BULL. AUSTRAL. MATH. SOC. 46A55, 26A21
VOL. 71 (2005) [235–258]

AFFINE BAIRE–ONE FUNCTIONS ON CHOQUET SIMPLEXES

JIŘÍ SPURNÝ

Metrisable Choquet simplexes with the set of extreme points being an F_σ-set are characterised by means of the behaviour of the space of affine Baire–one functions.

1. INTRODUCTION

Let X be a compact convex set in a locally convex space. According to the Choquet–Bishop–de Leeuw theorem (see [**1**, Theorem I.4.8]), for every $x \in X$ there exists a probability measure μ on X representing x which is maximal with respect to the Choquet ordering (see the next section for the definitions and notation not explained here). If this measure is uniquely determined, X is called a *Choquet simplex* (briefly *simplex*). If the set $\operatorname{ext} X$ of all extreme points of X is moreover closed, the set X is a *Bauer simplex*. There are a lot of conditions characterising Bauer simplexes. We list here conditions which are related to the structure of the space $\mathfrak{A}^c(X)$ of affine continuous functions on X.

For a compact convex set X the following conditions are equivalent:

(i) X is a Bauer simplex;

(ii) for every continuous function f on $\operatorname{ext} X$ there exists a continuous affine function h on X such that $f = h$ on $\operatorname{ext} X$;

(iii) for every continuous function f on X there exists a continuous affine function h on X such that $f = h$ on $\operatorname{ext} X$;

(iv) X is a simplex and the function $x \mapsto \delta_x(f)$, $x \in X$, is continuous for every continuous function f on X (here δ_x stands for the uniquely determined maximal measure representing $x \in X$);

(v) the upper envelope $f^* = \inf\{h : h \geqslant f, h \text{ is continuous affine}\}$ is affine and continuous for every continuous convex function f on X;

Received 20th October, 2004
Research supported in part by the grant GA ČR 201/03/0935, GA ČR 201/03/D120 and in part by the Research Project MSM 1132 00007 from the Czech Ministry of Education.
The author wishes to express his thanks to participants of Seminar on mathematical analysis and Seminar on real and abstract analysis held at the Charles University for their invaluable comments and helpful suggestions.

(vi) the space $\mathfrak{A}^c(X)$ of all affine continuous functions on X is a lattice in its natural ordering.

Proof of this theorem can be found in [**1**, Theorem II.4.1 and Theorem II.4.3] or in [**3**, Satz 2].

If X is a Choquet simplex and $\operatorname{ext} X$ is an F_σ–set in X, it is well–known that any bounded Baire–one function f on $\operatorname{ext} X$ can be extended to an affine Baire–one function h to the whole set X (a standard method of the proof can be found, for example, in [**18**, Theórème 37]). Hence it is natural to ask whether an analogue of the aforementioned theorem can be valid if we deal with affine Baire–one functions instead of continuous affine functions and with Choquet simplexes with the set of all extreme points being an F_σ–set instead of Bauer simplexes. This question is a generalisation of a problem posed by Jellett in [**11**].

The aim of the paper is to provide such a characterisation, at least for metrisable compact convex sets (see Corollary 3.5). In order to prove it we improve and generalise ideas contained in [**21**] where the equivalence (i) $\iff$ (ii) of Theorem 3.5 is shown for metrisable compact convex sets. We prove in Example 3.18 that a Choquet simplex constructed by Talagrand in [**23**] provides a counterexample to the implication (ii) $\implies$ (i) of Corollary 3.5 if we omit the assumption of metrisability. Thus the conjecture of Jellett posed in [**11**] is false in general.

We remark that the results of the paper are formulated in a more general context of *function spaces.*

2. Preliminaries

All topological space will be considered as Hausdorff. If K is a compact space, we denote by $\mathcal{C}(K)$ the space of all continuous functions on K. We shall identify the dual of $\mathcal{C}(K)$ with the space $\mathcal{M}(K)$ of all Radon measures on K. Let $\mathcal{M}^1(K)$ denote the set of all probability Radon measures on K and let ε_x stand for the Dirac measure at $x \in K$.

If K is a topological space, we write $\mathcal{B}^b(K)$ for the space of all *bounded Baire functions* on K, that is, the smallest space containing $\mathcal{C}(K)$ and closed with respect to taking pointwise limits of bounded sequences. The space of all *bounded Baire-one functions* on K, that is, the space of pointwise limits of bounded sequences of continuous functions, is denoted by $\mathcal{B}_1^b(K)$. (Baire–one functions are sometimes called *functions of the first Baire class.*) We shall need the following facts on Baire–one functions.

Theorem 2.1. *Let $f : K \to \mathbb{R}$ be a function on a topological space K.*

(a) *If f is a bounded Baire–one function, then there exists bounded sequences $\{u_n\}$ and $\{l_n\}$ such that each u_n, $-l_n$, is upper semicontinuous, $u_n \nearrow f$ and $l_n \searrow f$.*

(b) *If $f \in \mathcal{B}_1(K)$, the set D of all points of discontinuity of f is a set of the* *first category in K. In particular, the set of all points of continuity of f is a dense set provided K is a Baire space.*

(c) *The function f on a normal space K is of the first Baire class if and only if both sets $\{x \in K : f(x) < c\}$ and $\{x \in K : f(x) > c\}$ are F_σ-sets in K for every $c \in \mathbb{R}$.*

(d) *The space $\mathcal{B}_1^b(K)$ of bounded Baire–one functions on K is closed with respect to the uniform convergence.*

(e) *If f is a bounded Baire–one function and $\varepsilon > 0$, there exists a partition $\{A_1, \dots, A_n\}$ of K consisting of F_σ-sets and real numbers $c_1, \dots, c_n$ so that $\left\| f - \sum_{i=1}^{n} c_i \chi_{A_i} \right\| < \varepsilon$.*

(f) *If f is a Baire–one function, K is metrisable and $g : K \to \mathbb{R}$ is such that $\left\{x \in K : |f(x) - g(x)| > \varepsilon\right\}$ is finite for every $\varepsilon > 0$, then g is a Baire–one function as well.*

The proofs of assertions (a), (b), (c) and (d) can be found, for example, in [**16**, Lemma 3.5, Example 2.D.11, Example 3.A.1].

By virtue of the lack of suitable references, we include proofs of the remaining assertions. Starting with (e), let f be a bounded Baire–one function on a topological space K and $\varepsilon > 0$. Let $\{U_i\}_{i=1}^n$ be an open cover of $f(K)$ by sets of the diameter less than ε. Then $\{f^{-1}(U_i)\}_{i=1}^n$ is a cover of K consisting of sets expressible as a countable union of sets from $\mathcal{A}$, where $\mathcal{A}$ denotes the algebra of sets in K which are both F_σ and G_δ. Using the method of the reduction theorem [**13**, Section 26, II, Theorem 1] we find a disjoint cover $\{A_i\}_{i=1}^n$ of K such that $A_i \subset f^{-1}(U_i)$, $1 \leqslant i \leqslant n$, and each A_i is a countable union of sets from $\mathcal{A}$. If c_i is an arbitrary number from U_i, it is easy to verify that

$$\sup_{x \in K} \left| f(x) - \sum_{i=1}^{n} c_i \chi_{A_i}(x) \right| < \varepsilon \,.$$

For the proof of (f), we consider functions

$$g_n(x) := \begin{cases} g(x)\,, & |f(x) - g(x)| > \dfrac{1}{n}\,, \\ f(x)\,, & \text{otherwise}\,. \end{cases}$$

Then $\{g_n\}$ is a sequence of Baire–one functions which uniformly converges to g. Thus g is a Baire–one function likewise.

Throughout the paper we shall consider a *function space* $\mathcal{H}$ on a compact space K. By this we mean a (not necessarily closed) linear subspace of $C(K)$ containing the constant functions and separating the points of K. Let $\mathcal{M}_x(\mathcal{H})$ be the set of all *$\mathcal{H}$-representing measures* for $x \in K$, that is,

$$\mathcal{M}_x(\mathcal{H}) := \left\{\mu \in \mathcal{M}^1(K) : f(x) = \int_K f \, d\mu \text{ for any } f \in \mathcal{H}\right\}.$$

If $\mu \in \mathcal{M}_x(\mathcal{H})$, we say that x is a *barycenter of* μ and denote $x = r(\mu)$. Where no confusion can arise we simply say that μ represents x.

The set

$$Ch_{\mathcal{H}}K := \{x \in K : \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$$

is called the *Choquet boundary* of $\mathcal{H}$. It may be highly irregular from the topological point of view but it is a G_δ–set if K is metrisable (see [**1**, Corollary I.5.17]).

We say that a function $h \in \mathcal{H}$ is $\mathcal{H}$*–exposing* for $x \in K$ if h attains its maximum precisely at x. Obviously, any $\mathcal{H}$–exposed point is contained in the Choquet boundary of $\mathcal{H}$.

We introduce the following main examples of function spaces.

(a) In the "convex case", the function space $\mathcal{H}$ is the linear space $\mathfrak{A}^c(X)$ of all continuous affine functions on a compact convex subset X of a locally convex space. In this example, the Choquet boundary of $\mathfrak{A}^c(X)$ coincides with the set of all extreme points of X and is denoted by $\operatorname{ext} X$.

Hence the barycenter of a probability measure μ on X is a unique point $r(\mu) \in X$ for which

$$f(r(\mu)) = \int_X f\,d\mu \quad \text{for any} \quad f \in \mathfrak{A}^c(X),$$

that is, x is $\mathfrak{A}^c(X)$–represented by μ. A bounded Borel function f on X is said to satisfy the *barycentric formula* if $f(r(\mu)) = \mu(f)$ for any $\mu \in \mathcal{M}^1(X)$.

(b) In the "harmonic case", U is a bounded open subset of the Euclidean space $\mathbb{R}^m$ and the corresponding function space $\mathcal{H}$ is $\mathbf{H}(U)$, that is, the family of all continuous functions on $\overline{U}$ which are harmonic on U. In the "harmonic case", the Choquet boundary of $\mathbf{H}(U)$ coincides with the set $\partial_{\text{reg}}U$ of all regular points of U.

We define the space $\mathcal{A}(\mathcal{H})$ of all $\mathcal{H}$*–affine functions* as the family of all bounded Borel functions on K satisfying

$$f(x) = \int_K f\,d\mu \quad \text{for each} \quad x \in K \quad \text{and} \quad \mu \in \mathcal{M}_x(\mathcal{H}).$$

Further, let $\mathcal{A}^c(\mathcal{H})$ be the family of all continuous $\mathcal{H}$–affine functions on K. Then $\mathcal{A}^c(\mathcal{H})$ is a uniformly closed function space with $\mathcal{M}_x(\mathcal{H}) = \mathcal{M}_x(\mathcal{A}^c(\mathcal{H}))$ for every $x \in K$. It is easy to deduce that $\mathcal{A}^c(\mathcal{H})$ coincides with $\mathcal{H}$ in both "convex" and "harmonic" case.

We write $\mathcal{B}_1(\mathcal{H})$ for the set of all pointwise limits of sequences from $\mathcal{H}$ and by $\mathcal{B}_1^b(\mathcal{H})$ we understand the set of bounded elements from $\mathcal{B}_1(\mathcal{H})$. We denote by $\mathcal{B}_1^{bb}(\mathcal{H})$ the family of all functions on K which are pointwise limits of bounded sequence of functions from $\mathcal{H}$. Obviously we have the following inclusion

$$\mathcal{B}_1^{bb}(\mathcal{H}) \subset \mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K),$$

but the converse need not hold (see [**15**, Example 5.5]).

An upper bounded Borel function f is called *$\mathcal{H}$-convex* if $f(x) \leqslant \mu(f)$ for any $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. A function f is *$\mathcal{H}$-concave* if $-f$ is $\mathcal{H}$-convex. Let $\mathcal{K}^c(\mathcal{H})$ denote the family of all continuous $\mathcal{H}$-convex functions on K. Notice that the space $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$ is uniformly dense in $\mathcal{C}(K)$ due to the lattice version of the Stone–Weierstrass theorem.

The convex cone $\mathcal{K}^c(\mathcal{H})$ determines a partial ordering $\prec$ (called the *Choquet ordering*) on the space $\mathcal{M}^+(K)$ of all positive Radon measures on K:

$$\mu \prec \nu \quad \text{if} \quad \mu(f) \leqslant \nu(f) \quad \text{for each} \quad f \in \mathcal{K}^c(\mathcal{H}).$$

Lemma I.4.7 in [**1**] implies that for any measure $\mu \in \mathcal{M}^1(K)$ there exists a maximal measure ν with $\mu \prec \nu$. If we take μ to be the Dirac measure ε_x in a point $x \in K$, we obtain that for any point $x \in K$ there exists a maximal measure ν such that $f(x) = \nu(f)$ for every $f \in \mathcal{H}$. This is the content of the famous Choquet–Bishop–de-Leeuw theorem [**1**, Theorem I.4.8].

If K is metrisable, then a measure $\mu \in \mathcal{M}^+(K)$ is maximal if and only if $\mu(K \setminus Ch_{\mathcal{H}} K) = 0$. In nonmetrisable spaces every maximal measure μ satisfies $\mu(G) = 0$ for any G_δ–set disjoint from $Ch_{\mathcal{H}} K$ (see [**8**, Lemma 27.14]) and $\mu(B) = 0$ for any Baire set $B \subset K \setminus Ch_{\mathcal{H}} K$ (see [**1**, Corollary I.4.12 and the subsequent Remark]).

If a maximal measure representing $x \in K$ is uniquely determined for every $x \in K$, we say that $\mathcal{H}$ is a *simplicial function space*. In the "convex case" it is equivalent to say that X is a *Choquet simplex* (see [**1**, Theorem II.3.6]). As an example of a simplicial function space serves the space $\mathbf{H}(U)$ from the "harmonic case" (see [**5**], for a simple proof see [**17**]). We denote the unique maximal measure representing $x \in K$ by δ_x.

For a function $f : K \to \mathbb{R}$ we define the upper envelope f^* as

$$f^*(x) := \inf\{h(x) : h \geqslant f, h \in \mathcal{H}\}, \quad x \in K.$$

The lower envelope f_* is defined as $f_* := -(-f)^*$. In Theorem 3.1 we shall deal with an upper envelope generated by $\mathcal{H}$–affine Baire–one functions. This envelope is defined as

$$\widehat{f}(x) := \inf\{h(x) : h \geqslant f, h \in \mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)\}, \quad x \in K.$$

We remark that $\mathcal{H}$ is a simplicial function space if and only if f^* is an $\mathcal{H}$–affine function for every $f \in -\mathcal{W}(\mathcal{H})$ (see [**6**, Theorem 3.1] or [**1**, Theorem II.3.7 and the subsequent Remark]) where $\mathcal{W}(\mathcal{H})$ is the smallest family of functions containing $\mathcal{H}$ and closed with respect to taking infimum of finite families.

For a simplicial function space $\mathcal{H}$ we define an operator T by

$$Tf(x) := \delta_x(f), \quad x \in K, \quad f \in \mathcal{B}^b(K).$$

It is well–known (see for example [**15**, Proposition 6.1]) that $Tf \in \mathcal{A}(\mathcal{H})$ for any bounded Baire function f on K. Moreover, $Tf = f^*$ for every $\mathcal{H}$–convex bounded upper semicontinuous function f on K (see [**6**, Theorem 3.1]). Note also that $Tf = f$ on $Ch_{\mathcal{H}} K$ for every $f \in \mathcal{B}^b(K)$.

We write $\mathcal{H}^\perp$ for the space of all Radon measures μ on K which satisfies $\mu(h) = 0$ for every $h \in \mathcal{H}$. It follows from [**6**, Corollary 3.5] that $\mathcal{H}$ is simplicial if and only if there is no nonzero measure $\mu \in \left(\mathcal{A}^c(\mathcal{H})\right)^\perp$ such that its total variation $|\mu|$ is maximal.

If f and g are functions on a set X, we write $f \vee g$ for the pointwise maximum of f and g. The restriction of a function $f : X \to \mathbb{R}$ to a set F is denoted by $f \restriction F$. The characteristic function of a set $F \subset X$ is denoted by χ_F.

If x is a point of a metric space (X, ρ) and $\varepsilon > 0$, let $U(x, r) = \{y \in X : \rho(x, y) < \varepsilon\}$. We write $\operatorname{dist}(F, G)$ for the distance of sets $F, G \subset X$. For a set $F \subset X$ we denote by $U_\varepsilon(F) = \{y \in X : \operatorname{dist}(y, F) < \varepsilon\}$ the ε-neighbourhood of F. For a set $A \subset X$ we denote by $\operatorname{der} A$ the set of all accumulation points of A.

3. Results

The main result of the paper reads as follows.

Theorem 3.1. *Let $\mathcal{H}$ be a function space on a compact space K. Consider the following assertions:*

- (i) *$\mathcal{H}$ is simplicial and $Ch_{\mathcal{H}} K$ is an F_σ-set;*
- (ii) *for any bounded Baire-one function on $Ch_{\mathcal{H}} K$ there exists an $\mathcal{H}$-affine Baire-one function h such that $f = h$ on $Ch_{\mathcal{H}} K$;*
- (iii) *for any bounded Baire-one function f on K there exists an $\mathcal{H}$-affine Baire-one function h such that $f = h$ on $Ch_{\mathcal{H}} K$;*
- (iv) *$\mathcal{H}$ is simplicial and the operator T maps $\mathcal{B}_1^b(K)$ into $\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H})$;*
- (v) *$\widehat{f}$ is an $\mathcal{H}$-affine Baire-one function for every $\mathcal{H}$-convex function $f \in \mathcal{B}_1^b(K)$;*
- (vi) *$\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ is a lattice in the natural ordering.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi). *If K is supposed to be metrisable, then the assertions* (i)–(v) *are equivalent.*

Remark 3.2. For a simplicial function space $\mathcal{H}$, any function $f \in \mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H})$ is in fact a pointwise limit of a bounded sequence of functions from $\mathcal{A}^c(\mathcal{H})$, that is, $\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H}) = \mathcal{B}_1^{bb}(\mathcal{A}^c(\mathcal{H}))$. This assertion was proved in [**15**, Theorem 6.3].

Remark 3.3. If f is a Baire-one affine function on a compact convex set X, then f is a pointwise limit of a bounded sequence of affine continuous functions. The proof of this assertion can be found in [**18**, Théorème 80]. If we write $\mathfrak{A}(X)$ for the space of affine functions on X, we have the following equalities

$$\mathcal{A}(\mathfrak{A}^c(X)) \cap \mathcal{B}_1(X) = \mathfrak{A}(X) \cap \mathcal{B}_1(X) = \mathcal{B}_1^{bb}(\mathfrak{A}^c(X)) = \mathcal{B}_1^b(\mathfrak{A}^c(X)) = \mathcal{B}_1(\mathfrak{A}^c(X)) .$$

The first equality is the Choquet barycentric theorem [**7**] (see also [**1**, Theorem I.2.6]). The inclusion $\mathfrak{A}(X) \cap \mathcal{B}_1(X) \subset \mathcal{B}_1^{bb}(X)$ follows from the aforementioned [**18**, Théorème 80] and the remaining inclusions are trivial.

https://doi.org/10.1017/S0004972700038211 Published online by Cambridge University Press

Remark 3.4. If f is a bounded Baire–one function on a compact convex set X, $\mu(f) \geqslant f(r(\mu))$ for every $\mu \in \mathcal{M}^1(X)$ (see [20, Theorem 3]). In other words, f is an $\mathfrak{A}^c(X)$–convex function.

With these facts in mind, we can rewrite the preceding Theorem 3.1 for the "convex case" in the form laid down in Corollary 3.5.

Corollary 3.5. *Let X be a compact convex set in a locally convex space. Consider the following assertions:*

(i) *X is a Choquet simplex and* ext X *is an F_σ–set;*
(ii) *for any bounded Baire–one function on* ext X *there exists an affine Baire–one function h on X such that $f = h$ on* ext X;
(iii) *for any bounded Baire–one function on X there exists an affine Baire–one function h on X such that $f = h$ on* ext X;
(iv) *X is a Choquet simplex and the operator T maps $\mathcal{B}_1^b(X)$ into $\mathcal{B}_1(\mathfrak{A}^c(X))$;*
(v) *$\widehat{f}$ is an affine Baire–one function for every convex function $f \in \mathcal{B}_1^b(X)$;*
(vi) *$\mathcal{B}_1(\mathfrak{A}^c(X))$ is a lattice in the natural ordering.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi). *If X is supposed to be metrisable, then the assertions* (i)–(v) *are equivalent.*

We start with a preliminary well-known result called the minimum principle for Baire concave functions.

Proposition 3.6. *Let f be an $\mathcal{H}$–concave Baire function on K such that $f \geqslant 0$ on $Ch_{\mathcal{H}} K$. Then $f \geqslant 0$ on K.*

Proof: Let f be an $\mathcal{H}$–concave Baire–one function on K which is positive on the Choquet boundary $Ch_{\mathcal{H}} K$. Suppose that $f(x) < 0$ for some $x \in K$. Then

$$L := \{y \in K : f(y) \leqslant f(x)\}$$

is a Baire set not intersecting $Ch_{\mathcal{H}} K$. According to [1, Corollary I.4.12 and the subsequent Remark], $\mu(L) = 0$ where μ is a maximal measure representing x. Then the following inequalities

$$f(x) \geqslant \mu(f) = \int_{K \setminus L} f \, d\mu > \int_{K \setminus L} f(x) \, d\mu = f(x)$$

yields a contradiction and concludes the proof. ∎

Lemma 3.7. *Let $\mathcal{H}$ be a simplicial function space on a compact space K and f be a bounded $\mathcal{H}$–convex Baire–one function on K. Then $\widehat{f} = f$ on $Ch_{\mathcal{H}} K$.*

Proof: Let x be a point in the Choquet boundary of $\mathcal{H}$. We fix a strictly positive ε and set

$$l(y) := \begin{cases} f(x) + \varepsilon, & y = x, \\ C, & \text{otherwise}, \end{cases}$$

https://doi.org/10.1017/S0004972700038211 Published online by Cambridge University Press

where $C > 0$ is chosen so that $f + \varepsilon \leqslant C$ on K. Then l is a lower semicontinuous $\mathcal{H}$–concave function.

As f is a Baire–one function, we can find a bounded sequence $\{u_n\}$ of upper semicontinuous functions on K so that, for each $n \in \mathbb{N}$, $u_n < f$ and $u_n \nearrow f$. For $y \in K$ we find a measure $\mu \in \mathcal{M}_y(\mathcal{H})$ so that $\mu(l) = l_*(y)$ (see [**6**, Lemma 1.1]). Then

$$l(y) \geqslant l_*(y) = \mu(l) > \mu(f) \geqslant f(y) > u_n(y) , \quad n \in \mathbb{N} .$$

Thus $u_n < l_*$. An easy compactness argument gives the existence of a continuous $\mathcal{H}$–convex function k_n (k_n is even in $-\mathcal{W}(\mathcal{H})$) such that $u_n < k_n < l_*$.

Since $k_1 \leqslant l$ and $\mathcal{H}$ is simplicial, the analogue of Edwards' "in–between" theorem [**6**, Theorem 3.2] provides an $\mathcal{H}$–affine continuous function a_1 so that $k_1 \leqslant a_1 \leqslant l$. In the second step we construct an $\mathcal{H}$–affine continuous function a_2 so that $k_2 \vee a_1 \leqslant a_2 \leqslant l$. If we proceed with this inductive construction, we obtain an increasing sequence $\{a_n\}$ of $\mathcal{H}$–affine continuous functions satisfying $u_n \leqslant a_n \leqslant l$. By setting $a := \lim a_n$ we obtain an $\mathcal{H}$–affine Baire–one function such that $f \leqslant a \leqslant l$. Thus $\widehat{f} \leqslant a$, in particular

$$\widehat{f}(x) \leqslant a(x) \leqslant f(x) + \varepsilon .$$

As ε and x are arbitrary, $\widehat{f} = f$ on $Ch_{\mathcal{H}} K$. □

Proposition 3.8. *Suppose that $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ is a lattice in its natural ordering. Then $\mathcal{H}$ is a simplicial function space and $T(f \vee g)$ is the least upper bound in $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ for every couple f and g of $\mathcal{H}$–affine Baire–one functions.*

Conversely, let $\mathcal{H}$ be a simplicial function space such that $T(f \vee g)$ is a Baire–one function for every $f, g \in \mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$. Then $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ is a lattice in its natural ordering.

Proof: In order to prove the first assertion we need to verify that f^* is an $\mathcal{H}$–affine function for every $f \in -\mathcal{W}(\mathcal{H})$. Let $f = f_1 \vee \cdots \vee f_n$ where $f_1, \dots, f_n \in \mathcal{H}$. Thanks to the assumption there exists an $\mathcal{H}$–affine Baire–one function h such that $h \geqslant f$ and h is the least $\mathcal{H}$–affine Baire–one function with this property. In particular,

$$h \leqslant \inf\{g : g \in \mathcal{H}, g \geqslant f\} = f^* .$$

We are going to prove the reverse inequality. For a given $x \in K$ we use [**6**, Lemma 1.1] and find a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ so that $f^*(x) = \mu(f)$. Then

$$f^*(x) = \mu(f) \leqslant \mu(h) = h(x) ,$$

which gives the equality $h = f^*$. Thus the upper envelope f^* is $\mathcal{H}$–affine for every $f \in -\mathcal{W}(\mathcal{H})$ and $\mathcal{H}$ is simplicial according to the characterisation of simplicial spaces mentioned in Section 2.

Moreover, $Tf = f^* = h$ is a Baire–one function for every $\mathcal{H}$–convex continuous function $f \in -\mathcal{W}(\mathcal{H})$. It follows from the uniform density of $\mathcal{W}(\mathcal{H}) - \mathcal{W}(\mathcal{H})$ in $\mathcal{C}(K)$ that $T(\mathcal{C}(K)) \subset \mathcal{B}^b(K)$. Thus Tg is a Baire function for any bounded Baire function g on K.

Further, let f, g be $\mathcal{H}$–affine Baire–one functions on K and h be the least upper bound of f and g in $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$. According to the definition, $\widehat{(f \vee g)} = h$. Lemma 3.7 yields that $h = f \vee g$ on $Ch_{\mathcal{H}} K$. Hence $h = T(f \vee g)$ on $Ch_{\mathcal{H}} K$ and Proposition 3.6 applied to the functions $h - T(f \vee g)$ and $T(f \vee g) - h$ gives that $h = T(f \vee g)$ on K.

It remains to prove the converse assertion. Let $\mathcal{H}$ be a function space satisfying the assumption in the statement. If f and g are $\mathcal{H}$–affine Baire–one functions, then $T(f \vee g)$ is an $\mathcal{H}$–affine function because $T(\mathcal{B}^b(K)) \subset \mathcal{A}(\mathcal{H})$. Thanks to the hypothesis, it is a Baire–one function. It immediately follows from the minimum principle (see Proposition 3.6) that $f \vee g \leqslant T(f \vee g) \leqslant h$ for every $h \in \mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ satisfying $h \geqslant f \vee g$. Thus $\widehat{(f \vee g)} = T(f \vee g)$ and the space of all $\mathcal{H}$–affine Baire–one functions is a lattice in the natural ordering. ∎

In order to clarify the core of the proof of Theorem 3.1, we construct a simple example of a metrisable Choquet simplex X such that $\operatorname{ext} X$ is not an F_σ–set. This example serves as a guide for the proof of the most difficult part (the implication (v) $\Longrightarrow$ (i)) of Theorem 3.1. Namely, we suppose that $Ch_{\mathcal{H}} K$ is not an F_σ–set and try to find a closed set F which "looks" like X.

The standard technique is to construct a suitable function space $\mathcal{H}$ and then set X to be the *state space* $\mathsf{S}(\mathcal{H})$ of $\mathcal{H}$. It can be shown that $\mathsf{S}(\mathcal{H})$ shares with $\mathcal{H}$ a lot of properties and thus the behaviour of $\mathsf{S}(\mathcal{H})$ is determined by the function space $\mathcal{H}$. Below we briefly described this construction. Details can be found in [**1**, Chapter 2, Section 2], [**2**, Chapter 1, Section 4] or [**8**, Chapter, Section 29].

If $\mathcal{H}$ is a function space on a compact space, we set

$$\mathsf{S}(\mathcal{H}) := \left\{\varphi \in \mathcal{H}^* : \|\varphi\| = \varphi(1) = 1\right\}.$$

Then $\mathsf{S}(\mathcal{H})$ endowed with the weak–star topology is a compact convex set which is metrisable if K is metrisable. Let $\phi : K \to \mathsf{S}(\mathcal{H})$ be the evaluation mapping defined as $\phi(x) = s_x$, $x \in K$, where $s_x(h) = h(x)$ for $h \in \mathcal{H}$. Then ϕ is a homeomorphic embedding of K onto $\phi(K)$ and $\phi(Ch_{\mathcal{H}} K) = \operatorname{ext} \mathsf{S}(\mathcal{H})$.

Let $\Phi : \mathcal{H} \to \mathfrak{A}^c(\mathsf{S}(\mathcal{H}))$ be the mapping defined for $h \in \mathcal{H}$ by $\Phi(h)(s) := s(h)$, $s \in \mathsf{S}(\mathcal{H})$. Then Φ serves as an isometric isomorphism of $\mathcal{H}$ into $\mathfrak{A}^c(\mathsf{S}(\mathcal{H}))$, and Φ is onto if and only if the function space $\mathcal{H}$ is uniformly closed in $\mathcal{C}(K)$. In this case the inverse mapping is realised by

$$\Phi^{-1}(F) = F \circ \phi, \quad F \in \mathfrak{A}^c(\mathsf{S}(\mathcal{H})).$$

Further, according to [**4**, Theorem], $\mathsf{S}(\mathcal{A}^c(\mathcal{H}))$ is a Choquet simplex if and only if $\mathcal{H}$ is simplicial.

https://doi.org/10.1017/S0004972700038211 Published online by Cambridge University Press

Let X stand for the compact convex set $\mathrm{S}(\mathcal{A}^c(\mathcal{H}))$ and $\phi : K \to X$ and $\Phi : \mathcal{A}^c(\mathcal{H}) \to \mathfrak{A}^c(X)$ be the mappings defined above (here we deal with the function space $\mathcal{A}^c(\mathcal{H})$ instead of $\mathcal{H}$). If $\mathcal{H}$ is a simplicial function space, we can use methods of [**15**, Proposition 6.1 and Corollary 6.2] to deduce that any function $f \in \mathcal{A}(\mathcal{H})$ is even *completely* $\mathcal{A}^c(\mathcal{H})$*-affine*, that is, $\mu(f) = 0$ for every $\mu \in (\mathcal{A}^c(\mathcal{H}))^\perp$. According to [**22**, Theorem 4.3], there exists an isometric isomorphism I of the space $\mathrm{A}(\mathcal{H})$ of all completely $\mathcal{A}^c(\mathcal{H})$-affine functions onto the space of all bounded Borel functions on X satisfying the barycentric formula. Moreover, $I = \Phi$ on $\mathcal{A}^c(\mathcal{H})$ and $I^{-1}F = F \circ \phi$ for any bounded Borel function F on X satisfying the barycentric formula. The restriction of I onto the space $\mathrm{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ (denoted likewise) serves as an isometric isomorphism mapping $\mathrm{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ onto $\mathcal{B}_1(\mathfrak{A}^c(X))$. Since $I(1) = \|I\| = 1$, $If \geqslant 0$ if and only if $f \geqslant 0$. Hence I is even a lattice isomorphism between $\mathrm{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ and $\mathcal{B}_1(\mathfrak{A}^c(X))$.

From the view of the previous paragraphs the following proposition is not surprising.

Proposition 3.9. *Let $\mathcal{H}$ be a function space on a compact space K and X denotes the state space $\mathrm{S}(\mathcal{A}^c(\mathcal{H}))$. Then $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1(K)$ is a lattice in the natural ordering if and only if $\mathcal{B}_1(\mathfrak{A}^c(X))$ is a lattice in the natural ordering.*

Proof: Let $\phi : K \to X$ and $\Phi : \mathcal{A}^c(\mathcal{H}) \to \mathfrak{A}^c(X)$ be the mappings defined above.

If $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1(K)$ is a lattice, then $\mathcal{H}$ is a simplicial function space due to Proposition 3.8. According to Remark 3.2, $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K) = \mathrm{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$. Using the isometric lattice isomorphism $I : \mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1(K) \to \mathcal{B}_1(\mathfrak{A}^c(X))$ we easily deduce that $\mathcal{B}_1(\mathfrak{A}^c(X))$ is a lattice as well.

Conversely, if $\mathcal{B}_1(\mathfrak{A}^c(X))$ is a lattice, we use Proposition 3.8 and obtain that $\mathfrak{A}^c(X)$ is a simplicial function space, that is, X is a Choquet simplex. Hence $\mathcal{A}^c(\mathcal{H})$ and consequently $\mathcal{H}$ is a simplicial function space and we can use again the mapping $I : \mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K) \to \mathcal{B}_1(\mathfrak{A}^c(X))$ to verify that $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ is a lattice in the natural ordering (we remind that $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K) = \mathrm{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ again). ∎

Example 3.10. There exists a metrisable compact convex set X such that $\operatorname{ext} X$ is not an F_σ-set and $\mathcal{B}_1(\mathfrak{A}^c(X))$ is not a lattice in the natural ordering.

Proof: Let $\{q_n\}$ be an enumeration of rational numbers contained in $[0,1]$. We define a subset $K \subset \mathbb{R}^2$ as follows:

$$K := ([0,1] \times \{0\}) \cup \{(q_n, n^{-1}), (q_n, -n^{-1}) : n \in \mathbb{N}\}\,.$$

(We write (a,b) for a point in $\mathbb{R}^2$ with the coordinates a and b.) Obviously, K is a compact set in $\mathbb{R}^2$ (considered with the usual Euclidean topology). Let

$$\mathcal{H} = \left\{f \in \mathcal{C}(K) : f(q_n, 0) = \frac{1}{2}\big(f(q_n, -n^{-1}) + f(q_n, n^{-1})\big), n \in \mathbb{N}\right\}.$$

Then $\mathcal{H}$ is a correctly defined simplicial function space, $Ch_{\mathcal{H}} K \cap ([0,1] \times \{0\}) = \{(x,0) \in K : x \text{ is irrational}\}$ and $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$. The verification of these assertions is analogous to the one used in Example 3.17 where a similar construction is used as a counterexample to the implication (vi) $\Longrightarrow$ (i) of Theorem 3.1.

Unlike Example 3.17, the space $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ is not a lattice in the natural ordering. Indeed, let

$$f(a,b) := \begin{cases} 1\,, & b > 0\,, \\ 0\,, & b = 0\,, \\ -1\,, & b < 0\,. \end{cases}$$

Then f is an $\mathcal{H}$–affine Baire–one function but

$$T(f \vee -f)(a,b) = \begin{cases} 0\,, & b = 0 \text{ and } a \text{ is irrational}\,, \\ 1\,, & \text{otherwise} \end{cases}$$

is not a Baire–one function because $T(f \vee -f)$ has no point of continuity on $[0,1] \times \{0\}$. According to Proposition 3.8, $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ is not a lattice in the natural ordering.

The sought compact convex set X is defined as the state space $\mathsf{S}(\mathcal{H})$ of $\mathcal{H}$. It follows from the general properties of a state space cited above and in Proposition 3.9 that ext X is not an F_σ–set, X is a Choquet simplex and $\mathcal{B}_1\big(\mathfrak{A}^c(X)\big)$ is not a lattice in the natural ordering. ∎

The main construction needed in the proof of Theorem 3.1 begins with the following two lemmas.

Lemma 3.11. *Let K be a metrisable compact space and F be a G_δ–subset of K such that $\overline{F} = K = \overline{K \setminus F}$. Let $\{K_n\}$ be a sequence of compact subsets of F. Then $F \setminus \bigcup_n K_n$ is dense in K.*

Proof: We claim that each K_n is a nowhere dense subset of F. Indeed, let n be a fixed positive integer and suppose that K_n is not nowhere dense in F. Then we can find a nonempty open set $U \subset K$ such that $U \cap F \neq \emptyset$ and $U \cap F \subset K_n$. Since $K \setminus F$ is dense in K, we may find a point $x \in U \cap (K \setminus F)$. Due to density of F in K there is a sequence $\{x_k\}$ of points of F such that $x = \lim x_k$. Since $x \in U$ and U is open in K, we may assume that $x_k \in U \cap F$ for each integer k. As $U \cap F \subset K_n$ and K_n is a closed set, $x \in K_n \subset F$. This contradicts the fact that $x \in K \setminus F$.

Since $K \setminus F$ is dense in K, every K_n is nowhere dense in K. Since F is a residual subset of K as well as $K \setminus \bigcup_n K_n$, the set $F \setminus \bigcup_n K_n$ is residual in K. According to [**9**, Theorem 3.9.3], $F \setminus \bigcup_n K_n$ is dense in K. ∎

Lemma 3.12. *Let $\mathcal{H}$ be a function space on a metrisable compact set K, x be a point in $Ch_{\mathcal{H}} K$ and $\{x_n\}$ be a sequence of points converging to x. Then $\mu_n \to \varepsilon_x$ for every sequence $\{\mu_n\}$ where $\mu_n \in \mathcal{M}_{x_n}(\mathcal{H})$.*

Proof: If we suppose the contrary, then there exists a measure $\mu \neq \varepsilon_x$ and a subsequence $\{\mu_{n_k}\}$ so that $\mu_{n_k} \to \mu$. It is straightforward to verify that μ is an

$\mathcal{H}$–representing measure for x. Since μ is not the Dirac measure at x, we have arrived to a contradiction with the assumption that $x \in Ch_{\mathcal{H}} K$. ∎

Proposition 3.13. *Let $\mathcal{H}$ be a simplicial function space on a compact metric space (K, ρ) such that $Ch_{\mathcal{H}} K$ is not an F_σ–set. Then there exists a nonempty closed set $F \subset K$ with $C := F \setminus Ch_{\mathcal{H}} K$ being a countable set and compact sets $\{K_y : y \in C\}$ so that*

(a) $\overline{C} = \overline{F \cap Ch_{\mathcal{H}} K} = F$;
(b) $K_y \subset Ch_{\mathcal{H}} K$ *for every* $y \in C$;
(c) $K_y \cap F = \emptyset$ *if* $y \in C$ *and* $K_y \cap K_x = \emptyset$ *if* $y, x \in C$, $y \neq x$;
(d) $\{y \in C : \delta_y(K_y) \leqslant 1 - \varepsilon\}$ *is finite for every* $\varepsilon > 0$; *and*
(e) $\{y \in C : K_y \not\subset U_\varepsilon(F)\}$ *is finite for every* $\varepsilon > 0$.

Proof: Since $Ch_{\mathcal{H}} K$ is assumed not to be an F_σ–set, we can use Hurewicz' theorem (see [**12**, Theorem 21.22]) and find a closed subset H of K such that

$$\overline{H \cap Ch_{\mathcal{H}} K} = \overline{H \setminus Ch_{\mathcal{H}} K} = H$$

and $H \setminus Ch_{\mathcal{H}} K$ is countable. For every $y \in H \setminus Ch_{\mathcal{H}} K$ we find an increasing sequence $\{L_{y,m}\}$ of compact subsets of $Ch_{\mathcal{H}} K$ such that $\delta_y(\bigcup_m L_{y,m}) = 1$. We enumerate the countable family

$$\{L_{y,m} : m \in \mathbb{N}, y \in H \setminus Ch_{\mathcal{H}} K\}$$

into a single sequence $\{L_k\}$. We pick an arbitrary point $x_0 \in H \setminus \bigcup_k L_k$ and set

$$F_0 := \{x_0\} \quad \text{and} \quad I := H \cap Ch_{\mathcal{H}} K \cap \bigcap_{k=1}^{\infty} (H \setminus L_k) .$$

It follows from Lemma 3.11 that the set I is dense in H.

We construct by induction sets $U_k, V_k, A_k, B_k, F_k, K_{k,j}$, $k, j \in \mathbb{N}$, so that, for every $k \in \mathbb{N}$,

(i) $A_k = \{x_{k,j} : j \in \mathbb{N}\} \subset I$, $B_k = \{y_{k,j} : j \in \mathbb{N}\} \subset H \setminus Ch_{\mathcal{H}} K$, both sets consists of distinct elements, the set $F_k := A_k \cup B_k$ satisfies

$$F_k \cap (F_0 \cup \dots \cup F_{k-1}) = \emptyset \quad \text{and} \quad \operatorname{der} A_k = \operatorname{der} B_k = F_0 \cup \dots \cup F_{k-1} ,$$

(ii) U_k, V_k are open subsets of K,

$$U_k \supset L_k , \quad (V_1 \cap \dots \cap V_k) \supset (F_0 \cup \dots \cup F_k) \quad \text{and} \quad U_k \cap V_k = \emptyset ;$$

(iii) $K_{k,j}$, $j \in \mathbb{N}$, is a compact set,

$$K_{k,j} \subset \bigcup_{m=1}^{\infty} L_{y_{k,j},m} , \quad \text{and} \quad K_{k_1,j_1} \cap K_{k_2,j_2} = \emptyset$$

if $1 \leqslant k_1 < k_2 \leqslant k$ or if $j_1, j_2 \in \mathbb{N}$, $j_1 \neq j_2$;

https://doi.org/10.1017/S0004972700038211 Published online by Cambridge University Press

(iv) for every $j \in \mathbb{N}$,

$$\rho(x_{k,j}, y_{k,j}) < \frac{1}{jk}, \quad K_{k,j} \subset U\Big(x_{k,j}, \frac{1}{jk}\Big), \quad \text{and} \quad \delta_{y_{k,j}}(K_{k,j}) > 1 - \frac{1}{jk}.$$

In the first step of the construction we find a couple of disjoint open sets U_1, V_1 in K so that $L_1 \subset U_1$ and $x_0 \in V_1$. Using density of I we select a sequence $\{x_{1,j}\}$ of distinct points of $I \cap V_1$ such that $x_{1,j} \to x_0$. Let $r_j > 0$, $j \in \mathbb{N}$, be positive numbers such that

1. $r_j < 1/j$,
2. $U(x_{1,j}, r_j) \subset V_1$,
3. the family $\{U(x_{1,j}, r_j) : j \in \mathbb{N}\}$ is pairwise disjoint, and
4. $x_0 \notin U(x_{1,j}, r_j)$.

Now we use density of $H \setminus Ch_{\mathcal{H}} K$ in H and Lemma 3.12 to find points $y_{1,j} \in U(x_{1,j}, r_j) \cap (H \setminus Ch_{\mathcal{H}} K)$, $j \in \mathbb{N}$, such that $\delta_{y_{1,j}}(U(x_{1,j}, r_j)) > 1 - (1/j)$. Since

$$\delta_{y_{1,j}}\Big(U(x_{1,j}, r_j) \cap \bigcup_{m=1}^{\infty} L_{y_{1,j},m}\Big) = \delta_{y_{1,j}}(U(x_{1,j}, r_j)) > 1 - \frac{1}{j},$$

we can use the regularity of $\delta_{y_{1,j}}$ and find a compact set

$$K_{1,j} \subset U(x_{1,j}, r_j) \cap \bigcup_{m=1}^{\infty} L_{y_{1,j},m}$$

so that

$$\delta_{y_{1,j}}(K_{1,j}) > 1 - \frac{1}{j}.$$

Obviously,

$$\operatorname{der} A_1 = \operatorname{der}\{x_{1,j} : j \in \mathbb{N}\} = \{x_0\} = F_0,$$

as well as

$$\operatorname{der} B_1 = \operatorname{der}\{y_{1,j} : j \in \mathbb{N}\} = \{x_0\} = F_0.$$

Since the validity of the remaining properties (ii)–(iv) follows directly from the construction, the first step is finished.

Suppose now that the construction has been completed for every integer $i \leqslant k$. It easily follows from (i) that the set $F_0 \cup \cdots F_k$ is closed and does not intersect $\bigcup_k L_k$. Thus we can find a couple of disjoint open sets U_{k+1} and V_{k+1} so that $L_{k+1} \subset U_{k+1}$ and $F_0 \cup \cdots \cup F_k \subset V_{k+1}$. Since H contains no isolated points and $F_0 \cup \cdots \cup F_k$ is countable, this set has an empty interior in H. Now we are going to construct a countable set $A_{k+1} = \{x_{k+1,j} : j \in \mathbb{N}\}$ of distinct points of I so that

$$(1) \qquad A_{k+1} \cap (F_0 \cup \cdots \cup F_k) = \emptyset \quad \text{and} \quad \operatorname{der} A_{k+1} = F_0 \cup \cdots \cup F_k.$$

To this end, let $\{N_m\}$ be a sequence of finite $1/m$–net of $F_0 \cup \cdots \cup F_k$. We use density of I in H and the fact that $F_0 \cup \cdots \cup F_k$ has an empty interior in H and inductively find finite sets

$$M_m \subset (I \cap V_{k+1}) \setminus (F_0 \cup \cdots \cup F_k)\,, \quad m \in \mathbb{N}\,,$$

so that

$$N_m \subset \bigcup \Big\{ U\Big(x, \frac{1}{m}\Big) : x \in M_m \Big\} \quad \text{and} \quad M_m \cap (M_1 \cup \cdots \cup M_{m-1}) = \emptyset$$

for each $m \in \mathbb{N}$. If we enumerate points of $\bigcup_m M_m$ into a single sequence $\{x_{k+1,j}\}$ and define $A_{k+1} := \{x_{k+1,j} : j \in \mathbb{N}\}$, we obtain a set satisfying (1).

As above we get from (i) and (iv) that the set

$$F_0 \cup \cdots \cup F_k \cup \bigcup \{K_{i,j} : 1 \leqslant i \leqslant k, j \in \mathbb{N}\} \tag{2}$$

is closed and does not intersect A_{k+1}. Hence we can find strictly positive numbers r_j so that

1. $r_j < 1/(jk)$,
2. $U(x_{k+1,j}, r_j) \subset V_{k+1}$,
3. the family $\{U(x_{k+1,j}, r_j) : j \in \mathbb{N}\}$ is pairwise disjoint, and
4. $U(x_{k+1,j}, r_j)$ does not intersect the set (2).

As in the first step of the construction, for every integer j we pick a point

$$y_{k+1,j} \in U(x_{k+1,j}, r_j) \cap (H \setminus Ch_{\mathcal{H}} K)$$

and a compact set

$$K_{k+1,j} \subset U(x_{k+1,j}, r_j) \cap \bigcup_{m=1}^{\infty} L_{y_{k+1,j},m}$$

so that

$$\delta_{y_{k+1,j}}(K_{k+1,j}) > 1 - \frac{1}{jk}\,.$$

By setting $B_{k+1} := \{y_{k+1,j} : j \in \mathbb{N}\}$ we finish the inductive step of the construction.

We set

$$\widehat{F} := \overline{\bigcup_{k=0}^{\infty} F_k}\,, \text{ and } \quad \widehat{C} := \bigcup_{k=1}^{\infty} B_k\,.$$

Then (ii) yields

$$\bigcup_{k=0}^{\infty} F_k \subset \bigcap_{k=1}^{\infty} V_k \subset K \setminus \bigcup_{k=1}^{\infty} U_k\,.$$

Since the latter set is closed,

$$\widehat{F} \subset K \setminus \bigcup_{k=1}^{\infty} U_k$$

https://doi.org/10.1017/S0004972700038211 Published online by Cambridge University Press

as well. Thus $\widehat{F}$ does not intersect $\bigcup_k L_k$ which together with (iii) gives that

$$\widehat{F} \cap K_{k,j} = \emptyset\,, \quad k, j \in \mathbb{N}\,. \tag{3}$$

The property (i) implies that $\widehat{C}$ is a dense subset of $\bigcup_k F_k$ and thus also of $\widehat{F}$. By the same reasoning we get that $\widehat{F} \cap Ch_{\mathcal{H}} K$ is dense in $\widehat{F}$. We claim that $\widehat{F} \cap Ch_{\mathcal{H}} K$ is not F_σ-separated from $\widehat{C}$, that is, there exists no F_σ–set Z satisfying $\widehat{F} \cap Ch_{\mathcal{H}} K \subset Z \subset K \setminus \widehat{C}$.

Indeed, if $\widehat{F} \cap Ch_{\mathcal{H}} K$ were separated from $\widehat{C}$ by an F_σ–set Z, then the sets $\widehat{F} \cap Ch_{\mathcal{H}} K$ and $\widehat{F} \setminus Z$ would be a couple of disjoint dense G_δ–sets in $\widehat{F}$. But this is impossible as $\widehat{F}$ is a Baire space.

Hence we can use [**12**, Theorem 21.22] again and find a closed set

$$F \subset (\widehat{F} \cap Ch_{\mathcal{H}} K) \cup \widehat{C}$$

such that

$$\overline{F \cap Ch_{\mathcal{H}} K} = \overline{F \setminus Ch_{\mathcal{H}} K} = F\,. \tag{4}$$

Obviously, $C = F \setminus Ch_{\mathcal{H}} K$ is a countable set. For every $y \in F \setminus Ch_{\mathcal{H}} K$ we set $K_y := K_{y_{k,j}}$ if $y = y_{k,j}$, $k, j \in \mathbb{N}$.

We claim that F and compact sets $\{K_y : y \in F \setminus Ch_{\mathcal{H}} K\}$ posses all the required properties.

Indeed, property (a) is stated in (4) and properties (b) and (c) follows from (iii), (3) and from the choice of compact sets L_k.

It remains to verify (d) and (e). To this end, let $\varepsilon > 0$. Let k_0 be an integer satisfying $1/k_0 < \varepsilon$, If $y \in C$ then $y = y_{k,j}$ for some $k, j \in \mathbb{N}$. Then the condition (iv) implies that $\delta_{y_{k,j}}(K_{k,j}) > 1 - \varepsilon$ if $k \geqslant k_0$ and $j \in \mathbb{N}$ or if $j \geqslant k_0$ and $1 \leqslant k < k_0$. Hence the set $\{y \in C : \delta_y(K_y) \leqslant 1 - \varepsilon\}$ is finite.

Similarly we get from the condition (iv) that

$$K_{k,j} \subset U\Big(x_{k,j}, \frac{1}{jk}\Big) \subset U\Big(y_{k,j}, \frac{2}{kj}\Big) \subset U_\varepsilon(F)$$

if $k \geqslant 2k_0$ and $j \in \mathbb{N}$ or $1 \leqslant k < 2k_0$ and $j \geqslant 2k_0$. This observation completes the proof of the proposition. ∎

Now we are ready for the proof of the main theorem.

PROOF: [Proof of Theorem 3.1] For the proof of the implication (i) $\Longrightarrow$(ii), suppose that $\mathcal{H}$ is simplicial and $Ch_{\mathcal{H}} K$ is an F_σ–set. Thus we can write $Ch_{\mathcal{H}} K = \bigcup_n F_n$ where $\{F_n\}$ is an increasing sequence of compact sets. Let f be a bounded Baire–one function on $Ch_{\mathcal{H}} K$ and $\{f_n\}$ be a sequence of continuous functions on $Ch_{\mathcal{H}} K$ converging pointwise to f. We may assume that $\|f\|$, $\|f_n\|$ are bounded by a positive number M. According to [**6**, Corollary 3.6], there exists a sequence $\{h_n\}$ of $\mathcal{H}$–affine continuous functions on K such that $h_n = f_n$ on F_n and $\|h_n\| = \|f_n\|$.

The proof will be completed by showing that the sequence $\{h_n\}$ converges pointwise to the function $h(x) := \delta_x(f)$, $x \in K$. Notice that the definition is meaningful since maximal measures are carried by $Ch_{\mathcal{H}} K$ due to [8, Lemma 27.14].

For a fixed $x \in K$ and $\varepsilon > 0$ we find an integer n_0 such that $\int_{Ch_{\mathcal{H}} K} |f - f_n| \, d\delta_x < \varepsilon$ and $\delta_x(F_n) > 1 - \varepsilon$ for all $n \geqslant n_0$. Then, for $n \geqslant n_0$, we have

$$\begin{aligned} |h(x) - h_n(x)| &= \left| \int_K (f - h_n) \, d\delta_x \right| \\ &\leqslant \int_{F_{n_0}} |f - f_n| \, d\delta_x + \int_{K \setminus F_{n_0}} 2M \, d\delta_x \\ &\leqslant \varepsilon + \varepsilon 2M , \end{aligned}$$

which proves the required statement and concludes the first part of the proof.

Since the implication (ii) $\Longrightarrow$(iii) is obvious, we proceed to the proof of (iii) $\Longrightarrow$ (iv). Let $\mathcal{H}$ be a function space on a compact K satisfying the condition (iii). First of all we verify that $\mathcal{H}$ is simplicial.

Indeed, for a given continuous $\mathcal{H}$–convex function f on K we find an $\mathcal{H}$–affine Baire–one function h with $h = f$ on $Ch_{\mathcal{H}} K$. Thanks to Proposition 3.6, $h \geqslant f$ on K. For a given $x \in K$, [6, Lemma 1.1] yields the existence of a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $\mu(f) = f^*(x)$. Then

$$f^*(x) = \mu(f) \leqslant \mu(h) = h(x) .$$

On the other hand, let $g \in \mathcal{H}$ satisfy $g \geqslant f$. Then $g \geqslant h$ on $Ch_{\mathcal{H}} K$ and thus $g \geqslant h$ on K, which again follows from Proposition 3.6. Hence

$$h(x) \leqslant \inf\{g(x) : g \in \mathcal{H}, g \geqslant f\} = f^*(x) .$$

Thus $f^* = h$ is $\mathcal{H}$–affine for every $\mathcal{H}$–convex continuous function f and $\mathcal{H}$ is a simplicial function space.

It remains to check that Tf is a Baire–one function for every $f \in \mathcal{B}_1^b(K)$. Thanks to the previous paragraph, Tf is a Baire–one function for any $\mathcal{H}$–convex continuous function f. Hence $T(\mathcal{C}(K)) \subset \mathcal{B}^b(K)$ which gives that Tg is a Baire function for any bounded Baire function g on K.

If f is a bounded Baire–one function on K, let h be an $\mathcal{H}$–affine Baire one function on K with $f = h$ on $Ch_{\mathcal{H}} K$. Then $h = Tf$ on $Ch_{\mathcal{H}} K$ and the application of Proposition 3.6 yields that $h = Tf$ on K. Thus Tf is a Baire–one function as required.

As the implication (iv) $\Longrightarrow$ (iii) is obvious, the next step will be the proof of the implication (iv) $\Longrightarrow$ (v). Let f be an $\mathcal{H}$–convex Baire–one function on K. Due to the condition (iv), Tf is an $\mathcal{H}$–affine Baire–one function. Moreover, $Tf = f$ on $Ch_{\mathcal{H}} K$ and $Tf \geqslant f$ on K by Proposition 3.6. Thus $Tf \geqslant \widehat{f}$.

On the other hand, given an $\mathcal{H}$–affine Baire–one function h with $h \geqslant f$, the minimum principle laid down in Proposition 3.6 gives that $h \geqslant Tf$. Thus $Tf \leqslant \widehat{f}$ and $\widehat{f} = Tf$ is an $\mathcal{H}$–affine Baire–one function.

In order to prove (v) $\Longrightarrow$ (vi), let f and g be $\mathcal{H}$–affine Baire–one functions. Since $h = \widehat{(f \vee g)}$ is an $\mathcal{H}$–affine Baire–one function, h is obviously the least upper bound for the couple f and g in $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$. Thus $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ is a lattice in its natural ordering.

It remains to prove the implication (v) $\Longrightarrow$ (i) provided K is metrisable. Since (v) $\Longrightarrow$ (vi), we know from Proposition 3.8 that $\mathcal{H}$ is a simplicial function space. We fix on K a compatible metric ρ.

If we assume that $Ch_{\mathcal{H}} K$ is not an F_σ–set, let F, C and $\{K_x : x \in C\}$ be sets constructed in Proposition 3.13. By setting

$$f := \begin{cases} 1\,, & \text{on } \bigcup\{K_x : x \in C\}\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

we get a Baire–one function on K.

Indeed, the set $H := F \cup \bigcup\{K_x : x \in C\}$ is closed according to the condition (e) of Proposition 3.13. Thus $f = \chi_H \setminus \chi_F$ is a Baire–one function.

It follows directly from the definition that f is an $\mathcal{H}$–convex function on K. We conclude the proof by showing that $\widehat{f}$ is not a Baire–one function on F.

To this end we pick an arbitrary $\varepsilon \in (0,1)$. If $h \in \mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1(K)$ satisfies $h \geqslant f$, then

$$h(x) = \delta_x(h) \geqslant \delta_x(f) > 1 - \varepsilon$$

for all but finitely many points $x \in C$. We denote this exceptional set by C_ε.

On the other hand, given a point $z \in Ch_{\mathcal{H}} K \cap F$, the function $k_z := \chi_{K \setminus \{z\}}$ is an $\mathcal{H}$–concave lower semicontinuous function. Thus $Tk_z = (k_z)_*$ is a lower semicontinuous $\mathcal{H}$–affine function satisfying $f \leqslant Tk_z$ which gives $0 \leqslant \widehat{f}(z) \leqslant Tk_z(z) = 0$.

Hence we have obtained that

$$\widehat{f} = 0 \text{ on } F \cap Ch_{\mathcal{H}} K \quad \text{and} \quad \widehat{f} > 1 - \varepsilon \text{ on } C \setminus C_\varepsilon\,.$$

Since

$$\overline{F \cap Ch_{\mathcal{H}} K} = \overline{F \cap (C \setminus C_\varepsilon)} = F\,,$$

the function $\widehat{f}$ has no point of continuity on F and hence cannot be of the first Baire–class. This concludes the proof. ∎

REMARK 3.14. We note that the proof of (ii) $\Longrightarrow$ (i) in Theorem 3.1 for metrisable compact sets can be substantially simplified, namely we do not need Proposition 3.13. We briefly indicate this simplification.

We assume that K is metrisable and the condition (ii) of Theorem 3.1 holds. It is easy to check that $\mathcal{H}$ is a simplicial function space. If we assume that $Ch_{\mathcal{H}} K$ is not an F_σ–set, we proceed as in the proof of Proposition 3.13 and find a closed set H so that $\overline{H \cap Ch_{\mathcal{H}} K} = \overline{H \setminus Ch_{\mathcal{H}} K} = H$ and $H \setminus Ch_{\mathcal{H}} K$ is countable. For every $x \in H \setminus Ch_{\mathcal{H}} K$ we find an F_σ–set $K_x \subset Ch_{\mathcal{H}} K$ such that $\delta_x(K_x) = 1$. Then

$$A := H \cap Ch_{\mathcal{H}} K \setminus \bigcup \{K_x : x \in H \setminus Ch_{\mathcal{H}} K\}$$

is not F_σ-separated from $H \setminus Ch_{\mathcal{H}} K$, that is, there is no F_σ–set $Z \subset K$ satisfying $A \subset Z$ and $Z \cap (H \setminus Ch_{\mathcal{H}} K) = \emptyset$. (Otherwise $H \cap Ch_{\mathcal{H}} K$ would be an F_σ–set which is impossible.) Another use of [**12**, Theorem 21.22] provides a compact set $F \subset A \cup (H \setminus Ch_{\mathcal{H}} K)$ so that $\overline{F \cap A} = \overline{F \cap (H \setminus Ch_{\mathcal{H}} K)} = F$. Then χ_F is a bounded Baire–one function on K and

$$T(\chi_F) = \begin{cases} 1\,, & \text{on } F \cap Ch_{\mathcal{H}} K\,, \\ 0\,, & \text{on } F \setminus Ch_{\mathcal{H}} K\,. \end{cases}$$

Thus the function $T(\chi_F)$, lacking a point of continuity on F, is not of the first Baire class. Since $T(\chi_F)$ is the only possible $\mathcal{H}$–affine extension of $\chi_F \restriction Ch_{\mathcal{H}} K$, the function $\chi_F \restriction Ch_{\mathcal{H}} K$ has no $\mathcal{H}$–affine Baire–one extension on K.

REMARK 3.15. We remark that for a function space $\mathcal{H}$ on a metrisable compact space K another "in–between" condition equivalent to (i) in Theorem 3.1 can be formulated. Namely, Theorem 3.1 (i) holds if and only if for every couple $f, -g$ of bounded Baire–one $\mathcal{H}$–convex functions with $f \leqslant g$ there exists an $\mathcal{H}$–affine Baire–one function h so that $f \leqslant h \leqslant g$. (This condition is a Baire–one analogue of Edwards' "in–between" theorem [**6**, Theorem 3.2].)

We sketch the proof of the assertion. If $\mathcal{H}$ is a simplicial function space and $Ch_{\mathcal{H}} K$ is an F_σ–set, Theorem 3.1 (iv) and Proposition 3.6 easily yields the validity of the condition cited above.

Conversely, suppose that the "in–between" condition holds. First we have to prove that $\mathcal{H}$ is simplicial. To this end, let f be an $\mathcal{H}$–convex continuous function on K. If $\mu \in \mathcal{M}_x(\mathcal{H})$ represents $x \in K$ and $\varepsilon > 0$, we use the generalisation of the Lebesgue monotone convergence theorem (see [**10**, Theorem 12.46]) and find an $\mathcal{H}$–concave continuous function k such that $f \leqslant k$ and $\mu(f^*) \geqslant \mu(k) - \varepsilon$. An appeal to the "in–between" property provides an $\mathcal{H}$–affine Baire–one function h so that $f \leqslant h \leqslant k$. As in the proof of Proposition 3.8 we get that $h(x) \geqslant f^*(x)$. Thus

$$\mu(f^*) \geqslant \mu(k) - \varepsilon \geqslant \mu(h) - \varepsilon = h(x) - \varepsilon \geqslant f^*(x) - \varepsilon\,.$$

As ε is arbitrary, $\mu(f^*) \geqslant f^*(x)$. Since the converse inequality is obvious, $\mu(f^*) = f^*(x)$ and f^* is $\mathcal{H}$–affine. According to the characterisation of simplicial spaces cited in Section 2, $\mathcal{H}$ is simplicial.

To finish the proof we have to verify that $Ch_{\mathcal{H}} K$ is an F_σ–set. Assuming the contrary, Proposition 3.13 provides sets F, C and K_x, $x \in C$, with the corresponding properties. We split C into a couple of disjoint dense sets C_1 and C_2. By setting

$$f = \begin{cases} 1 & \text{on } \bigcup\{K_x : x \in C_1\}\,, \\ 0 & \text{otherwise}\,, \end{cases} \quad \text{and} \quad g = \begin{cases} 0 & \text{on } \bigcup\{K_x : x \in C_2\}\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

we obtain a couple of Baire–one functions (see the proof of (v) $\Longrightarrow$ (i) of Theorem 3.1) such that $f \leqslant g$, f is $\mathcal{H}$–convex and g is $\mathcal{H}$–concave. Obviously, any $\mathcal{H}$–affine function h satisfying $f \leqslant h \leqslant g$ has no point of continuity on F and thus cannot be of the first Baire–class. This contradiction finishes the proof of the remark.

REMARK 3.16. We consider X to be Poulsen's simplex (see [**2**, Chapter 3.7] for its construction and properties). Then $\overline{\operatorname{ext} X} = X$ and thus $\operatorname{ext} X$ cannot be an F_σ–set. We remark that $\mathcal{B}_1(\mathfrak{A}^c(X))$ is not a lattice in the natural ordering.

Indeed, the compact convex set constructed in Example 3.10 is affinely homeomorphic to a closed face F of X (see [**2**, Theorem 7.6]). Thus $\mathcal{B}_1(\mathfrak{A}^c(F))$ is not a lattice in the natural ordering. We use [**14**, Theorem 3.6] and find an affine retraction r of X onto F, that is, $r : X \to F$ is an affine continuous mapping and $r(x) = x$ for every $x \in F$. If f, g are affine Baire–one functions on F, the functions $f \circ r, g \circ r$ are affine Baire–one functions on X. Assuming that $\mathcal{B}_1(\mathfrak{A}^c(X))$ is a lattice, we can find an affine Baire–one function h on X so that $h \geqslant f \vee g$ and h is the least affine Baire–one function with this property. It does no harm to verify that $h \restriction F$ is the least affine Baire–one function on F which is greater or equal to $f \vee g$. But this contradicts the fact that $\mathcal{B}_1(\mathfrak{A}^c(F))$ is not a lattice in the natural ordering.

The following example shows that the implication (vi) $\Longrightarrow$ (i) of Theorem 3.1 need not hold in general. The construction is a slight modification of Example 3.10.

EXAMPLE 3.17. There exists a metrisable Choquet simplex X such that $\mathcal{B}_1(\mathfrak{A}^c(X))$ is a lattice in the natural ordering but $\operatorname{ext} X$ is not an F_σ–set.

PROOF: First of all we shall construct a function space $\mathcal{H}$ on a metrisable compact space K such that:

(a) $Ch_{\mathcal{H}} K$ is not an F_σ–set;
(b) $\mathcal{H}$ is a simplicial function space;
(c) $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$; and
(d) $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ is a lattice in the natural ordering.

Let $\{q_n\}$ be an enumeration of rational numbers contained in $[0,1]$. We define a subset $K \subset \mathbb{R}^2$ as follows:

$$K := \big([0,1] \times \{0\}\big) \cup \big\{(q_n, n^{-1}), (q_n, -n^{-1}) : n \in \mathbb{N}\big\}\,.$$

https://doi.org/10.1017/S0004972700038211 Published online by Cambridge University Press

Obviously, K is a compact set in $\mathbb{R}^2$. Given a natural number n, for the sake of brevity we shall write q_n^0, q_n^+ and q_n^- instead of $(q_n, 0)$, $(q_n, 1/n)$ and $\big(q_n, -(1/n)\big)$, respectively.

The function space $\mathcal{H}$ will consist of all continuous functions f on K which satisfies

(5) $$f(q_n^0) = \frac{1}{n} f(q_n^-) + \Big(1 - \frac{1}{n}\Big) f(q_n^+)\,, \quad n \in \mathbb{N}\,.$$

Obviously, $\mathcal{H}$ contains the constant functions. In order to check that $\mathcal{H}$ separates points of K we can consider the following family of functions:

$$h_{x_0}(x,y) := |x - x_0|\,, \quad x_0 \in [0,1]\,;$$

$$h_{q_n^+}(x,y) := \begin{cases} 0\,, & (x,y) = q_n^+\,, \\ n\,, & (x,y) = q_n^-\,, \\ 1\,, & \text{otherwise}\,, \end{cases} \qquad n \in \mathbb{N}\,;$$

$$h_{q_n^-}(x,y) := \begin{cases} 0\,, & (x,y) = q_n^-\,, \\ \frac{n}{n-1}\,, & (x,y) = q_n^+\,, \\ 1\,, & \text{otherwise}\,, \end{cases} \qquad n \in \mathbb{N}\,.$$

Thus $\mathcal{H}$ is a function space.

We claim that $Ch_{\mathcal{H}} K = K \setminus \{q_n^0 : n \in \mathbb{N}\}$. Indeed, no point of the set $\{q_n^0 : n \in \mathbb{N}\}$ lies in the Choquet boundary $\mathcal{H}$ of K. On the other hand, functions defined above show that for every point in $K \setminus \{q_n^0 : n \in \mathbb{N}\}$ there exists an $\mathcal{H}$–exposing function and thus $K \setminus \{q_n^0 : n \in \mathbb{N}\} = Ch_{\mathcal{H}} K$. It follows that $Ch_{\mathcal{H}} K$ is not an F_σ–set and the property (a) is proved.

Concerning the property (b), it is enough to prove that, for every $n \in \mathbb{N}$, the measure $(1/n)\varepsilon_{q_n^-} + \big((n-1)/n\big)\varepsilon_{q_n^+}$ is the only maximal measure $\delta_{q_n^0}$ representing the point q_n^0. For $n \in \mathbb{N}$ it immediately follows from the definition of h_{q_n} that any measure representing q_n^0 is supported by the set $\{q_n^0, q_n^-, q_n^+\}$. Clearly,

$$\delta_{q_n^0} = \frac{1}{n}\varepsilon_{q_n^-} + \frac{n-1}{n}\varepsilon_{q_n^+}$$

and $\mathcal{H}$ is simplicial.

For the proof of (c), let f be an $\mathcal{H}$-affine continuous function. Since any $\mathcal{H}$–representing measure for a point q_n^0 is supported by $\{q_n^0, q_n^-, q_n^+\}$, we get that

(6) $$\mathcal{M}_{q_n^0}(\mathcal{H}) = \mathrm{co}\{\varepsilon_{q_n^0}, \delta_{q_n^0}\}\,, \quad n \in \mathbb{N}\,.$$

Thus f, being an $\mathcal{H}$–affine function, satisfies the equalities (5) and $f \in \mathcal{H}$ according to the definition.

In order to check the last assertion (d), it is enough to prove that $T(f \vee g)$ is a Baire–one function for every couple f and g of $\mathcal{H}$–affine Baire–one functions (see Proposition 3.8). Let f and g be such functions with values in $[0,1]$ and set $h := f \vee g$. We claim that

$$|h(q_n^+) - h(q_n^0)| \leqslant \frac{2}{n} \quad \text{for every } n \in \mathbb{N}\,. \tag{7}$$

Indeed, for a fixed integer n we have

$$\begin{aligned} |f(q_n^+) - f(q_n^0)| &= \Big|f(q_n^+) - \frac{1}{n}f(q_n^-) - \Big(1 - \frac{1}{n}\Big)f(q_n^+)\Big| \\ &\leqslant \frac{1}{n} \cdot |f(q_n^+) - f(q_n^-)| \leqslant \frac{2}{n}\,. \end{aligned}$$

By the same argument, $|g(q_n^+) - g(q_n^0)| \leqslant 2/n$. We need to check this inequality for the function h. The only nontrivial case is when $h(q_n^+) = f(q_n^+)$ and $h(q_n^0) = g(q_n^0)$ (or vice versa). Then

$$\begin{aligned} h(q_n^+) = f(q_n^+) &= f(q_n^+) - f(q_n^0) + f(q_n^0) \leqslant f(q_n^+) - f(q_n^0) + g(q_n^0) \\ &\leqslant \frac{2}{n} + g(q_n^0) = \frac{2}{n} + h(q_n^0)\,, \end{aligned}$$

and

$$\begin{aligned} h(q_n^0) = g(q_n^0) &= g(q_n^0) - g(q_n^+) + g(q_n^+) \leqslant g(q_n^0) - g(q_n^+) + f(q_n^+) \\ &\leqslant \frac{2}{n} + f(q_n^+) = \frac{2}{n} + h(q_n^+)\,. \end{aligned}$$

Combining these inequalities together we get (7).

Applying this inequality (7) we obtain

$$\begin{aligned} |Th(q_n^0) - h(q_n^0)| &= \Big|\frac{1}{n}h(q_n^-) + \Big(1 - \frac{1}{n}\Big)h(q_n^+) - h(q_n^0)\Big| \\ &= \Big|\frac{1}{n}\big(h(q_n^-) - h(q_n^+)\big) + h(q_n^+) - h(q_n^0)\Big| \\ &\leqslant \frac{2}{n} + \frac{2}{n} = \frac{4}{n}\,. \end{aligned}$$

Hence the set

$$\Big\{x \in K : |Th(x) - h(x)| \geqslant \varepsilon\Big\} = \Big\{q_n^0 \in [0,1] \times \{0\} : |Th(q_n^0) - h(q_n^0)| \geqslant \varepsilon, n \in \mathbb{N}\Big\}$$

is finite for every $\varepsilon > 0$. By virtue of Theorem 2.1 (f) and Proposition 3.8, Th is a Baire–one function and the space $\mathcal{A}(\mathcal{H}) \cap \mathcal{B}_1^b(K)$ is a lattice in the natural ordering.

According to Proposition 3.9, $X := \mathbf{S}(\mathcal{H})$ is a compact convex set such that $\operatorname{ext} X = \phi(Ch_{\mathcal{H}} K)$ is not an F_σ–set and $\mathcal{B}_1\big(\mathfrak{A}^c(X)\big)$ is a lattice in the natural ordering. ∎

The following example shows that Theorem 3.1 is not true in general if we omit the assumption of the metrisability of the compact space K. Namely, we verify that the simplex constructed by Talagrand in **[23]** satisfies the condition (ii) of Theorem 3.1 but the set of all extreme points is not even a $\mathcal{K}$–Borel set (the smallest family containing all compact sets and closed with respect to taking countable unions and intersections).

EXAMPLE 3.18. There exists a simplex X such that $\operatorname{ext} X$ is not a $\mathcal{K}$–Borel set and every bounded Baire–one function defined on $\operatorname{ext} X$ can be extended to an affine Baire–one function defined on X.

PROOF: We recall M. Talagrand's construction from [**23**]. Let $T := \mathbb{N}^{\mathbb{N}} \cup \{\omega\}$ where ω is a point not belonging to $\mathbb{N}^{\mathbb{N}}$. Let $\mathcal{A}$ be a family of sets in $\mathbb{N}^{\mathbb{N}}$ such that

1. every $A \in \mathcal{A}$ is a closed discrete set in $\mathbb{N}^{\mathbb{N}}$ considered with the usual topology;
2. the family $\mathcal{A}$ is almost disjoint, that is, $A \cap B$ is at most finite for every couple $A, B \in \mathcal{A}$ of distinct sets.

We consider T endowed with a topology τ that makes each point of $\mathbb{N}^{\mathbb{N}}$ open and the neighbourhoods of ω are of the form $T \setminus B$, where B is the union of a finite set and finitely many elements from $\mathcal{A}$.

Talagrand proved that T is a completely regular space which is $K_{\sigma\delta}$ in its Stone–Čech compactification. In particular, T is a $\mathcal{K}$–analytic set, that is, it is the image of $\mathbb{N}^{\mathbb{N}}$ under an upper semicontinuous compact–valued map (see [**19**, 2.1]). According to [**19**, Theorem 2.7.1], T is a Lindelöf space.

Let K be the compactification of T such that closed sets in K can be identified to the algebra $\mathcal{L}$ generated by $\mathcal{A}$ and finite sets of $\mathbb{N}^{\mathbb{N}}$. (The compactification K is obtained as the closure of $\varphi(T)$ in $\{0,1\}^{\mathcal{L}}$, where $\varphi(x) = \{\chi_L(x)\}_{L \in \mathcal{L}}$, $x \in T$.)

Then every set $\overline{A}^K$ is clopen in K (here $\overline{A}^K$ stands for the closure of A in K) and $T \setminus \{\omega\}$ is an open subset of K. It follows from almost disjointness of $\mathcal{A}$ that every set $A \in \mathcal{A}$ determines a unique point $\{a_A\} = \overline{A}^K \setminus T$ and vice versa, every point $x \in K \setminus T$ is of the form a_A for some $A \in \mathcal{A}$.

The most important step in the construction is a careful choice of the family $\mathcal{A}$ which ensures that T is not a $\mathcal{K}$–Borel set in K.

For every $A \in \mathcal{A}$ a couple of points $b_A, c_A \in \mathbb{N}^{\mathbb{N}}$ is chosen so that these points are all distinct and they do not belong to any member of $\mathcal{A}$. Let

$$\mathcal{H} := \left\{ f \in \mathcal{C}(K) : f(a_A) = \frac{1}{2}\big(f(b_A) + f(c_A)\big), A \in \mathcal{A} \right\}.$$

It is easy to show that $Ch_{\mathcal{H}} K = T$ and $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$ is a simplicial function space.

After recalling M. Talagrand's construction we have to verify that every bounded Baire-one function on T can be extended to an $\mathcal{H}$–affine Baire–one function.

To this end we prove the following claim: *Any countable set $S \subset K \setminus T$ is a G_δ–set in K.*

Given a countable set $S \subset K \setminus T$, $S = \{a_n : n \in \mathbb{N}\}$, let A_n, $n \in \mathbb{N}$, be sets in $\mathcal{A}$ such that $\{a_n\} = \overline{A_n}^K \setminus T$. Then $G := \bigcup_n \overline{A_n}^K$ is an open subset of K. If $\{x_k\}$ is an enumeration of $\bigcup_n A_n$, we define $G_k := G \setminus \{x_1, \ldots, x_k\}$. Then G_k are open subsets of K and $S = \bigcap_k G_k$. Thus S is a G_δ–subset of K as desired.

https://doi.org/10.1017/S0004972700038211 Published online by Cambridge University Press

Let f be a bounded Baire–one function on T, and Tf the extension of f to K defined by saying that $(Tf)(a_A) = \big(f(b_A) + f(c_A)\big)/2$ for $A \in \mathcal{A}$. We claim that $Tf \in \mathcal{B}_1^b(K)$.

According to Theorem 2.1 (d), we may suppose that f is the characteristic function of a set $F \subset T$. We can also assume that $\omega \in F$. Since $T \setminus F = \{x \in T : f(x) = 0\}$ is an F_σ–set in T and T is Lindelöf, $T \setminus F$ is a Lindelöf space as well. As $T \setminus F$ is a discrete space, it is a countable set.

Obviously,

$$\{x \in K : Tf(x) = 0\} = (T \setminus F) \cup \{a_A \in K \setminus T : b_A, c_A \in T \setminus F\}$$

is a countable and thus also an F_σ–set. Similarly,

$$\Big\{x \in K : Tf(x) = \frac{1}{2}\Big\} = \{a_A \in K \setminus T : b_A \in T \setminus F, c_A \in F\} \\ \cup \{a_A \in K \setminus T : b_A \in F, c_A \in T \setminus F\}$$

is countable likewise. As the set

$$G := \Big\{x \in K \setminus T : Tf(x) = 0\} \cup \{x \in K \setminus T : Tf(x) = \frac{1}{2}\Big\}$$

is a countable subset of $K \setminus T$, the italicised claim yields that G is a G_δ–subset of K. Since $T \setminus F$ is an open set in K, we get that

$$\{x \in K : Tf(x) = 1\} = K \setminus \big((T \setminus F) \cup G\big)$$

is an F_σ–set in K. Due to Theorem 2.1 (c), Tf is a Baire–one function on K and we have proved that any bounded Baire–one function on $Ch_{\mathcal{H}} K$ can be extended to an $\mathcal{H}$–affine Baire–one function on K.

As in the previous examples, the required compact convex set X will be the state space $\mathrm{S}(\mathcal{H})$ of $\mathcal{H}$. Then X is a simplex and $\operatorname{ext} X = \phi(Ch_{\mathcal{H}} K)$ is a $\mathcal{K}$–analytic set which is not $\mathcal{K}$–Borel. Let F be a bounded Baire–one function on $\operatorname{ext} X$. We find an $\mathcal{H}$–affine Baire–one function g on K such that $g = F \circ \phi$ on $Ch_{\mathcal{H}} K$. As was mentioned in the paragraph above Proposition 3.9, any $\mathcal{H}$–affine function in a simplicial function space is completely $\mathcal{A}^c(\mathcal{H})$–affine function. Since $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$, [**22**, Theorem 4.3] yields the existence of an affine Baire–one function G on X such that $g = G \circ \phi$. Then G is the desired affine Baire–one extension of F and the proof is finished. ∎

References

[1] E.M. Alfsen, *Compact convex sets and boundary integrals* (Springer–Verlag, Berlin, Heidelberg, New York, 1971).

[2] L. Asimow and A.J. Ellis, *Convexity theory and its applications in functional analysis*, London Mathematical Society Monographs **16** (Academic Press, London, New York, 1980).

https://doi.org/10.1017/S0004972700038211 Published online by Cambridge University Press

[3] H. Bauer, 'Šilowscher Rand und Dirichletsches Problem', *Ann. Inst. Fourier (Grenoble)* **11** (1961), 89–136.

[4] H. Bauer, 'Simplicial function spaces and simplexes', *Exposition Math.* **3** (1985), 165–168.

[5] J. Bliedtner and W. Hansen, 'Simplicial cones in potential theory', *Inventiones Math.* **29** (1975), 83–110.

[6] N. Boboc and A. Cornea, 'Convex cones of lower semicontinuous functions on compact spaces', *Rev. Roumaine Math. Pures Appl.* **12** (1967), 471–525.

[7] G. Choquet, 'Remarque à propos de la démonstration de l'unicité de P.A. Meyer', *Séminaire Brelot–Choquet–Deny (Théorie de Potentiel)* **8** (1961/62), 6 e année.

[8] G. Choquet, *Lectures on analysis I - III* (W. A. Benjamin, Inc., New York, Amsterdam, 1969).

[9] R. Engelking, *General Topology* (Heldermann Verlag, Berlin, 1989).

[10] E. Hewitt and K. Stromberg, *Real and abstract analysis* (Springer–Verlag, Berlin, Heidelberg, New York, 1969).

[11] F. Jellett, 'On affine extensions of continuous functions defined on the extreme points of a Choquet simplex', *Quart. J. Math. Oxford Ser. (2)* **36** (1985), 71–73.

[12] A.S. Kechris, *Classical descriptive set theory* (Springer-Verlag, New York, 1995).

[13] K.Kuratowski, *Topology, Vol. I* (Academic Press, New York, 1966).

[14] A. Lazar, 'Spaces of affine continuous functions on simplexes', *Trans. Amer. Math. Soc.* **134** (1968), 503–525.

[15] J. Lukeš, J. Malý, I. Netuka, M. Smrčka and J. Spurný, 'On approximation of affine Baire-one functions', *Israel Jour. Math.* **134** (2003), 255–289.

[16] J. Lukeš, J. Malý and L. Zajíček, *Fine topology methods in real analysis and potential theory*, Lecture Notes in Math. **1189** (Springer–Verlag, Berlin, Heldelberg, New York, 1986).

[17] I. Netuka, 'The Dirichlet problem for harmonic functions', *Amer. Math. Monthly* **87** (1980), 621–628.

[18] M. Rogalski, 'Opérateurs de Lion, projecteurs boréliens et simplexes analytiques', *J. Funct. Anal.* **2** (1968), 458–488.

[19] C.E. Rogers and J.E. Jayne, *K-analytic sets* (Academic Press, London, New York, 1980).

[20] J. Saint Raymond, 'Fonctions convexes de première classe', *Math. Scand.* **54** (1984), 121–129.

[21] J. Spurný, 'On the Dirichlet problem for the functions of the first Baire class', *Comment. Math. Univ. Carolin.* **42** (2001), 721-728.

[22] J. Spurný, 'Representation of abstract affine functions', *Real. Anal. Exchange* **28** (2002/2003), 337–354.

[23] M. Talagrand, 'Choquet simplexes whose set of extreme points is $\mathcal{K}$–analytic', *Ann. Inst. Fourier (Grenoble)* **35** (1985), 195–206.

Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Praha 8
Czech Republic
e-mail: spurny@karlin.mff.cuni.cz