}$ in Y, a pseudo-subsequence $\{g_i\}_{i\geqslant 1}$ of $\{x_j^*|_X\}_{j\geqslant 1}$ in $\ell_\infty(X)$ and $g_0 \in \operatorname{co}_\sigma\{g_i\colon\ i \geqslant 1\}$ such that

(14.4) $$\operatorname{argmax}_X(g_0 - \widetilde{g} - \rho\psi) = \emptyset, \quad \text{for all } \widetilde{g} \in \operatorname{liminfsup}_i g_i.$$

Lemma 2 implies that $g_0 \in \operatorname{co}_\sigma\{x_j^*|_X\colon\ j \geqslant 1\}$. Thus there exists $\{\lambda_j\}_{j\geqslant 1} \in \mathcal{P}$ such that $g_0 = \sum_{j=1}^{\infty} \lambda_j x_j^*|_X$ in $\ell_\infty(X)$. Fix $n_0 \geqslant 1$ so that $\lambda_{n_0} > 0$. For $n \geqslant n_0$, let

$$y_n^* := \sum_{j=1}^{n} \lambda_j x_j^* \Big/ \sum_{j=1}^{n} \lambda_j \in Y.$$

From (14.1), $\{y_n^*\}_{n\geqslant n_0}$ has a $w(E^*,X)$–cluster point $y^* \in E^*$. Since $y_n^*|_X \to g_0$ in $\ell_\infty(X)$, it follows that $g_0 = y^*|_X$. Now suppose that $i \geqslant 1$ and examine g_i. The argument above shows that there exists $z_i^* \in Y$ such that $\|z_i^*|_X - g_i\| \leqslant 1/i$ and, from (14.1), $\{z_i^*\}_{i\geqslant 1}$ has a $w(E^*,X)$–cluster point $z^* \in E^*$. It follows from all this that

$$\liminf_i g_i = \liminf_i z_i^* \leqslant z^* \leqslant \limsup_i z_i^* = \limsup_i g_i \quad \text{on } X,$$

and so $z^*|_X \in \operatorname{liminfsup}_i g_i$. (14.4) now gives $\operatorname{argmax}_X\big(y^*|_X - z^*|_X - \rho\psi\big) = \emptyset$, which would contradict (14.2) since $x^* := (y^* - z^*)/\rho \in E^*$. ∎

The following converse minimax theorem was proved in [**7**, Theorem 15, p.717]. However, we include in Lemma 18 in the appendix a proof (using Goldstine's theorem) of Lemma 15 that is valid for the special case when E is normed space.

Lemma 15. *Suppose that X is bounded and complete in E and, for all nonempty convex equicontinuous subsets Y of E^*, $\inf_Y \sup_X \delta = \sup_X \inf_Y \delta$. Then X is $w(E, E^*)$-compact.*

If we now combine Theorem 14 and Lemma 15, we obtain the following nonlinear version of James's theorem. (If Y is equicontinuous in E^* then $\langle\cdot,\cdot\rangle$ is bounded on $X \times Y$ and, further, Y is $w(E^*, E)$ relatively compact in E^*, from which (14.1) is satisfied.)

Theorem 16. *Suppose that X is bounded and complete in E and there exists $\psi \in \ell_\infty(X)$ such that*

$$\text{(14.2)} \qquad \text{for all } x^* \in E^*, \quad \operatorname{argmax}_X(x^*|_X - \psi) \neq \emptyset.$$

Then X is $w(E, E^*)$-compact.

Remark 17. James gave an example in [**3**] of an incomplete normed space E such that every bounded linear functional on E attains its norm on the unit ball, X, of E. X clearly satisfied the condition (14.2) with $\psi = 0$. However, since E is not reflexive, X is not weakly compact.

Appendix

While the preceding analysis is valid for locally convex spaces, it is possible to give another proof of Lemma 15 for normed spaces that uses more standard techniques. For the convenience of the reader, we give details of this proof in Lemma 18 below.

Lemma 18. *Let E be a real normed space and suppose that X is bounded and complete in E and, for all nonempty convex equicontinuous subsets Y of E^*, $\inf_Y \sup_X \delta = \sup_X \inf_Y \delta$. Then X is $w(E, E^*)$-compact.*

Proof: Let $\widehat{\ }$ stand for the canonical map from E into E^{**} or the canonical map from E^* into E^{***}, as the case may be. Write $\widetilde{X}$ for the $w(E^{**}, E^*)$-closure of $\widehat{X}$ in E^{**}. We first prove that

$$\text{(18.1)} \qquad x^{**} \in \widetilde{X} \text{ and } x^{***} \in E^{***} \quad \Longrightarrow \quad \langle x^{**}, x^{***}\rangle \leqslant \sup_{x \in X} \langle \widehat{x}, x^{***}\rangle.$$

To this end, suppose that $x^{**} \in \widetilde{X}$ and $x^{***} \in E^{***}$. Then

$$\text{(18.2)} \qquad x^* \in E^* \quad \Longrightarrow \quad \langle x^*, x^{**}\rangle \leqslant \sup_{x \in X} \langle x^*, \widehat{x}\rangle = \sup_{x \in X} \langle x, x^*\rangle$$

and Goldstine's theorem (see, for instance, [**6**, Section 28.40, p.777]), provides us with a bounded net $\{x_\lambda^*\}_{\lambda \in \Lambda}$ in E^* such that $\widehat{x_\lambda^*} \to x^{***}$ in $w(E^{***}, E^{**})$. Let $\lambda_0 \in \Lambda$, and $Y = \operatorname{co}\{x_\lambda^* \colon \lambda \geqslant \lambda_0\} \subset E^*$. Then, from (18.2)

$$\inf_{\lambda \geqslant \lambda_0} \langle x^{**}, \widehat{x_\lambda^*}\rangle = \inf_{\lambda \geqslant \lambda_0} \langle x_\lambda^*, x^{**}\rangle = \inf_{x^* \in Y} \langle x^*, x^{**}\rangle \leqslant \inf_Y \sup_X \delta.$$

Since Y is equicontinuous, by hypothesis,

$$\inf_{\lambda \geqslant \lambda_0} \langle x^{**}, \widehat{x^*_\lambda} \rangle \leqslant \sup_X \inf_Y \delta = \sup_{x \in X} \inf_{x^* \in Y} \langle \widehat{x}, \widehat{x^*} \rangle$$
$$= \sup_{x \in X} \inf_{\lambda \geqslant \lambda_0} \langle \widehat{x}, \widehat{x^*_\lambda} \rangle \leqslant \sup_{x \in X} \lim_\lambda \langle \widehat{x}, \widehat{x^*_\lambda} \rangle = \sup_{x \in X} \langle \widehat{x}, x^{***} \rangle.$$

This completes the proof of (18.1), since $\inf_{\lambda \geqslant \lambda_0} \langle x^{**}, \widehat{x^*_\lambda} \rangle \to \lim_\lambda \langle x^{**}, \widehat{x^*_\lambda} \rangle = \langle x^{**}, x^{***} \rangle$ as λ_0 runs through Λ. Now the canonical map is a norm–isometry from X onto $\widehat{X}$, and so $\widehat{X}$ is norm–closed and (by convexity) also $w(E^{**}, E^{***})$–closed. Thus (18.1) implies that $\widetilde{X} \subset \widehat{X}$, from which $\widehat{X} = \widetilde{X}$. The Banach–Alaoglu theorem now gives us that $\widehat{X}$ is $w(E^{**}, E^*)$–compact. On the other hand, the canonical map is also a $w(E, E^*)$ $-w(E^{**}, E^*)$ homeomorphism from X onto $\widehat{X}$, and so X is $w(E, E^*)$–compact, as required. ∎

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Departamento de Matemática Aplicada
E. U. Arquitectura Técnica
Universidad de Granada
18071 Granada
Spain

Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
United States of America