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The Minimal Free Resolution of Fat Almost Complete Intersections in $\mathbb{P}^1 \times \mathbb{P}^1$

Giuseppe Favacchio and Elena Guardo

Abstract. A current research theme is to compare symbolic powers of an ideal I with the regular powers of I. In this paper, we focus on the case where $I = I_X$ is an ideal defining an almost complete intersection (ACI) set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, we describe a minimal free bigraded resolution of a non-arithmetically Cohen-Macaulay (also non-homogeneous) set $\mathcal{Z}$ of fat points whose support is an ACI, generalizing an earlier result of Cooper et al. for homogeneous sets of triple points. We call $\mathcal{Z}$ a fat ACI. We also show that its symbolic and ordinary powers are equal, *i.e,* $I_{\mathcal{Z}}^{(m)} = I_{\mathcal{Z}}^m$ for any $m \geq 1$.

1 Introduction

A research problem of interest regarding which symbolic powers of ideals are contained in a given ordinary power of the ideal has recently been studied in [1–3, 12], with a focus on ideals defining 0-dimensional subschemes of projective space.

Inspired by recent papers of [5, 7–9], we focus on the case where I is an ideal defining a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, since, in particular, I can be considered as a set of particular lines in $\mathbb{P}^3$.

Throughout this paper, the polynomial ring $R := k[x_0, x_1, x_2, x_3]$ with the bigrading given by $\deg x_0 = \deg x_1 = (1,0)$ and $\deg x_2 = \deg x_3 = (0,1)$ is the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. A point is denoted by $P = [a_0 : a_1] \times [b_0 : b_1]$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and it is defined by the bihomogeneous ideal $I_P = (a_1x_0 - a_0x_1, b_1x_2 - b_0x_3)$. A set of points $X = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is then associated with the bihomogeneous ideal $I_X = \bigcap_{P \in X} I_P$. If we only consider the standard grading of this ideal, then I_X defines a union X of lines in $\mathbb{P}^3$. Given a set of distinct points $X = \{P_1, \dots, P_s\}$ and positive integers $m_1, \dots, m_s$, we call $Z = m_1P_1 + \cdots + m_sP_s$ a *set of fat points supported at* X.

Given a homogeneous ideal $I \subset R$, the m-th symbolic power of I is the ideal $I^{(m)} = R \cap (\bigcap_{P \in Ass(I)} (I^m R_P))$. Following [3], an ideal of the form $I = \bigcap_i (I_{P_i}^{m_i})$ where $P_1, \dots, P_n$ are distinct points of $\mathbb{P}^1 \times \mathbb{P}^1$, I_{P_i} is the ideal generated by all forms vanishing at P_i, and each m_i is a non-negative integer, $I^{(m)}$ turns out to be $\bigcap_i (I_{P_i}^{mm_i})$. If I^m is the usual power, then there is clearly a containment $I^m \subseteq I^{(m)}$, and a much more difficult problem is to determine when there are containments of the form $I^{(m)} \subseteq I^r$. Furthermore, the m-th symbolic power of I_X has the form $I_X^{(m)} = \bigcap_{i=1}^s I_{P_i}^m$. The scheme defined by $I_X^{(m)}$ is sometimes referred to as a *homogeneous* set of fat points and denoted by $mP_1 + \cdots + mP_s$.

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We say that a set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$ is *arithmetically Cohen-Macaulay* (ACM) if its coordinate ring R/I_X is Cohen-Macaulay. A set of points X is a *complete intersection* if I_X is a complete intersection. We write that $X = CI(a,b)$ if I_X is generated by a form of degree $(a,0)$ and a form of degree $(0,b)$. The set X is an *almost complete intersection* (ACI) if the number of minimal generators is one more than the codimension of X; *i.e.,* X has three minimal generators.

Let X be an almost complete intersection in $\mathbb{P}^1 \times \mathbb{P}^1$ and let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points supported at X. We call $\mathcal{Z}$ a *fat almost complete intersection*.

A classification of reduced and fat ACM sets of points of $\mathbb{P}^1 \times \mathbb{P}^1$ can be found in [10, Theorems 4.11 and 6.21].

In this paper, we focus on the study of special sets of fat points $\mathcal{Z}$ whose support is either ACM or non-ACM. In particular, we give a minimal free bigraded resolution of $\mathcal{Z}$ in both cases (see Theorems 3.4 and 3.5).

In [8, Theorem 1.1], the authors proved the following theorem.

Theorem 1.1 ([8, Theorem 1.1]) *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be an ACM set of points. Then $I_X^m = I_X^{(m)}$ for all $m \geq 1$ if and only if $I_X^3 = I_X^{(3)}$.*

In [5], S. Cooper et al. proposed a classification of the sets of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ satisfying $I_X^3 = I_X^{(3)}$. We require the following notation. Let $\pi_1: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the natural projection $P = A \times B \mapsto A$. If $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a finite set of reduced points, let $\pi_1(X) = \{H_1, \dots, H_r\}$ be the set of distinct first coordinates that appear in X. For $i = 1, \dots, h$, set $\overline{\alpha_i} = |X \cap \pi_1^{-1}(H_i)|$, *i.e.,* the number of points in X whose first coordinate is H_i. After relabeling the H_i's so that $\overline{\alpha_i} \geq \overline{\alpha_{i+1}}$ for $i = 1, \dots, r-1$, we set $\alpha_X = (\overline{\alpha_1}, \dots, \overline{\alpha_r})$. In particular, they proved the following two results.

Corollary 1.2 ([5, Corollary 4.4]) *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be any ACM set of points. Then*

(i) $I_X^2 = I_X^{(2)}$.

(ii) *The following are equivalent:*
   (a) I_X^2 *defines an ACM scheme;*
   (b) $I_X^3 = I_X^{(3)}$ *is the saturated ideal of an ACM scheme;*
   (c) X *is a complete intersection;*
   (d) $\alpha_X = (a, a, \dots, a)$ *for some integer* $a \geq 1$.

(iii) *The following are equivalent:*
   (a) $I_X^3 = I_X^{(3)}$ *is the saturated ideal of a non-ACM scheme;*
   (b) I_X *is an almost complete intersection;*
   (c) $\alpha_X = (a, \dots, a, b, \dots, b)$ *for integers* $a > b \geq 1$.

Corollary 1.3 ([5, Corollary 4.6]) *Let $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a homogeneous set of triple points (*i.e., *where every point has multiplicity three) and let X be the support of Z. If I_X is an almost complete intersection with*

$$\alpha_X = (\underbrace{a, \dots, a}_{c}, \underbrace{b, \dots, b}_{d}),$$

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then I_Z has a bigraded minimal free resolution of the form

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I_Z \longrightarrow 0,$$

where

$$\begin{aligned}
F_0 =& R(-3c-3d,0) \oplus R(-3c-2d,-b) \oplus R(-2c-2d,-a)\\
&\oplus R(-3c-d,-2b) \oplus R(-2c-d,-b-a) \oplus R(-c-d,-2a)\\
&\oplus R(-3c,-3b) \oplus R(-2c,-2b-a) \oplus R(-c,-b-2a)\\
&\oplus R(0,-3a),\\
F_1 =& R(-c,-3a) \oplus R(-2c,-2a-b) \oplus R(-3c,-a-2b) \oplus R(-c-d,-2a-b)\\
&\oplus R(-2c-d,-a-2b) \oplus R(-3c-d,-3b) \oplus R(-2c-d,-2a)\\
&\oplus R(-3c-d,-a-b) \oplus R(-2c-2d,-a-b) \oplus R(-3c-2d,-2b)\\
&\oplus R(-3c-2d,-a) \oplus R(-3c-3d,-b),\\
F_2 =& R(-3c-2d,-b-a) \oplus R(-3c-d,-a-2b) \oplus R(-2c-d,-2a-b).
\end{aligned}$$

Here, we generalize Corollary 1.3 for a special set $\mathcal{Z}$ of fat points whose support is an almost complete intersection (ACI), *i.e.,* for a special fat almost complete intersection. We note that we do not require that $\mathcal{Z}$ be homogeneous. To shorten the notation we will say $\mathcal{Z}$ is a fat ACI.

Let X be an ACI set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$ such that

$$\alpha_X = (\underbrace{a,\dots,a}_{\alpha_1},\underbrace{b,\dots,b}_{\alpha_2})$$

for two integers $a > b \geq 1$. Set $a := \beta_1 + \beta_2$, $b := \beta_1$ and $r = \alpha_1 + \alpha_2$ so that

$$\alpha_X = (\underbrace{\beta_1+\beta_2,\dots,\beta_1+\beta_2}_{\alpha_1},\underbrace{\beta_1,\dots,\beta_1}_{\alpha_2}).$$

Let H_i be horizontal lines of type $(1,0)$ and let V_j be vertical lines of type $(0,1)$. Then a point in $\mathbb{P}^1 \times \mathbb{P}^1$ can be denoted by $P_{ij} := H_i \times V_j$. If $\pi_1(X) = \{H_1,\dots,H_r\}$ and $\pi_2(X) = \{V_1,\dots,V_a\}$, then $X \subset W = \{P_{ij} \mid i = 1,\dots,r \text{ and } j = 1,\dots,a\}$. Note that W is a complete intersection of reduced points.

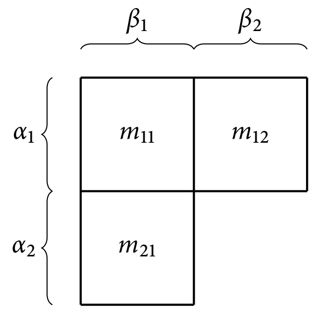
Define $\mathcal{Z} := \sum w_{ij}P_{ij}$ to be a fat ACI of $\mathbb{P}^1 \times \mathbb{P}^1$ where

$$w_{ij} = \begin{cases} m_{11} & \text{if } (i,j) \leq (\alpha_1,\beta_1),\\ m_{21} & \text{if } (\alpha_1+1,1) \leq (i,j) \leq (\alpha_1+\alpha_2,\beta_1),\\ m_{12} & \text{if } (1,\beta_1+1) \leq (i,j) \leq (\alpha_1,\beta_1+\beta_2),\\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

for some non-negative integers m_{11}, m_{12}, m_{21}. Renumbering the lines H_i or V_j, we can always assume that $m_{21} \leq m_{12}$.

The following picture shows what $\mathcal{Z}$ looks like.

$\mathcal{Z} =$



We denote by $\mathcal{Z}_1 := \sum \overline{w}_{ij} P_{ij}$ a set of fat points of $\mathbb{P}^1 \times \mathbb{P}^1$, where

$$\overline{w}_{ij} = \begin{cases} (m_{11} - 1)_+ & \text{if } (i, j) \le (\alpha_1, \beta_1), \\ (m_{21} - 1)_+ & \text{if } (\alpha_1 + 1, 1) \le (i, j) \le (\alpha_1 + \alpha_2, \beta_1), \\ m_{12} & \text{if } (1, \beta_1 + 1) \le (i, j) \le (\alpha_1, \beta_1 + \beta_2), \\ 0 & \text{otherwise,} \end{cases}$$

for m_{11}, m_{12}, m_{21} as in $\mathcal{Z}$ and $(n)_+ := \max\{n, 0\}$.

The following theorem is the main result of this paper.

***Theorem* 3.5** *Let $0 \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0 \to R \to R/I_{\mathcal{Z}_1} \to 0$ be a minimal free resolution of $I_{\mathcal{Z}_1}$. Then a minimal free resolution of a fat ACI of type (1.1) $I_{\mathcal{Z}}$ is*

$$\begin{aligned} 0 \longrightarrow & \bigoplus_{(a,b-\beta_1) \in \mathcal{A}_1(\mathcal{Z})} R(-a, -b) \oplus \mathcal{L}_2(0, -\beta_1) \\ \longrightarrow & \bigoplus_{(a,b-\beta_1) \in \mathcal{A}_0(\mathcal{Z})} R(-a, -b) \bigoplus_{(a,b) \in \mathcal{A}_1(\mathcal{Z})} R(-a, -b) \oplus \mathcal{L}_1(0, -\beta_1) \\ \longrightarrow & \bigoplus_{(a,b) \in \mathcal{A}_0(\mathcal{Z})} R(-a, -b) \oplus \mathcal{L}_0(0, -\beta_1) \longrightarrow I_{\mathcal{Z}} \longrightarrow 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_0(\mathcal{Z}) &= \big\{ (\alpha_1(m_{11} + i) + \alpha_2 m_{21}, ((m_{12} - m_{11})_+ - i)\beta_2) \mid i = 0, \dots, (m_{12} - m_{11})_+ \big\}, \\ \mathcal{A}_1(\mathcal{Z}) &= \big\{ (\alpha_1(m_{11} + i + 1) + \alpha_2 m_{21}, ((m_{12} - m_{11})_+ - i)\beta_2) \mid \\ & \qquad\qquad\qquad\qquad i = 0, \dots, (m_{12} - m_{11})_+ - 1 \big\}. \end{aligned}$$

That is, if we set $\mu = \min(m_{11}, m_{21})$ recursively, we find a minimal bigraded free resolution of non-homogeneous sets of fat points $\mathcal{Z}_i \subset \mathcal{Z}$ whose support is an almost complete intersection for all $i = 0, \dots, \mu$ but $\mathcal{Z}_\mu$. In particular, $\mathcal{Z}_0 = \mathcal{Z}$ and the base case $\mathcal{Z}_\mu$ can be of two types

$$(1)\quad \begin{array}{|c|c|}\hline m_{11}-m_{21} & m_{12} \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \text{or} \qquad (2)\quad \begin{array}{|c|c|}\hline 0 & m_{12} \\ \hline m_{21}-m_{11} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

(a) If $m_{11} > m_{21}$, then $\mathcal{Z}_\mu$ is an ACM set fat points supported on a complete intersection $CI(\alpha_1, \beta)$. From [10, Theorem 6.21] we can recover its minimal bigraded free resolution.

(b) If $m_{11} < m_{21}$, then $\mathcal{Z}_\mu$ is not ACM. In this case, Lemma 3.4 gives a minimal free bigraded resolution of $\mathcal{Z}_\mu$. In particular, in this second case, the support X of $\mathcal{Z}_\mu$ is the disjoint union of two complete intersections $X_1 = CI(\alpha_1, \beta_2)$ and $X_2 = CI(\alpha_2, \beta_1)$.

(c) The case $m_{11} = m_{21}$ is shown in Corollary 3.7. In this case, the support of $\mathcal{Z}_\mu$ is a $CI(\alpha_1, \beta_2)$.

We also note that Theorem 3.5 in the case where $m_{11} = m_{12} = m_{21} = 3$ gives Corollary 1.3 proved in [5].

In Theorem 4.2, we prove that $I_{\mathcal{Z}}^{(m)} = I_{\mathcal{Z}}^m$ for any positive integer m where $\mathcal{Z}$ is a fat ACI of type (1.1). This result gives a new class of non-ACM set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose symbolic and regular powers are equals.

2 Background and Notation

In this section, we recall some well-known facts about ACM sets of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then we begin the study of a set $\mathcal{W}$ of three non-collinear fat points of $\mathbb{P}^1 \times \mathbb{P}^1$. We observe that $\operatorname{Supp}(\mathcal{W})$ of $\mathcal{W}$ is ACI but $\mathcal{W}$ can be either ACM or not ACM. Proposition 2.5 extends a property of the ACM set of points to our case of interest.

Lemma 2.1 *Let $P \in \mathbb{P}^1 \times \mathbb{P}^1$ be a point. Then the bigraded minimal free resolution of $I(P)^m$ is*

$$0 \longrightarrow \bigoplus_{t=1}^{m} R(t-m-1, -t) \longrightarrow \bigoplus_{t=0}^{m} R(t-m, -t) \longrightarrow I(P)^m \longrightarrow 0$$

Proof This follows, for instance, from [10, Theorem 6.27]. ■

From [11, Theorems 5.4 and 4.11], the following two results hold.

Lemma 2.2 *In $\mathbb{P}^1 \times \mathbb{P}^1$, let $\mathcal{Z}$ be*

$$\mathcal{Z} := \sum_{(1,1)\le(i,j)\le(\alpha,\beta_1)} m_{11} P_{ij} + \sum_{(1,\beta_1+1)\le(i,j)\le(\alpha,\beta_1+\beta_2)} m_{12} P_{ij}$$

a set of fat points whose support is $X = CI(\alpha, \beta)$ where $\beta = \beta_1 + \beta_2$.

Set $M := \max\{m_{11}, m_{12}\}$; *then a minimal free resolution of* $I_{\mathcal{Z}}$ *is*

$$0 \longrightarrow \bigoplus_{t=1}^{M} R(-\alpha t, -\beta_1(m_{11}-t+1)_+ - \beta_2(m_{12}-t+1)_+)$$
$$\longrightarrow \bigoplus_{t=0}^{M} R(-\alpha t, -\beta_1(m_{11}-t)_+ - \beta_2(m_{12}-t)_+) \longrightarrow I_{\mathcal{Z}} \longrightarrow 0$$

Proof We have that $\mathcal{Z}$ is ACM, and the associated tuple is

$$\alpha_{\mathcal{Z}} = (\underbrace{\gamma_0, \dots, \gamma_0}_{\alpha}, \underbrace{\gamma_1, \dots \gamma_1}_{\alpha}, \dots, \underbrace{\gamma_M, \dots \gamma_M}_{\alpha}).$$

where $\gamma_i := (m_{11}-i)_+\beta_{11} + (m_{12}-i)_+\beta_{12}$. ∎

Corollary 2.3 *With the notation as above, if* $m_{11} = m_{12}$, i.e., $\mathcal{Z}$ *is a homogeneous set of fat points whose support is* $X = CI(\alpha,\beta)$, *then a minimal free resolution is*

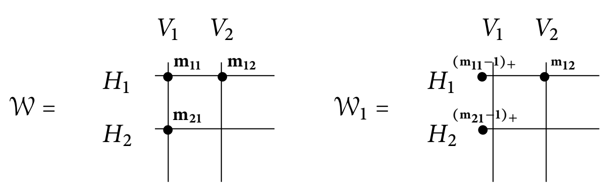
$$0 \longrightarrow \bigoplus_{i=0}^{m-1} R\big(-(i+1)\alpha, -(m-i)\beta\big) \longrightarrow \bigoplus_{i=0}^{m} R\big(-i\alpha, -(m-i)\beta\big) \longrightarrow I_{\mathcal{Z}} \longrightarrow 0$$

To describe a minimal free bigraded resolution of a fat ACI $\mathcal{Z}$ of type (1.1), we need to describe the minimal free bigraded resolution of a particular case of a fat ACI.

We set our notation.

Notation 2.4 Let $\mathcal{W}$ be a fat ACI consisting only of three non-collinear fat points $P_{ij} := H_i \times V_j$ with H_i horizontal lines of type $(1,0)$ and V_j vertical lines of type $(0,1)$ for $i, j = 1, 2$.

We will assume that $m_{21} \le m_{12}$ and $(a)_+ := \max\{a, 0\}$. Then $\mathcal{W} := m_{11}P_{11} + m_{21}P_{21} + m_{12}P_{12}$, and $\mathcal{W}_1 := (m_{11}-1)_+P_{11} + (m_{21}-1)_+P_{21} + m_{12}P_{12}$ is the set of points obtained from $\mathcal{W}$ by decreasing by 1 the multiplicity of each point on V_1.



If $m_{21} = 0$, then $\mathcal{W}$ is an ACM set of collinear points and everything is known ([11, Corollary 4.9 and Theorem 4.11]).

In order to describe the homological invariants of $\mathcal{W}$, we start by proving a proposition that holds for ACM finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$; see for instance [10, Theorem 7.12].

Proposition 2.5 *With the notation as above, let* $\mathcal{W} = m_{11}P_{11} + m_{21}P_{21} + m_{12}P_{12}$ *be a set of three non-collinear fat points in* $\mathbb{P}^1 \times \mathbb{P}^1$; *then* $I_{\mathcal{W}}$ *is minimally generated by a set of forms such that each of them is a product of powers of lines.*

Proof We claim that $I_{\mathcal{W}}$ is generated by the set of bihomogeneous forms

$$\mathcal{G}(\mathcal{W}) = \{H_1^{a_1}H_2^{a_2}V_1^{b_1}V_2^{b_2} \mid a_1+b_2 \geq m_{12},\ a_2+b_1 \geq m_{21},\ a_1+b_1 \geq m_{11}\}.$$

It is easy to check that $H_1^{a_1}H_2^{a_2}V_1^{b_1}V_2^{b_2} \in \mathcal{G}(\mathcal{W})$ if and only if $H_1^{a_1}H_2^{a_2}V_1^{b_1}V_2^{b_2} \in I_{\mathcal{W}}$. On the other hand, we distinguish the following cases:

(a) If either $m_{12} = 0$ or $m_{21} = 0$, then $\mathcal{W}$ is ACM, and so the statement is true.

(b) Suppose $m_{12} > 0$ and $m_{21} > 0$ and let $F \in I_{\mathcal{W}}$ be a bihomogeneous form of bidegree (a,b). Since $F \in (H_1, V_2)^{m_{12}}$ we get $F = \sum_i Q_i H_1^i V_2^{m_{12}-i}$, where either $Q_i = 0$ or $\deg(Q_i) = (a-i, b-m_{12}+i)$. Moreover, $F \in (H_2, V_1)^{m_{21}}$, but $H_1^i V_2^{m_{12}-i} \notin (H_2, V_1)^{m_{21}}$, and, since $I_{\mathcal{W}}$ is bihomogeneous, Q_i have to belong to $(H_2, V_1)^{m_{21}}$ for each i, which means $Q_i = \sum_j T_{ij} H_2^{m_{21}-j} V_1^j$. Therefore,

$$\begin{aligned} F &= \sum_i \sum_j T_{ij} H_2^{m_{21}-j} V_1^j H_1^i V_2^{m_{12}-i} \\ &= \underbrace{\sum_{i+j<m_{11}} T_{ij} H_1^i V_1^j H_2^{m_{21}-j} V_2^{m_{12}-i}}_{F'} + \underbrace{\sum_{i+j\geq m_{11}} T_{ij} H_1^i V_1^j H_2^{m_{21}-j} V_2^{m_{12}-i}}_{F^*}. \end{aligned}$$

Note that $F^* \in (\mathcal{G}(\mathcal{W}))$, so the claim follows if we also prove that $F' \in (\mathcal{G}(\mathcal{W}))$. Then

(i) if $m_{11} = 0$ we get $F' = 0$, and we are done;

(ii) if $m_{11} > 0$, we proceed by induction on $s := m_{12} + m_{21}$. If $s \leq m_{11} + 1$ then $\mathcal{W}$ is ACM, by [11, Theorem 4.8], and the statement is true. Suppose $s > m_{11} + 1$. Denoted by $w_1 = \min\{m_{12}, m_{11} - 1\}$, and by $w_2 = \min\{m_{21}, m_{11} - 1\}$ then

$$F' = H_2^{m_{21}-w_2} V_2^{m_{12}-w_1} \cdot \underbrace{\sum_{i+j<m_{11}} T_{ij} H_1^i V_1^j H_2^{w_2-j} V_2^{w_1-i}}_{F''}.$$

From $F' \in (H_1, V_1)^{m_{11}}$ we have $F'' \in (H_1, V_1)^{m_{11}}$. If $m_{12} > m_{11}-1$, then $w_1+w_2 < s$ and $F'' \in I(\mathcal{W}'')$, where $\mathcal{W}'' = m_{11}P_{11} + w_1 P_{12} + w_2 P_{21}$. By inductive hypothesis, the forms in $\mathcal{G}(\mathcal{W}'')$ generate $I_{\mathcal{W}''}$, and, for some bihomogeneus polynomial C_l, $F'' = \sum C_l H_1^{a_1} H_2^{a_2} V_1^{b_1} V_2^{b_2}$. Then

$$F' = \sum C_l H_1^{a_1} H_2^{a_2+m_{21}-w_2} V_1^{b_1} V_2^{b_2+m_{12}-w_1}$$

with the exponents satisfying the systems

$$\begin{cases} a_1 + b_1 \geq m_{11}, \\ a_1 + b_2 \geq w_1, \\ a_2 + b_1 \geq w_2, \end{cases} \quad \text{and then} \quad \begin{cases} a_1 + b_1 \geq m_{11}, \\ a_1 + b_2 + m_{12} - w_1 \geq m_{12}, \\ a_2 + b_1 + m_{21} - w_2 \geq m_{21}, \end{cases}$$

as we need.

In order to conclude the proof, we have to consider $m_{12} < m_{11} < s - 1$. In this case, note that $F' \in I(\widehat{\mathcal{W}})$, where $\widehat{\mathcal{W}} = m_{11}P_{11} + m_{12}P_{12} + m_{21}P_{21} + (s - m_{11} - 1)P_{22}$ that is an ACM set of points, by [11, Theorem 4.8]. So $F' \in (\mathcal{G}(I_{\mathcal{W}}))$. ∎

Notation 2.6 From now on we will denote by $\mathcal{G}(I_{\mathcal{W}})$ a minimal set of generators of $I_{\mathcal{W}}$ as in Proposition 2.5.

The next results are immediate consequences of Proposition 2.5. Since $I_{\mathcal{W}_1}$ is still in the hypothesis of Proposition 2.5, it suffices to prove them only for the product of powers of H_1, H_2, V_1 and V_2.

Proposition 2.7 *With the notation as above,*

$$I_{\mathcal{W}} = V_1 I_{\mathcal{W}_1} + H_1^{m_{11}} H_2^{m_{21}} \cdot (H_1, V_2)^{(m_{21}-m_{11})_+}.$$

Proposition 2.8 *With the notation as above,*

$$V_1 I_{\mathcal{W}_1} \cap H_1^{m_{11}} H_2^{m_{21}} \cdot (H_1, V_2)^{(m_{12}-m_{11})_+} = V_1 H_1^{m_{11}} H_2^{m_{21}} \cdot (H_1, V_2)^{(m_{12}-m_{11})_+}.$$

The following proposition will give us a way to construct a free resolution of $I_{\mathcal{W}}$.

Proposition 2.9 *The following sequence is exact:*

$$0 \longrightarrow V_1 H_1^{m_{11}} H_2^{m_{21}} \cdot (H_1, V_2)^{(m_{12}-m_{11})_+} \longrightarrow \\ V_1 I_{\mathcal{W}_1} \oplus H_1^{m_{11}} H_2^{m_{21}} \cdot (H_1, V_2)^{(m_{12}-m_{11})_+} \longrightarrow I_{\mathcal{W}} \longrightarrow 0.$$

Proof This follows from the exact sequence

$$0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow I + J \longrightarrow 0$$

(where I, J are R-modules), Proposition 2.7, and Proposition 2.8. ∎

Remark 2.10 As a consequence of Proposition 2.9 and the mapping cone construction, if $0 \to L_2 \to L_1 \to L_0$ is a minimal free resolution of $I_{\mathcal{W}_1}$, then it is easy to compute that a free resolution for $I_{\mathcal{W}}$ is

$$\begin{aligned} 0 &\longrightarrow \bigoplus_{(a,b)\in A_2(\mathcal{W})} R(-a,-b) \oplus L_2(0,-1) \\ &\longrightarrow \bigoplus_{(a,b)\in A_1(\mathcal{W})} R(-a,-b)^2 \oplus R(-m_{11}-m_{21}, -(m_{12}-m_{11})_+ - 1) \oplus L_1(0,-1) \\ &\longrightarrow \bigoplus_{(a,b)\in A_0(\mathcal{W})} R(-a,-b) \oplus L_0(0,-1) \longrightarrow I_{\mathcal{W}} \longrightarrow 0, \end{aligned}$$

where

$$\begin{aligned} A_0(\mathcal{W}) &:= \{ (a,b) \mid a+b = m_{11}+m_{21}+(m_{12}-m_{11})_+ \text{ and } 0 \le b \le (m_{12}-m_{11})_+ \} \\ A_1(\mathcal{W}) &:= \{ (a,b) \mid a+b = 1+m_{11}+m_{21}+(m_{12}-m_{11})_+ \text{ and } 1 \le b \le (m_{12}-m_{11})_+ \} \\ A_2(\mathcal{W}) &:= \{ (a,b) \mid a+b = 2+m_{11}+m_{21}+(m_{12}-m_{11})_+ \\ &\qquad \text{and } 2 \le b \le (m_{12}-m_{11})_+ + 1 \} \end{aligned}$$

We will show in Theorem 2.12 that the resolution will be minimal.

From Remark 2.10 we can describe the bigraded Betti numbers of $I_{\mathcal{W}}$ when $m_{11} = 0$; *i.e.*, $\mathcal{W}$ is a non-ACM set of two non-collinear fat points. We note that in this case the support of $\mathcal{W}$ is not an ACI.

Lemma 2.11 *Let $\mathcal{W} = m_{12}P_{12} + m_{21}P_{21}$ be a set of two non-collinear fat points; then the minimal free resolution of $I_{\mathcal{W}}$ is*

$$0 \longrightarrow \bigoplus_{\substack{a+b=m_{12}+m_{21}+2\\ a,b\geq 2}} R(-a,-b)^{\beta_2(a,b)} \longrightarrow \bigoplus_{\substack{a+b=m_{12}+m_{21}+1\\ a,b\geq 1}} R(-a,-b)^{\beta_1(a,b)}$$
$$\longrightarrow \bigoplus_{\substack{a+b=m_{12}+m_{21}\\ a,b\geq 0}} R(-a,-b)^{\beta_0(a,b)} \longrightarrow I_{\mathcal{W}} \longrightarrow 0,$$

where

$$\begin{aligned}
\beta_0(a,b) &:= \min\{a,b,m_{21}\}+1,\\
\beta_1(a,b) &:= \min\{a,b-1,m_{21}\}+\min\{a-1,b,m_{21}\}+1,\\
\beta_2(a,b) &:= \min\{a,b,m_{21}\}.
\end{aligned}$$

Proof If $m_{21}=0$ then $\mathcal{W}$ consists of only one fat point and the statement is true by Lemma 2.1. Let us suppose $m_{21}>0$ and the statement true for $\mathcal{W}_1$. From Remark 2.10 we get that no cancellation is numerically allowed in the resolution arising from the mapping cone construction, then by inductive hypothesis

$$\begin{aligned}
\beta_0(a,b) &= \begin{cases}
\min\{m_{21}-1+m_{12}-(b-1),b-1,m_{21}-1\}+2 \\
\qquad\qquad \text{if } a+b-1=m_{21}-1+m_{12},\ b\leq m_{12},\\
\min\{m_{21}-1+m_{12}-(b-1),b-1,m_{21}-1\}+1 \\
\qquad\qquad \text{if } a+b-1=m_{21}-1+m_{12},\ b> m_{12},\\
0 \qquad\qquad \text{otherwise,}
\end{cases}\\
&= \begin{cases}
\min\{b,m_{21}\}+1 & \text{if } a+b=m_{21}+m_{12},\ b\leq m_{12},\\
a+1 & \text{if } a+b=m_{21}+m_{12},\ b> m_{12},\\
0 & \text{otherwise}
\end{cases}\\
&= \begin{cases}
\min\{a,b,m_{21}\}+1 & \text{if } a+b=m_{21}+m_{12},\\
0 & \text{otherwise,}
\end{cases}
\end{aligned}$$

as required.

Analogously, we can compute $\beta_1(a,b)$ and $\beta_2(a,b)$. ∎

Theorem 2.12 *Let $0\to F_2\to F_1\to F_0$ be the free resolution of $I_{\mathcal{W}}$ as in Remark 2.10; then no cancellation is allowed.*

Proof Let $0\to\bar F_2\to\bar F_1\to\bar F_0$ be a minimal free resolution of $I_{\mathcal{W}}$. Then we observe that $\dim_k(\bar F_0)_{(a,b)}=\dim_k(F_0)_{(a,b)}$, *i.e.*, $\mathcal{G}(I_{\mathcal{W}})=V_1\cdot\mathcal{G}(I_{\mathcal{W}_1})\cup H_1^{m_{11}}H_2^{m_{21}}\cdot\mathcal{G}((H_1,V_2)^{(m_{12}-m_{11})_+})$, and it is a minimal set of generators for $I_{\mathcal{W}}$. From Proposition 2.5, it is easy to check that $\mathcal{G}(I_{\mathcal{W}})\subseteq V_1\cdot\mathcal{G}(I_{\mathcal{W}_1})\cup H_1^{m_{11}}H_2^{m_{21}}\cdot\mathcal{G}((H_1,V_2)^{(m_{12}-m_{11})_+})$. On the other hand , take $W\in I_{\mathcal{W}_1}$ and $G\in((H_1,V_2)^{(m_{12}-m_{11})_+})$ such that $V_1W+H_1^{m_{11}}H_2^{m_{21}}G=0$, then $G\in(V_1)$. Hence let $G=\sum_{i,j}T_{ij}H_1^iV_2^j$, for some $T_{ij}\neq 0$, and let $P:=(H_u\times V_1)\notin\mathcal{W}$ be such that $T_{ij}\notin(H_u)$. We set $H_1^iV_2^j(P)=\alpha_{ij}\neq 0$ so we get $\sum T_{ij}\alpha_{ij}\in(V_1)$ and, because the bihomogenity of $I_{\mathcal{W}}$, this implies that all

$T_{ij} \in (V_1)$. Then $G = V_1 G'$ and $W = -H_1^{m_{11}} H_2^{m_{21}} G'$. Thus, if a cancellation is allowed it has to involve F_2 and F_1. If $m_{21} - m_{12} + 1 \geq m_{21}$, then $\mathcal{W}$ is ACM, and we are done. We will show that no cancellation is numerically allowed also in the not ACM case. We proceed by induction on m_{11}. If $m_{11} = 0$, then the statement is true from Lemma 2.11. Now we suppose $m_{11} > 0$. If for some (a', b') we have $\dim_k(F_1)_{(a',b')} \neq 0$ and $\dim_k(F_2)_{(a',b')} \neq 0$ then two cases can be distinguished

(a) $\dim_k(L_1)_{(a',b'-1)} \neq 0$ and $\dim_k(L_2)_{(a',b'-1)} = 0$
(b) $\dim_k(L_1)_{(a',b'-1)} = 0$ and $\dim_k(L_2)_{(a',b'-1)} \neq 0$

where $0 \to L_2 \to L_1 \to L_0$ is a minimal free resolution of $I_{\mathcal{W}_1}$. By Remark 2.10 and using the same notation, the first case happens if $(a', b') \in A_2(\mathcal{W}) \neq \varnothing$ so it must be $m_{12} > m_{11}$ and $a' + b' = 2 + m_{21} + m_{12}$. If $m_{11} = 1$, then we get a contradiction, since in this case, by Lemma 2.11, we get $\dim_k(L_1)_{(a',b'-1)} \neq 0$ if and only if $a' + b' - 1 = m_{12} + (m_{21} - 1) + 1$. We can assume $m_{11} > 1$, and we set $\mathcal{W}_2 := (m_{11} - 2)_+ P_{11} + (m_{21} - 2)_+ P_{21} + m_{12} P_{12}$. From $\dim_k(L_2)_{(a',b'-1)} = 0$, we have $(a', b' - 1) \notin A_2(\mathcal{W}_1)$, but $(a', b') \in A_2(\mathcal{W})$, and then the only case we need to consider is $(a', b') = (m_{12} + m_{21}, 2)$. Since $(a', b' - 1) \notin A_1(\mathcal{W}_1)$, we have $\dim_k(L_1)_{(a',1)} \neq 0$, and again, since $(a', 0) \notin A_1(\mathcal{W}_2)$. In the second case, we can proceed in a similar way. First note that $(a', b') \in A_1 \cup \{(m_{11} + m_{21}, (m_{12} - m_{11})_+)\}$ *i.e.*,

$$\begin{cases} a' + b' = 1 + m_{11} + m_{21} + (m_{12} - m_{11})_+, \\ 1 \leq b' \leq (m_{12} - m_{11})_+ + 1. \end{cases}$$

Moreover, since $\dim_k(L_1)_{(a',b'-1)} = 0$ then $(a', b'-1) \notin A_1(\mathcal{W}_1)$ *i.e.*, either $a'+b' \neq m_{11} + m_{21} + (m_{12} - m_{11} + 1)_+$ or $b' \notin \{2, \dots, (m_{12} - m_{11} - 1)_+ + 2\}$. Since the second condition always holds, we get $m_{12} < m_{11}$, and then $(a', b') = (m_{11} + m_{12}, 1)$. Then $\dim_k(L_2)_{(a',0)} \neq 0$, which is not allowed for a finite set of points. ■

The next example shows how to compute inductively a minimal bigraded resolution of $I_{\mathcal{W}}$.

Example 2.13 Let be $\mathcal{W} = 2P_{11} + 4P_{12} + 3P_{21}$, we set $\mathcal{W}_k := (2 - k)P_{11} + 4P_{12} + (3 - k)P_{21}$, for $k = 1, 2$. We use Lemma 2.11 to compute the resolution of $I_{\mathcal{W}_2}$ where $\mathcal{W}_2 = 4P_{12} + P_{21}$ is a set of two non-collinear fat points.

$$\begin{aligned} (2.1)\quad 0 &\longrightarrow R(-5,-2) \oplus R(-4,-3) \oplus R(-3,-4) \oplus R(-2,-5) \\ &\longrightarrow R(-5,-1)^2 \oplus R(-4,-2)^3 \oplus R(-3,-3)^3 \oplus R(-2,-4)^3 \oplus \longrightarrow R(-1,-5)^2 \\ &\longrightarrow R(-5,0) \oplus R(-4,-1)^2 \oplus R(-3,-2)^2 \oplus \\ &\qquad\qquad R(-2,-3)^2 \oplus R(-1,-4)^2 \oplus R(0,-5) \\ &\longrightarrow I_{\mathcal{W}_2} \longrightarrow 0 \end{aligned}$$

The next step is to compute a minimal free resolution for $I_{\mathcal{W}_1}$ where $\mathcal{W}_1 = P_{11} + 4P_{12} + 2P_{21}$. First, we shift all the degrees of the modules in resolution (2.1) by $(0, -1)$; then we compute all the pairs (i, j) in $\mathcal{A}_0(\mathcal{W}_1)$ and add $R(-i, -j)$ among the generators' module; we compute all the pairs (i, j) in $\mathcal{A}_1(\mathcal{W}_1)$ and add $R(-i, -j)$ among the first syzygies' module and, as last step, we compute all the pairs (i, j) in $\mathcal{A}_2(\mathcal{W}_1)$ and add $R(-i, -j)$ among the second syzygies's module of $\mathcal{W}_2$. Thus, a minimal free resolution for $I_{\mathcal{W}_1}$ is

$$
\begin{aligned}
(2.2)\qquad 0 &\longrightarrow R(-6,-2)\oplus R(-5,-3)^2\oplus R(-4,-4)^2\oplus R(-3,-5)\oplus R(-2,-6)\\
&\longrightarrow R(-6,-1)^2\oplus R(-5,-2)^4\oplus R(-4,-3)^5\oplus R(-3,-4)^4\oplus R(-2,-5)^3\\
&\longrightarrow R(-1,-6)^2 \longrightarrow R(-6,0)\oplus R(-5,-1)^2\oplus R(-4,-2)^3\oplus\\
&\qquad\qquad R(-3,-3)^3\oplus R(-2,-4)^2\oplus R(-1,-5)^2\oplus R(0,-6)\\
&\longrightarrow I_{\mathcal{W}_1}\longrightarrow 0.
\end{aligned}
$$

Finally, repeating the same procedure as above, *i.e.,* shifting all the modules' degrees in the resolution (2.2) by $(0,-1)$ and adding $R(-i,-j)$ with (i,j) all the pairs in $\mathcal{A}_0(\mathcal{W})$, $\mathcal{A}_1(\mathcal{W})$, $\mathcal{A}_2(\mathcal{W})$ among the generators' module, first syzygies' module, and second syzygies's module of $\mathcal{W}_1$, respectively, we get a minimal free resolution of $I_{\mathcal{W}}$:

$$
\begin{aligned}
0 &\longrightarrow R(-7,-2)\oplus R(-6,-3)^2\oplus R(-5,-4)^2\oplus R(-4,-5)^2\oplus R(-3,-6)\oplus\\
&\qquad\qquad R(-2,-7)\\
&\longrightarrow R(-7,-1)^2\oplus R(-6,-2)^4\oplus R(-5,-3)^5\oplus R(-4,-4)^5\oplus R(-3,-5)^4\oplus\\
&\qquad\qquad R(-2,-6)^3\oplus R(-1,-7)^2\\
&\longrightarrow R(-7,0)\oplus R(-6,-1)^2\oplus R(-5,-2)^3\oplus R(-4,-3)^3\oplus R(-3,-4)^3\oplus\\
&\qquad\qquad R(-2,-5)^2\oplus R(-1,-6)^2\oplus R(0,-7)\\
&\longrightarrow I_{\mathcal{W}}\longrightarrow 0.
\end{aligned}
$$

3 The Minimal Free Resolution of a Fat Almost Complete Intersection in $\mathbb{P}^1\times\mathbb{P}^1$

As stated in the introduction, in this section we prove the main result of the paper that generalizes Theorem 2.12 for any fat almost complete intersection $\mathcal{Z}$. Recall our notation.

Notation 3.1 Let $\alpha_1,\alpha_2,\beta_1,\beta_2$ be positive integers. We denote by $\mathcal{Z} := \sum w_{ij}P_{ij}$ a fat ACI of $\mathbb{P}^1\times\mathbb{P}^1$ where

$$
w_{ij}=\begin{cases}
m_{11} & \text{if } (i,j)\le(\alpha_1,\beta_1),\\
m_{21} & \text{if } (\alpha_1+1,1)\le(i,j)\le(\alpha_1+\alpha_2,\beta_1),\\
m_{12} & \text{if } (1,\beta_1+1)\le(i,j)\le(\alpha_1,\beta_1+\beta_2),\\
0 & \text{otherwise,}
\end{cases}
$$

for some non-negative integers m_{11},m_{12},m_{21}, and we denote by $\mathcal{Z}_1 := \sum \overline{w}_{ij}P_{ij}$ a set of fat points of $\mathbb{P}^1\times\mathbb{P}^1$ where

$$
\overline{w}_{ij}=\begin{cases}
(m_{11}-1)_+ & \text{if } (i,j)\le(\alpha_1,\beta_1),\\
(m_{21}-1)_+ & \text{if } (\alpha_1+1,1)\le(i,j)\le(\alpha_1+\alpha_2,\beta_1),\\
m_{12} & \text{if } (1,\beta_1+1)\le(i,j)\le(\alpha_1,\beta_1+\beta_2),\\
0 & \text{otherwise,}
\end{cases}
$$

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for m_{11}, m_{12}, m_{21} as in $\mathcal{Z}$. We set $Q_1 := H_1 \cdots H_{\alpha_1}$, $Q_2 := H_{\alpha_1+1} \cdots H_{\alpha_1+\alpha_2}$ and $U_1 := V_1 \cdots V_{\beta_1}$, $U_2 := V_{\beta_1+1} \cdots V_{\beta_1+\beta_2}$.

We have the following lemma.

Lemma 3.2 $I_{\mathcal{Z}} = (Q_1, U_1)^{m_{11}} \cap (Q_1, U_2)^{m_{12}} \cap (Q_2, U_1)^{m_{21}}$.

Proof $I_{\mathcal{Z}}$ is the intersection of three powers of homogeneous complete intersection ideals and $I^m = I^{(m)}$ where I is the ideal defining a complete intersection from [13, Appendix 6, Lemma 5]. We have

$$
\begin{aligned}
I_{\mathcal{Z}} &= \bigcap_{(i,j)\le(\alpha_1,\beta_1)} (H_i, V_j)^{m_{11}} \cap \bigcap_{(\alpha_1+1,1)\le(i,j)\le(\alpha_1+\alpha_2,\beta_1)} (H_i, V_j)^{m_{21}} \\
&\quad \cap \bigcap_{(1,\beta_1+1)\le(i,j)\le(\alpha_1,\beta_1+\beta_2)} (H_i, V_j)^{m_{12}} \\
&= \Big(\bigcap_{(i,j)\le(\alpha_1,\beta_1)} (H_i, V_j) \Big)^{m_{11}} \cap \Big(\bigcap_{(\alpha_1+1,1)\le(i,j)\le(\alpha_1+\alpha_2,\beta_1)} (H_i, V_j) \Big)^{m_{21}} \\
&\quad \cap \Big(\bigcap_{(1,\beta_1+1)\le(i,j)\le(\alpha_1,\beta_1+\beta_2)} (H_i, V_j) \Big)^{m_{12}}. \qquad \blacksquare
\end{aligned}
$$

Remark 3.3 All the results given in Section 2 can be generalized by replacing H_i by Q_i and V_j by U_j.

The following lemma generalizes Lemma 2.11. That is, we compute a minimal free resolution of $\mathcal{Z}$ whose support is the disjoint union of two fat complete intersections, and it is never ACM. As pointed out in the introduction, this is one of the starting base cases to describe a minimal free resolution of $I_{\mathcal{Z}}$ by induction when $m_{11} < m_{21}$.

Lemma 3.4 *In $\mathbb{P}^1 \times \mathbb{P}^1$, let*

$$
\mathcal{Z} := \sum_{(1,\beta_1+1)\le(i,j)\le(\alpha_1,\beta_1+\beta_2)} m_{12} P_{ij} + \sum_{(\alpha_1+1,1)\le(i,j)\le(\alpha_1+\alpha_2,\beta_1)} m_{21} P_{ij}
$$

be a set of fat points whose support is the disjoint union of two fat complete intersections. Then a minimal free resolution of $I_{\mathcal{Z}}$ is

$$
\begin{aligned}
0 &\longrightarrow \bigoplus_{(a,b,c,d)\in\mathcal{D}_2} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2) \\
&\longrightarrow \bigoplus_{(a,b,c,d)\in\mathcal{D}_1} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2) \\
&\longrightarrow \bigoplus_{(a,b,c,d)\in\mathcal{D}_0} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2) \longrightarrow I_{\mathcal{Z}} \longrightarrow 0,
\end{aligned}
$$

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where

$$\mathcal{D}_0 := \{(a,b,c,d) \mid 0 \le a,d \le m_{12},\ 0 \le b,c \le m_{21},\ a+d = m_{12},\ b+c = m_{21}\},$$
$$\mathcal{D}_1 := \{(a,b,c,d) \mid 0 \le a,d \le m_{12},\ 0 \le b,c \le m_{21},\\ (a+d = m_{12}+1,\ b+c = m_{21}) \vee (a+d = m_{12},\ b+c = m_{21}+1)\},$$
$$\mathcal{D}_2 := \{(a,b,c,d) \mid 0 \le a,d \le m_{12},\ 0 \le b,c \le m_{21},\\ a+d = m_{12}+1,\ b+c = m_{21}+1\}.$$

Proof This follows by induction on m_{21}, using Lemma 2.3 and the mapping cone construction. ∎

Theorem 3.5 *With Notation 3.1, let $0 \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0$ be a minimal free resolution of $I_{\mathcal{Z}_1}$. Then a minimal free resolution of a fat ACI $I_{\mathcal{Z}}$ is*

$$\begin{aligned} 0 \longrightarrow & \bigoplus_{(a,b-\beta_1)\in\mathcal{A}_1(\mathcal{Z})} R(-a,-b) \oplus \mathcal{L}_2(0,-\beta_1) \\ \longrightarrow & \bigoplus_{(a,b-\beta_1)\in\mathcal{A}_0(\mathcal{Z})} R(-a,-b) \bigoplus_{(a,b)\in\mathcal{A}_1(\mathcal{Z})} R(-a,-b) \oplus \mathcal{L}_1(0,-\beta_1) \\ \longrightarrow & \bigoplus_{(a,b)\in\mathcal{A}_0(\mathcal{Z})} R(-a,-b) \oplus \mathcal{L}_0(0,-\beta_1) \longrightarrow I_{\mathcal{Z}} \longrightarrow 0, \end{aligned}$$

where

$$\mathcal{A}_0(\mathcal{Z}) = \{(\alpha_1(m_{11}+i) + \alpha_2 m_{21}, ((m_{12}-m_{11})_+ - i)\beta_2) \mid i = 0,\dots,(m_{12}-m_{11})_+\},$$
$$\mathcal{A}_1(\mathcal{Z}) = \{(\alpha_1(m_{11}+i+1) + \alpha_2 m_{21}),\\ ((m_{12}-m_{11})_+ - i)\beta_2) \mid i = 0,\dots,(m_{12}-m_{11})_+ - 1\}$$

Proof The proof uses Lemma 2.3, Remark 3.3, and Remark 2.10. Note that, by induction, Lemma 3.4, and Lemma 2.2, the number of elements in a minimal set of generators for the modules in the resolution does not depend on $\alpha_1, \alpha_2, \beta_1, \beta_2$. Moreover, using Remark 2.10, if $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, we get $|\mathcal{A}_0(\mathcal{Z})| = |A_0(\mathcal{W})|$, $|\mathcal{A}_0(\mathcal{Z})| + |\mathcal{A}_1(\mathcal{Z})| = |A_1(\mathcal{W})|$, and $|\mathcal{A}_1(\mathcal{Z})| = |A_2(\mathcal{W})|$. Therefore, by induction and Theorem 2.12, no cancellation is allowed in the resolution arising from mapping cone. This follows, since the maps of the mapping cone cannot have invertible entries; otherwise, by Remark 3.3, the maps of the mapping cone used in Theorem 2.12 would also have invertible entries. ∎

Example 3.6 Consider the following set of fat points with $\alpha_1 = \beta_1 = \beta_2 = 2$ and $\alpha_2 = 1$:

$$\begin{aligned} \mathcal{Z} := \ & 2P_{11} + 2P_{12} + 4P_{13} + 4P_{14} + \\ & +2P_{21} + 2P_{22} + 4P_{23} + 4P_{24} + \\ & +3P_{31} + 3P_{32} \end{aligned}$$

Note that $\mathcal{A}_0(\mathcal{Z}) = \{(7,4),(9,2),(11,0)\}$ and $\mathcal{A}_1(\mathcal{Z}) = \{(9,4),(11,2)\}$.

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Set, for $i = 0, 1, 2$

$$\begin{aligned}\mathcal{Z}_i := \quad &(2-i)P_{11} + (2-i)P_{12} + 4P_{13} + 4P_{14} + \\ &+ (2-i)P_{21} + (2-i)P_{22} + 4P_{23} + 4P_{24} + \\ &+ (3-i)P_{31} + (3-i)P_{32}\end{aligned}$$

We start by computing the resolution of $\mathcal{Z}_2$. By Lemma 3.4 we get the following degrees for a minimal set of generators, first and second syzygies where $(a,b)^n$ indicates

$$\begin{array}{ll}\text{Generators:} & \{(9,0),(8,2),(7,2),(6,4),(5,4),(4,6),(3,6),(2,8),\\ & \quad (1,8),(0,10)\}\\ \text{First Syzygies:} & \{(9,2)^2,(8,4),(7,4)^2,(6,6),(5,6)^2,(4,8),(3,8)^2,\\ & \quad (2,10),(1,10)\}\\ \text{Second Syzygies:} & \{(9,4),(7,6),(5,8),(3,10)\}\end{array}$$

that the set contains n elements of degree (a,b). Now, by Theorem 3.5, and mimicking the procedure used in Example 2.13, we can compute the resolution of $I_{\mathcal{Z}_1}$ where the degrees of a minimal set of generators, first and second syzygies are respectively:

$$\begin{array}{ll}\text{Generators :} & \{(9,2),(8,4),(7,4),(6,6),(5,6),(4,8),(3,8),(2,10),(1,10),\\ & \quad (0,12),(10,0),(8,2),(6,4),(4,6)\}\\ \text{First Syzygies :} & \{(9,4)^2,(8,6),(7,6)^2,(6,8),(5,8)^2,(4,10),(3,10)^2,(2,12),\\ & \quad (1,12)(10,2)^2,(8,4)^2,(6,6)^2,(4,8)\}\\ \text{Second Syzygies :} & \{(9,6),(7,8),(5,10),(3,12)\} \cup \{(10,4),(8,6),(6,8)\}\end{array}$$

Finally, applying Theorem 3.5 again, we get a minimal resolution of $I_{\mathcal{Z}} = I_{\mathcal{Z}_0}$:

$$\begin{aligned}0 \longrightarrow &[R(-10,-6) \oplus R(-9,-8) \oplus R(-8,-8) \oplus R(-7,-10) \oplus \\ &\quad R(-6,-10) \oplus R(-5,-12) \oplus R(-3,-14)] \oplus [R(-9,-6) \oplus R(-11,-4)]\\ \longrightarrow &[R(-10,-4)^2 \oplus R(-9,-6)^2 \oplus R(-8,-8) \oplus R(-8,-6)^2 \oplus R(-7,-8)^2 \oplus \\ &\quad R(-6,-10) \oplus R(-6,-8)^2 \oplus R(-5,-10)^2 \oplus R(-4,-10) \oplus \\ &\quad R(-4,-12) \oplus R(-3,-12)^2 \oplus R(-2,-14) \oplus R(-1,-14)] \oplus \\ &\quad [R(-7,-6) \oplus R(-9,-4) \oplus R(-11,-2)] \oplus [R(-9,-4) \oplus R(-11,-2)]\\ \longrightarrow &[R(-10,-2) \oplus R(-9,-4) \oplus R(-8,-6) \oplus R(-8,-4) \oplus R(-7,-6) \oplus \\ &\quad R(-6,-8) \oplus R(-6,-6) \oplus R(-5,-8) \oplus R(-4,-10) \oplus R(-4,-8) \oplus \\ &\quad R(-3,-10) \oplus R(-2,-12) \oplus R(-1,-12) \oplus R(0,-14)] \oplus \\ &\quad [R(-7,-4) \oplus R(-9,-2) \oplus R(-11,0)] \longrightarrow I_{\mathcal{Z}} \longrightarrow 0\end{aligned}$$

The next corollary better describes the resolution of $I_{\mathcal{Z}}$ when $m_{11} = m_{21}$.

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Corollary 3.7 *With Notation 3.1, suppose $m_{11} = m_{21} = n$ and $m_{12} = m$. Then a minimal free resolution of $I_{\mathcal{Z}}$ is*

$$\begin{aligned}0 \longrightarrow & \bigoplus_{(a,b,c,d)\in\mathcal{B}_2(\mathcal{Z})} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2)\\ \longrightarrow & \bigoplus_{(a,b,c,d)\in\mathcal{B}_1(\mathcal{Z})} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2)\\ \longrightarrow & \bigoplus_{(a,b,c,d)\in\mathcal{B}_0(\mathcal{Z})} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2) \longrightarrow I_{\mathcal{Z}} \longrightarrow 0,\end{aligned}$$

$$\begin{aligned}\mathcal{B}_0(\mathcal{Z}) &:= \big\{(a,b,c,d) \mid a+d = m, b+c = n, 0 \le b \le \min\{a,n\} \le a \le m\big\},\\ \mathcal{B}_1(\mathcal{Z}) &:= \{(a,b,c,d) \mid (a+d = m+1, b+c = n)\\ &\qquad \vee (a+d = m,\ b+c = n+1), 0 \le b \le \min\{a,n\} \le a \le m\big\},\\ \mathcal{B}_2(\mathcal{Z}) &:= \big\{(a,b,c,d) \mid a+d = m+1, b+c = n+1, 0 \le b \le \min\{a,n\} \le a \le m\big\}.\end{aligned}$$

Proof We proceed by induction on n. If $n = 0$ then $\mathcal{Z}$ is homogeneous and its support is a complete intersection so, by Lemma 2.3, we are done. Assume now that $n > 0$ and take $\mathcal{Z}_1$ as in Notation 3.1. Then we get

$$\begin{aligned}\cdots \longrightarrow & \bigoplus_{(a,b,c+1,d)\in\mathcal{B}_0(\mathcal{Z}_1)} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2) \oplus\\ & \qquad\qquad \bigoplus_{(u,v)\in\mathcal{A}_0(\mathcal{Z})} R(-u,-v) \longrightarrow I_{\mathcal{Z}} \longrightarrow 0,\end{aligned}$$

$$\begin{aligned}\mathcal{B}_0(\mathcal{Z}_1) &:= \big\{(a,b,c,d) \mid a+d = m,\ b+c = n-1, 0 \le b \le \min\{a,n-1\} \le a \le m\big\},\\ \mathcal{A}_0(\mathcal{Z}) &= \big\{((\alpha_1(n+i) + \alpha_2 n), \beta_2(m-n-i)) \mid i = 0,\dots,m-n\big\},\end{aligned}$$

i.e.,

$$\begin{aligned}\cdots \longrightarrow & \bigoplus_{(a,b,c+1,d)\in\mathcal{B}_0(\mathcal{Z}_1)} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2) \oplus\\ & \qquad \bigoplus_{(a,b,c,d)\in\mathcal{A}'_0(\mathcal{Z})} R(-\alpha_1 a - \alpha_2 b, -\beta_1 c - \beta_2 d) \longrightarrow I_{\mathcal{Z}} \longrightarrow 0,\end{aligned}$$

where $\mathcal{A}'_0(\mathcal{Z}) := \{(a,b,c,d) \mid b = n,\ c = 0,\ a+d = m,\ n \le a \le m\}$.
Then $\mathcal{B}_0(\mathcal{Z}) = \mathcal{B}_0(\mathcal{Z}_1) \cup \mathcal{A}'_0(\mathcal{Z})$. Analogously, we get $\mathcal{B}_1(\mathcal{Z})$ and $\mathcal{B}_2(\mathcal{Z})$. ■

Consequently, if $\mathcal{Z}$ is a homogeneous set of fat points, then a minimal free resolution is easy to describe.

Corollary 3.8 *With Notation 3.1, suppose $m_{11} = m_{12} = m_{21} = m$,* i.e., *the support of $\mathcal{Z}$ is an almost complete intersection with associated tuple*

$$\alpha_{\mathcal{Z}} := (\underbrace{\beta_1+\beta_2,\dots,\beta_1+\beta_2}_{\alpha_1+\alpha_2}, \underbrace{\beta_1,\dots,\beta_1}_{\alpha_1})$$

for some $m \in \mathbb{N}$.

Then a minimal free resolution of $I_{\mathcal{Z}}$ *is*

$$\begin{aligned} 0 &\longrightarrow \bigoplus_{(a,b,c,d)\in\mathcal{B}_2(\mathcal{Z})} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2) \\ &\longrightarrow \bigoplus_{(a,b,c,d)\in\mathcal{B}_1(\mathcal{Z})} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2) \\ &\longrightarrow \bigoplus_{(a,b,c,d)\in\mathcal{B}_0(\mathcal{Z})} R(-a\alpha_1 - b\alpha_2, -c\beta_1 - d\beta_2) \longrightarrow I_{\mathcal{Z}} \longrightarrow 0, \end{aligned}$$

$$\begin{aligned} \mathcal{B}_0(\mathcal{Z}) &:= \big\{ (a,b,c,d) \mid a+d=m,\ b+c=m,\ 0\le b\le a\le m \big\} \\ \mathcal{B}_1(\mathcal{Z}) &:= \big\{ (a,b,c,d) \mid (a+d=m+1,\ b+c=m) \\ &\qquad\qquad \vee (a+d=m,\ b+c=m+1),\ 0\le b\le a\le m \big\} \\ \mathcal{B}_2(\mathcal{Z}) &:= \big\{ (a,b,c,d) \mid a+d=m+1, b+c=m+1, 0\le b\le a\le m \big\}. \end{aligned}$$

Proof Just use Corollary 3.7. ∎

Remark 3.9 Recently, using Theorem 1.1 and Corollary 1.2, it was proved in [5], that if $\mathcal{Z}$ is a homogeneous set of fat points whose support is an almost complete intersection, then

$$I_{\mathcal{Z}} = (Q_1, U_1)^m \cap (Q_1, U_2)^m \cap (Q_2, U_1)^m = J^m$$

where we set $J := (Q_1, U_1) \cap (Q_1, U_2) \cap (Q_2, U_1)$. That is, the symbolic powers of J and the regular powers are the same. Therefore a proof of Corollary 3.8 could be given by induction on m, since $J^m = J \cdot J^{m-1}$.

In the next section we look at the symbolic powers of $I_{\mathcal{Z}}$ in the non-homogeneous case.

4 Symbolic vs Regular Powers of a Particular Almost Complete Intersection

As said in the introduction, given a homogeneous ideal I, the m-th symbolic power of I is the ideal $I^{(m)} = R \cap \big(\bigcap_{P\in Ass(I)} (I^m R_P)\big)$. Following [3], for an ideal of the form $I_X = \bigcap_{P_{ij}\in X} (I_{P_{ij}}^{m_{ij}})$ where $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a finite set of points, $I_{P_{ij}}$ is the ideal generated by all forms vanishing at P_{ij} and each m_{ij} is a non-negative integer; $I_X^{(m)}$ turns out to be $\cap_{P_{ij}\in X} (I_{P_{ij}}^{m m_{ij}})$. If I_X^m is the usual power, then we have the containment $I_X^m \subseteq I_X^{(m)}$ and it is a difficult problem to determine when there are containments of the form $I_X^{(m)} \subseteq I_X^r$. Furthermore, the m-th symbolic power of I_X has the form $I_X^{(m)} = \cap_{ij} I_{P_{ij}}^m$.

In this section we prove that if $\mathcal{Z}$ is a fat ACI of type (1.1), then $I_{\mathcal{Z}}^{(m)} = I_{\mathcal{Z}}^m$. We start with the three non-collinear points case by comparing the ideal $I_{\mathcal{W}}^m$ with $I_{\mathcal{W}}^{(m)}$, where we denote by $I_{\mathcal{W}}^{(m)} := I(m \cdot m_{11}P_{11} + m \cdot m_{12}P_{12} + m \cdot m_{21}P_{21})$.

Theorem 4.1 *Let* $\mathcal{W} = m_{11}P_{11} + m_{12}P_{12} + m_{21}P_{21}$ *be a fat ACI of three non-collinear points in* $\mathbb{P}^1 \times \mathbb{P}^1$*. Then* $I_{\mathcal{W}}^m = I_{\mathcal{W}}^{(m)}$.

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Proof First note that $I_{\mathcal{W}}^{(m)}$, by Proposition 2.5 is generated by a set, $\mathcal{G}(I_{\mathcal{W}}^{(m)})$, of forms which are product of lines. Thus, take such a form $F := H_1^{a_1} H_2^{a_2} V_1^{b_1} V_2^{b_2}$ in $\mathcal{G}(I_{\mathcal{W}}^{(m)})$; we have to show that $F \in I_{\mathcal{W}}^{m}$. We will show that we can decompose the form F as $F_1 \cdot F_2$ with $F_1 \in I_{\mathcal{W}}$ and $F_2 \in I_{\mathcal{W}}^{(m-1)}$; therefore, the theorem will follows by induction. Let us consider the Euclidean division in $\mathbb{N}$ of a_i, b_j with m, say $a_i = c_i m + r_i$, $b_j = d_j m + s_j$, for $i, j \in \{1,2\}$. We get, for $(i,j) \in \{(1,1),(1,2),(2,1)\}$, $c_i m + r_i + d_j m + s_j = a_i + b_j \geq m \cdot m_{ij}$, *i.e.*, $c_i + d_j + (r_i + s_j)/m \geq m_{ij}$; then $c_i + d_j + \lfloor (r_i + s_j)/m \rfloor \geq m_{ij}$. Let

$$\delta_i = \begin{cases} 1 & \text{if } s_i \neq 0, \\ 0 & \text{if } s_i = 0, \end{cases}$$

and set

$$F_1 := H_1^{c_1} H_2^{c_2} V_1^{d_1+\delta_1} V_2^{d_2+\delta_2} \quad \text{and} \quad F_2 := H_1^{a_1-c_1} H_2^{a_2-c_2} V_1^{b_1-d_1-\delta_1} V_2^{b_2-d_2-\delta_2}.$$

For $(i,j) \in \{(1,1),(1,2),(2,1)\}$, we have $c_i + d_j + \delta_j \geq c_i + d_j + \lfloor (r_i + s_j)/m \rfloor \geq m_{ij}$. This guarantees that $F_1 \in I_{\mathcal{W}}$. Analogously,

$$\begin{aligned} a_i - c_i + b_j - d_j - \delta_j &= c_i(m-1) + r_i + d_j(m-1) + (s_j - 1)_+ \\ &= (m-1)(c_1 + d_1 + (r_i + (s_j-1)_+)/(m-1)). \end{aligned}$$

Since $(r_i + (s_j - 1)_+)/(m-1) \geq \lfloor (r_i + s_j)/m \rfloor$, we are done. ∎

We are ready to prove the main result of this section. Set $I_{\mathcal{Z}}^{(m)} := (Q_1, U_1)^{m_{11}m} \cap (Q_1, U_2)^{m_{12}m} \cap (Q_2, U_1)^{m_{21}m}$; *i.e.*, $\mathcal{Z}^{(m)}$ is

$$\mathcal{Z}^{(m)} = \begin{array}{|c|c|} \hline m_{11}m & m_{12}m \\ \hline m_{21}m & \\ \cline{1-1} \end{array}$$

Theorem 4.2 *Let $\mathcal{Z}$ be a fat ACI of type (1.1). Then $I_{\mathcal{Z}}^{m} = I_{\mathcal{Z}}^{(m)}$.*

Proof By Lemma 3.2 and Remark 3.3, we can repeat the same argument as in Theorem 4.1. ∎

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Dipartimento di Matematica e Informatica, Viale A. Doria, 6 - 95100 - Catania, Italy
e-mail: favacchio@dmi.unict.it guardo@dmi.unict.it