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WEAK SOLUTIONS FOR SEMI-MARTINGALES

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1. Introduction. The fundamental theorem of this paper is stated in Section 8. In this theorem, the stochastic differential equation $dX = a(X)dZ$ is studied when Z is a^*-dominated (cf. [**15**]) Banach space valued process and a is a predictable functional which is continuous for the uniform norm.

For such an equation, the existence of a "weak solution" is stated; actually, the notion of weak solution here considered is more precise than this one introduced by Strook and Varadhan (cf. [**30**], [**31**], [**23**]).

Namely, this weak solution is a probability, so-called "rule," defined on $(D^{\mathbf{H}} \times \Omega)$, $D^{\mathbf{H}}$ being the classical Skorohod space of all the cadlag sample paths and Ω is the initial space which Z is defined on: the marginal distribution of R on Ω is the given probability P on Ω. This concept of rule is defined in Section 3.

For a family of such rules a sequential compactness property is stated in Section 4: this is a generalization of the classical Prokhoroff' theorem. This compactness property is fulfilled for the family $\{V : V = \int Y\,dZ\}$ (for all the predictable processes Y bounded in norm by 1): this is stated in Section 7.

A sequence $(R(n))_{n>0}$ of rules is said to be *convergent in rule to* R if R is a rule and

$$E_R(f) = \lim_{n\to\infty} E_{R(n)}(f)$$

for some functions f which are defined on $(D^{\mathbf{H}} \times \Omega)$ and τ_s-continuous on $D^{\mathbf{H}}$ (i.e., continuous for the Skorohod topology). In Section 6, this same property is stated for τ_u-*continuous functions* f on $D^{\mathbf{H}}$ (i.e., for the functions which are continuous for the uniform norm) for some sequences of rules.

The fundamental step of our proof (see the proof of theorem in Section 8) is much different from the one introduced by Strook and Varadhan (cf. [**30**] or [**31**]) inasmuch as it does not need to solve first a "martingale problem".

2. Hypotheses and notations. For the classical definitions such as stochastic basis, adapted, cadlag, quadratic variation, etc. . . . we refer to [**15**].

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Through all this paper, we consider:

a probabilized stochastic basis $\mathbf{B}^I := (\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t\in T})$ with $T = [0, t_m]$, $t(m) := t_m < +\infty$; this basis is assumed to be complete and right continuous; it will be called the initial basis

a Banach space $\mathbf{K}$ and a finite dimensional vector space $\mathbf{H}$; the norm on $\mathbf{H}$ is assumed to be associated with a scalar product denoted by $\langle\ ,\ \rangle$; $\mathbf{K}'$ will denote the topological dual of $\mathbf{K}$

a complete subspace $\mathbf{L}$ of the vector space of the continuous linear operators from $\mathbf{K}$ into $\mathbf{H}$; the norm on $\mathbf{L}$ is such that

$$\|l(k)\|_{\mathbf{H}} \leqq \|l\|_{\mathbf{L}} \cdot \|k\|_{\mathbf{K}}$$

a positive cadlag increasing process Q adapted with respect to the initial basis such that $Q_{t(m)} = Q_{t(m)-}$

a $\mathbf{K}$-valued cadlag process Z adapted with respect to the initial basis $\mathbf{B}^I$ which is dominated by Q in the following sense:

(2.1) Z is $\mathbf{L}$-* dominated and $\mathbf{K}'$-* dominated by Q, i.e., for every $\mathbf{B}^I$-predictable bounded process Y with values in $\mathbf{L}$ or in $\mathbf{K}'$, one has:

$$E\left\{\sup_{t<u} \cdot \left\|\int_{]0,t]} Y_s dZ_s\right\|^2\right\} \leqq E\left\{Q_{u-} \int_{]0,u[} \|Y_s\|^2 dQ_s\right\}$$

(2.2) the variation of the quadratic variation $[Z]$ of Z is less than the variation of Q, i.e., for s, t elements of T, $s < t$, one has

$$[Z]_t - [Z]_s \leqq Q_t - Q_s \quad (P\text{-a.e.}).$$

The following notations are in force through the paper:

$D^{\mathbf{H}}$ is the Skorohod space of all the $\mathbf{H}$-valued cadlag functions defined on T

τ_s (resp. τ_u) is the Skorohod topology as defined in [**4**] (resp. the uniform topology): these topologies are defined on $D^{\mathbf{H}}$.

$\mathscr{D}_t^{\mathbf{H}}$ is the σ-algebra of subsets of $D^{\mathbf{H}}$ generated by the cylinders

$$\{f : f \in D^{\mathbf{H}}, f(s) \in B\}$$

with $s \leqq t$ and B a borelian subset of $\mathbf{H}$; let us define

$$\mathscr{D}^{\mathbf{H}} := \mathscr{D}^{\mathbf{H}}_{t(m)+} := \mathscr{D}^{\mathbf{H}}_{t(m)}$$

and, for $t < t(m)$,

$$\mathscr{D}_{t+}{}^{\mathbf{H}} := \bigcap_{s>t} \mathscr{D}_s{}^{\mathbf{H}}.$$

$\mathbf{B}^{\mathbf{H}} := (D^{\mathbf{H}} \times \Omega, \mathscr{D}^{\mathbf{H}} \otimes \mathscr{F}, (\mathscr{D}_{t+}{}^{\mathbf{H}} \otimes \mathscr{F}_t)_{t\in T})$ and this family will be called the *canonical basis* (for the $\mathbf{H}$-valued processes).

$G_t^{\mathbf{R}}$ (resp. $G_t^{\mathbf{H}}$, $G_t^{\mathbf{L}}$, $G_t^{\mathbf{K}'}$) is the set of all the real (resp. $\mathbf{H}$-valued, $\mathbf{L}$-valued, $\mathbf{K}'$-valued) (uniformly) bounded functions, defined on $(D^{\mathbf{H}} \times \Omega)$, $(\mathscr{D}_{t+}{}^{\mathbf{H}} \otimes \mathscr{F}_t)$-measurable and such that, for every element ω of Ω, the function $f \rightsquigarrow g(f, \quad)$ is τ_s-continuous on $D^{\mathbf{H}}$.

$G^{\mathbf{H}} : = G_{t(m)}{}^{\mathbf{H}}$.

$\mathscr{E}^{\mathbf{L}}$ (resp. $\mathscr{E}^{\mathbf{K}'}$) is the set of all the $\mathbf{L}$-valued (resp. $\mathbf{K}'$-valued) processes b, defined on the canonical basis $\mathbf{B}^{\mathbf{H}}$, uniformly bounded in norm by 1 and which can be written as follows:

$$b : = \sum_{i=1}^{n-1} g_i 1_{]s(i), s(i+1)]}$$

where $(s(i))_{1 \leqq i \leqq n}$ is an increasing family of elements of T and, for every i, g_i belongs to $G_{s(i)}{}^{\mathbf{L}}$ (resp. $G_{s(i)}{}^{\mathbf{K}'}$).

In other words, b is an $\mathbf{L}$-valued (resp. $\mathbf{K}'$-valued) $\mathbf{B}^{\mathbf{H}}$-predictable step process such that, for every element (ω, t) of $(\Omega \times T)$ the mapping $f \rightsquigarrow b(f, \omega, t)$ is τ_s-continuous (on $D^{\mathbf{H}}$).

It is easily seen that $\mathscr{E}^{\mathbf{L}}$ (or $\mathscr{E}^{\mathbf{K}'}$) generates the σ-algebra of predictable sets with respect to the canonical basis $\mathbf{B}^{\mathbf{H}}$.

3. Convergence in rule.

Definitions. R will be called a *rule* (on $(D^{\mathbf{H}} \times \Omega, \mathscr{D}^{\mathbf{H}} \otimes \mathscr{F}, P)$) if R is a probability defined on $(D^{\mathbf{H}} \times \Omega, \mathscr{D}^{\mathbf{H}} \otimes \mathscr{F})$ such that, for every element A of $\mathscr{F}$, $R(D^{\mathbf{H}} \times A) = P(A)$

Let $(R(n))_{n>0}$ be a sequence of rules: it will be said to be *convergent in rule* if there exists a rule R such that, for all the elements g of $G^{\mathbf{R}}$,

$$E_R(g) = \lim_{n\to\infty} E_{R(n)}(g)$$

(of course, $E_R(g)$ denotes the mathematical expectation of g with respect to the probability R).

Let X be an $\mathbf{H}$-valued cadlag process defined on the initial basis $\mathbf{B}^I$; let us consider the rule defined by

$$R(B \times F) : = P(X^{-1}(B) \cap F)$$

for every element $(B \times F)$ of $(\mathscr{D}^{\mathbf{H}} \times \mathscr{F})$, X being considered as a $D^{\mathbf{H}}$-valued function defined on Ω: this rule will be called the rule *associated with* X.

LEMMA. *Let $G_e{}^{\mathbf{R}}$ be the set of all the functions which belong to $G^{\mathbf{R}}$ and which are step functions in the following sense:*

$$(3.1) \quad g : = \sum_{i \in I} g_i{}^*(f) g_i{}^{**}(\omega)$$

where I is a finite set and, for any element i of I, $g_i{}^*$ is a real bounded function defined and continuous on $D^{\mathbf{H}}$ and $g_i{}^{**}$ belongs to $L_{\mathbf{R}}{}^{\infty}(\Omega, \mathscr{F}, P)$.

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Let $\mathscr{K}$ be a τ_s-compact subset of $D^{\mathbf{H}}$. Then, for every $\epsilon > 0$ and for every element g of $G^{\mathbf{R}}$, there exists an element g_e of $G_e^{\mathbf{R}}$ such that

$$P\{\omega : \sup_{f \in \mathscr{K}} |g_e(\omega, f) - g(\omega, f)| > \epsilon\} \leqq \epsilon.$$

Proof. Let $\epsilon > 0$ and g be an element of $G^{\mathbf{R}}$. The τ_s-separability of $D^{\mathbf{H}}$ and the compactness of $\mathscr{K}$ (in the polish space $D^{\mathbf{H}}$) imply (Ascoli-Arzelà theorem) that there exists a sequence $(h_n)_{n>0}$ of real functions, defined and τ_s-continuous on $\mathscr{K}$, which is dense, for the uniform topology, in the set of all the real functions, defined and τ_s-continuous on $\mathscr{K}$. Let us define:

$$A_n := \{\omega : \exists\, f \in \mathscr{K} \text{ such that } |g(f, \omega) - h_n(f, \omega)| > \epsilon\}$$
$$B(n) := (\Omega \backslash A_n) \cap (\textstyle\bigcap_{k<n} A_k).$$

$(B(n))_{n>0}$ is a partition of Ω; for every $n > 0$, let ω_n be an element of $B(n)$; let j be such that

$$P[\textstyle\bigcup_{n \leqq j} B(n)] \geqq 1 - \epsilon.$$

The lemma is proved by defining

$$g_e(f, \omega) := \sum_{n \leqq j} 1_{B(n)}(\omega) g(f, \omega_n).$$

4. Sequential compactness for the convergence in rule.

THEOREM. *Let $(R(n))_{n>0}$ be a sequence of rules. This sequence admits a subsequence which is convergent in rule to a rule R if, for every $\epsilon > 0$, there exists a τ_s-compact subset $\mathscr{K}$ of $D^{\mathbf{H}}$ such that, for every integer n,*

$$R(n)(\mathscr{K} \times \Omega) \geqq 1 - \epsilon.$$

Conversely, if the sequence $(R(n))_{n>0}$ converges in rule to R, the property above holds.

Proof. This theorem is clearly a generalization of Prokhoroff's theorem (cf. [**4**]). Noticing that the convergence in rule implies the convergence in distribution of the sequence $(R(n)(. \times \Omega))_{n>0}$, the second part of the theorem is a corollary of Prokhoroff's theorem.

Conversely, let $(R(n))_{n>0}$ be a sequence of rules such that, for every $\epsilon > 0$, there exists a τ_s-compact subset $\mathscr{K}$ of $D^{\mathbf{H}}$ with

$$R(n)(\mathscr{K} \times \Omega) \geqq 1 - \epsilon$$

(for every integer n). Let us define

$$R' := \sum_{n>0} 2^{-n} R(n);$$

let $\mathscr{F}^*$ be a separable σ-algebra of subsets of $\mathscr{F}$ such that, for every

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integer n, the Radon-Nikodym derivative $dR(n)/dR'$ is $(\mathscr{D}^{\mathbf{H}} \otimes \mathscr{F}^*)$-measurable.

Let $\mathscr{A}$ be a countable algebra which generates $\mathscr{F}^*$.

For every element A of $\mathscr{A}$, let $\bar{R}_A(n)$ be the positive measure defined on $(D^{\mathbf{H}}, \mathscr{D}^{\mathbf{H}})$ by

$$\bar{R}_A(n)(A') := R(n)(A' \times A).$$

The inequality $\bar{R}_A(n) \leqq \bar{R}_\Omega(n)$ implies that the sequence $(\bar{R}_A(n))_{n>0}$ is tight; then, for every element A of $\mathscr{A}$, Prokhoroff's theorem holds (cf. [**4**]) for the sequence $(\bar{R}_A(n))_{n>0}$. According to the countability of $\mathscr{A}$ and using the classical diagonal procedure, there exists a subsequence $(R(n(k)))_{k>0}$ extracted from the sequence $(R(n))_{n>0}$ such that, for every element A of $\mathscr{A}$, the sequence $(\bar{R}(n(k)))_{k>0}$ is weakly convergent to a positive measure $\bar{R}_A$ defined on $(D^{\mathbf{H}}, \mathscr{D}^{\mathbf{H}})$ and such that $\bar{R}_A(D^{\mathbf{H}}) = P(A)$.

For every element (A', A) of $(\mathscr{D}^{\mathbf{H}} \times \mathscr{A})$, let us define $R(A' \times A) := \bar{R}_A(A')$. One has

$$R(A' \times A) \leqq R(D^{\mathbf{H}} \times A) = P(A).$$

The function R is a positive function defined on $(\mathscr{D}^{\mathbf{H}} \times \mathscr{A})$ which is σ-additive separately on $\mathscr{D}^{\mathbf{H}}$ and on $\mathscr{A}$. This function admits a (unique) extension which is an additive function defined on the algebra generated by the "rectangles" $(A' \times A)$ with (A', A) element of $(\mathscr{D}^{\mathbf{H}} \times \mathscr{F}^*)$: let us call R this extension.

This extension has the following two properties:

(i) $R(A' \times A) \leqq P(A)$ for every element (A', A) of $(\mathscr{D}^{\mathbf{H}} \times \mathscr{F}^*)$;

(ii) for every $\epsilon > 0$, there exists a compact subset $\mathscr{K}$ of $D^{\mathbf{H}}$ such that

$$R((D^{\mathbf{H}} \backslash \mathscr{K}) \times A) \leqq \epsilon$$

(for every element A of $\mathscr{F}^*$). Then, it can be proved, exactly as in 3.5 of [**18**] or in 8.4 of [**15**] that R is σ-additive: that implies that R admits a (unique) extension which is a σ-additive function on the σ-algebra $(\mathscr{D}^{\mathbf{H}} \otimes \mathscr{F}^*)$. Let us call R this extension.

For every element A of $\mathscr{F}$, let A^* be an element of $\mathscr{F}^*$ such that 1_{A^*} is the orthogonal projection of 1_A, considered as an element of $L^2(\Omega, \mathscr{F}, P)$, onto $L^2(\Omega, \mathscr{F}^*, P)$. For every element A' of $\mathscr{D}^{\mathbf{H}}$, let us define $R(A' \times A) = R(A' \times A^*)$. Then, for every real bounded function g defined and τ_s-continuous on $D^{\mathbf{H}}$ and for every element A of $\mathscr{F}$, one has:

$$E_R(g\, 1_A) = E_R(g\, 1_{A^*}) \text{ (by definition of } R)$$

and

$$E_{R(n)}(g\, 1_A) = E_{R(n)}(g\, 1_{A^*}).$$

Indeed, $dR(n)/dR'$ is $(\mathscr{D}^{\mathbf{H}} \times \mathscr{F}^*)$-measurable and

$$E_{R'}(1_A | \mathscr{D}^{\mathbf{H}} \otimes \mathscr{F}^*) = 1_{A^*}.$$

Moreover

$$\lim_k E_{R(n(k))}(g\, 1_{A^*}) = E_R(g\, 1_{A^*}).$$

These three equalities imply

$$E_R(g\, 1_A) = \lim_k E_{R(n(k))}(g\, 1_A).$$

Now, according to the lemma of Section 3, the subsequence $(R(n(k)))_{k>0}$ converges in rule to the rule R.

5. τ_u-continuous functions. Let us recall that the topologies τ_u and τ_s have been defined in Section 2.

LEMMA. *Let A be an element of $(\mathscr{D} \otimes \mathscr{F})$ such that, for every element ω of Ω, the set $A(\omega)$: $= \{f : (f, \omega) \in A\}$ is a τ_s-compact subset of $D^{\mathbf{H}}$. Let ϕ be the function defined by:*

$$\phi(f, \omega) := \textit{Skorohod's distance from } f \textit{ to } A(\omega).$$

Then ϕ is $(\mathscr{D}^{\mathbf{H}} \otimes \mathscr{F})$ measurable.

Proof. Throughout this proof δ will denote the Skorohod's distance,

$$\delta' := \frac{\delta}{1+\delta} \quad \text{and} \quad \phi' := \frac{\phi}{1+\phi}.$$

Let x be an element which does not belong to $D^{\mathbf{H}}$ and let us denote $D_x^{\mathbf{H}} := D^{\mathbf{H}} \cup \{x\}$; $\mathscr{D}_x^{\mathbf{H}}$ is the σ-algebra of subsets of $D_x^{\mathbf{H}}$ generated by $\mathscr{D}^{\mathbf{H}}$. For every integer k, let $(h_{k,n})_{n>0}$ be the sequence of functions and $(F_{k,n})_{n>0}$ be the associated sequence of sets defined recurrently as follows:

$$F_{k,0} := A$$

$h_{k,n+1}$ is a measurable mapping from $(\Omega, \mathscr{F})$ into $(D_x^{\mathbf{H}}, \mathscr{D}_x^{\mathbf{H}})$ such that:

$$h_{k,n+1}(\omega) \in F_{k,n} \text{ if and only if } \{f : (f, \omega) \in F_{k,n}\} \neq \phi$$

$$h_{k,n+1}(\omega) = x \text{ if and only if } \{f : (f, \omega) \in F_{k,n}\} = \phi$$

(such a measurable mapping exists according to section theorem: see, for example, [**7**])

$$F_{k,n+1} := \{(f, \omega) : (f, \omega) \in F_{k,n+1} \text{ and } \delta'(f, h_{k,n+1}(\omega)) \geqq 1/k\}.$$

For every element f of $D^{\mathbf{H}}$, let us define $\delta'(f, x) := 1$. It is easily seen that

$$\phi'(f, \omega) := \inf_{k,n} \delta'(f, h_{k,n}(\omega)).$$

Thus ϕ' and ϕ are $(\mathscr{D}^{\mathbf{H}} \otimes \mathscr{F})$-measurable.

THEOREM. *Let g be a (uniformly) bounded function, with values in a finite dimensional vector space $\mathbf{J}$, defined on $(D^{\mathbf{H}} \times \Omega)$, $(\mathscr{D}^{\mathbf{H}} \otimes \mathscr{F})$-*

measurable and such that, for every element ω of Ω, the mapping $f \rightsquigarrow g(f, \omega)$ is τ_u-continuous.

1. *Let $\mathscr{K}$ be an element of $(\mathscr{D}^{\mathbf{H}} \otimes \mathscr{F})$ such that, for every element ω of Ω $\mathscr{K}(\omega) := \{ f : (f, \omega) \in \mathscr{K} \}$ is a τ_u-compact subset of $D^{\mathbf{H}}$. Then, for every $\epsilon > 0$, there exists a $\mathbf{J}$-valued $(D^{\mathbf{H}} \otimes \mathscr{F})$-measurable function g_s (defined on $(D^{\mathbf{H}} \times \Omega)$) such that, for every element ω of Ω, $f \rightsquigarrow g_s(f, \omega)$ is τ_s-continuous and such that:*

$$\sup_{(f,\omega) \in \mathscr{K}} \|g(f, \omega) - g_s(f, \omega)\| < \epsilon.$$

2. *Let $(R(n))_{n>0}$ be a sequence of rules which converges in rule to the rule R. Let us assume that, for every $\epsilon > 0$, there exists an element $\mathscr{K}_\epsilon$ of $(\mathscr{D}^{\mathbf{H}} \otimes \mathscr{F})$ such that, for all the elements ω of Ω,*

$$\mathscr{K}_\epsilon(\omega) := \{ f : (f, \omega) \in \mathscr{K}_\epsilon \}$$

is a τ_u-compact subset of $D^{\mathbf{H}}$, and such that, for every integer n,

$$R(n)(\mathscr{K}_\epsilon) \geqq 1 - \epsilon.$$

Then

$$\lim_n E_{R(n)}(g) = E_R(g).$$

Proof. 1. It is sufficient to consider the case in which g is a real function such that $0 \leqq g \leqq 1$. Let $\epsilon < 0$ and n be such that $n\epsilon \geqq 1 \geqq (n - 1)\epsilon$. For every integer k with $k \leqq n$, let us define:

$$A(k) := \{(f, \omega) : k\epsilon \leqq g(f, \omega)\} \cap \mathscr{K}$$

$$A'(k) := \{(f, \omega) : k\epsilon \geqq g(f, \omega)\} \cap \mathscr{K}$$

$$A(k)(\omega) := \{ f : (f, \omega) \in A(k)\}$$

$$A'(k)(\omega) := \{ f : (f, \omega) \in A'(k)\}.$$

For all the elements ω of Ω, $A(k + 1)(\omega)$ and $A'(k)(\omega)$ are τ_u-compact (and τ_s-compact) disjoint subsets of $D^{\mathbf{H}}$.

For every integer k, let ϕ_k and ϕ'_k be the functions defined by:

$$\phi_k(f, \omega) := \text{Skorohod's distance from } f \text{ to } A(k)(\omega)$$

$$\phi_k'(f, \omega) := \text{Skorohod's distance from } f \text{ to } A'(k)(\omega).$$

These functions ϕ_k and ϕ_k' are $(\mathscr{F} \otimes \mathscr{D}^{\mathbf{H}})$-measurable (see the previous lemma). Let g_s be the function defined by:

$$g_s := \sup_{k<n} \{(k + 1) \wedge (\phi_k'/\phi_{k+1})\}.$$

It is easily seen that g_s fulfills the properties of the proposition.

2. Let $\epsilon > 0, \mathscr{K}_\epsilon$ associated and g_s as constructed in 1 above. Let

$$\alpha := \sup_{f,\omega} \|g(f, \omega)\|.$$

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We have: 1

$$\|(E_R - E_{R(n)})(g)\| \leqq 2\alpha\epsilon + \|(E_R - E_{R(n)})(g_s)\|$$

and this last quantity goes to zero when n goes to infinity (convergence in rule).

6. A sequentially compact family.

THEOREM. *Using the hypotheses and notation of Section 2, let Q' be the positive increasing $\mathbf{B}^I$-adapted process defined by $Q' := \alpha Q$. Let $\mathscr{C}(Q')$ be the set of all the $\mathbf{H}$-valued cadlag processes such that:*

(i) $X_0 = 0$ *and X is $\mathbf{B}^I$-adapted.*

(ii) *The quadratic variation $[X]$ of X is such that*

$$[X]_t - [X]_s \leqq Q_t' - Q_s' \text{ for } s < t.$$

(iii) *For every $\mathbf{B}^I$-stopping time and for every $\mathbf{H}$-valued (uniformly) bounded predictable process Y, we have:*

$$E\left\{\sup_{t<u}\left\|\int_{]0,t]} \langle Y, dX\rangle^2\right\|\right\} \leqq E\left\{Q_{u-}'\int_{]0,u[} \|Y_t\|^2 dQ_t\right\}.$$

(iv) *For every pair (u, v) of $\mathbf{B}^I$-stopping times with $u \leqq v$, we have:*

$$E\{\sup_{t-v} \|X_t - X_u\|^2\} \leqq E\{Q_{v-}'(Q_{v-}' - Q_u')\, 1_{[u<v]}\}.$$

Let $(q_j)_{j>0}$ be an increasing sequence of positive numbers such that, for every integer j,

$$P[Q_{t(m)}' \geqq q_j] \leqq 1/j^2$$

and let $(v_j)_{j>0}$ be the associated sequence of $\mathbf{B}^I$-stopping times defined by $v_j := \inf\{t : Q_t' > q_j\}$.

For every pair (j, k) of integers, let $(w(n, j, k))_{n>0}$ be the sequence of $\mathbf{B}^I$-stopping times defined recursively as follows:

$$w(1, j, k) := v_{j-1}$$
$$w(n+1, j, k) := v_j \wedge \inf\{t : Q_t' - Q_{w'(n,j,k)} > q_j/k^3\}$$

(we remark that $w(k^3, j; k) = v_j$).
Let us define:

$$\lambda_{j,k} := \left[\frac{1}{k} j^2 q_j^2 (2 + 8q_j^2)\right]^{1/4}.$$

For every integer m, let $\mathscr{K}_m'$ be the set of all the elements (f, m) of $(D^{\mathbf{H}} \times \Omega)$ for which the following two properties are fulfilled:

(i)′ *for every triple of integers (n, j, k), with $n > 0$, $j \leqq m$ and $k \geqq m$,* *and for every element t of $]w(n, j, k)(\omega), w(n+1, j, k)(\omega)]$, we have*

$$\| f(t) - f(w(n, j, k)(\omega)) \| \leqq \lambda_{j,k}.$$

(ii)′ $\sup_{t < vm(\omega)} \| f(t) \| \leqq mq_m$.

Let

$$\epsilon_m := \sum_{k \geqq m} \frac{1}{k^2}.$$

Then, for every element X of $\mathscr{C}(Q')$, one has:

$$P(\{\omega : (X(\omega), \omega) \in \mathscr{K}_m'\}) \geqq 1 - 3\epsilon_m.$$

Moreover, there exists a τ_s-compact subset $\mathscr{K}_m$ of $D^{\mathbf{H}}$ such that:

$$P(\{\omega : \exists f \notin \mathscr{K}_m \text{ and } (f, \omega) \in \mathscr{K}_m'\}) \leqq 3\epsilon_m.$$

This implies (for any element X of $\mathscr{C}(Q')$):

$$P(X^{-1}(\mathscr{K}_m)) \geqq 1 - 6\epsilon_m.$$

Proof. Let us define $\mathscr{C} := \mathscr{C}(Q')$ and $x^2 := \langle x, x \rangle$. Let X be an element of $\mathscr{C}$; let (u, v) be a pair of stopping times with $u \leqq v$. Let β be defined by:

$$\beta := E\{\sup_{u<t<v} \|X_t - X_u\|^4\}.$$

According to property (iii), the following Ito formula holds (see [15]):

$$(\mathbf{\Lambda}_t - X_u)^2 = 2\int_{]u,t]} \langle X_{s-} - X_u, dX_s \rangle + [X]_t - [X]_u$$

which implies:

$$\beta \leqq 2E\left\{\sup_{t<v} \left\| \int_{]u,t]} 2\langle X_{s-} - X_u, dX_s \rangle \right\|^2 \right\}$$
$$+ 2E\{([X]_{v-} - [X]_u)^2 1_{[u<v]}\}$$
$$\leqq 8E\left\{Q_{v-}' \int_{]u,v[} (X_{s-} - X_u)^2 dQ_s' \right\} + 2E\{(Q_{v-}' - Q_u')^2 1_{[u<v]}\}.$$

Let $\alpha > 0$ and $q \geqq 0$; let us assume that, for every element ω of Ω, $Q_{v-}' \leqq q$ and $Q_{v-}' - Q_u' \leqq \alpha$; in such a situation, we have:

$$\beta \leqq 8\alpha q E\{\sup_{u<t<v} (X_t - X_u)^2\} + 2\alpha^2.$$

Moreover, property (iv) implies:

$$E\{\sup_{u<t<v} (X_t - X_u)^2\} \leqq E\{Q_{v-}'(Q_{v-}' - Q_u') 1_{[u<v]}\}$$

and, finally,

$$\beta \leqq \alpha^2(2 + 8q^2).$$

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We remark that the important fact in this inequality is the fact that β/α goes to zero when α goes to zero.

Let X be an element of $\mathscr{C}$ and (j, k) be a pair of integers; through this part of the proof, this pair (j, k) is fixed; thus, let us put $u(n) := w(n, j, k)$.

According to the definition of $w(n + 1, j, k)$ and the above, we have:

$$E\{\sup_{u(n)<t<u(n+1)} \|X_t - X_{u(n)}\|^4\} \leqq q_j^2(2 + 8q_j^2)/k^6$$

which implies

$$P\{\sup_{u(n)<t<u(n+1)} \|X_t - X_{u(n)}\| > \lambda_{j,k}\} \leqq 1/k^5j^2.$$

Let us define the set $B_{j,k}$ as follows:

$$B_{j,k} := \{\omega: \exists\ n \leqq k^3 \text{ such that } \sup_{u(n)<t<u(n+1)} \|X_t - X_{u(n)}\| > \lambda_{j,k}\}.$$

Remembering that $u(k^3) = v_j$, we have:

$$B_{j,k} = \{\omega: \exists\ n > 0 \text{ such that } \sup_{u(n)<t<u(n+1)} \|X_t - X_{u(n)}\| > \lambda_{j,k}\}.$$

The previous inequality implies that $P(B_{j,k}) \leqq 1/j^2k^2$.

Through the following, X is an element (fixed) of $\mathscr{C}$ but the pair (j, k) is not fixed. Let us define:

$$C_m' := \{\omega: \sup_{t<v_m(\omega)} \|X_t(\omega)\| > mq_m\}.$$

Property (iv) implies:

$$E\{\sup_{t<v_m} \|X_t\|^2\} \leqq E(Q_{v_m-}) \leqq q_m^2$$

and

$$P(C_m') \leqq 1/m^2.$$

Let ω be an element of Ω such that $(X(\omega), \omega)$ does not belong to $\mathscr{K}_m'$; this implies that either

$$\omega \in C_m' \text{ or } \omega \in \bigcup_{j>0} \bigcup_{k>m} B_{j,k}.$$

Thus, the probability of such an event is less than

$$\frac{1}{m^2} + \sum_{j>0} \sum_{k \geqq m} \frac{1}{j^2k^2} \leqq 3\epsilon_m$$

which proves the first inequality of the theorem.

Now, let us define $\mathscr{K}_m$. For every triple (n, j, k) of integers with $n \leqq k^3$, let $\rho(n, j, k)$ be a positive number such that

$$P(\{w(n + 1, j, k) - w(n, j, k) \leqq \rho(n, j, k) \text{ and } w(n, j, k) < v_j\}) \leqq 1/j^2k^5.$$

Let us define:

$$\delta_{m,k} := \inf \rho(n, j, k)$$

this infimum being taken for all the pairs (n, j) with $n \leqq k^3$ and $j \leqq m$. Let $\mathscr{K}_m$ be the set of all the elements f of $D^{\mathbf{H}}$ such that

$$\sup_t \|f(t)\| \leqq mq_m$$

and, for every $k \geqq m$,

$$w_f'(\delta_{m,k}) \leqq \lambda_{m,k}$$

where $w_f'(\delta)$ is defined as in 14.6 p. 110 of [**4**] (right modulus of continuity for the function f).

This set $\mathscr{K}_m$ is a τ_s-compact subset of $D^{\mathbf{H}}$ (see [**4**], Theorem 14.3). Let us define:

$$C_m: = \{\omega : v_m(\omega) < t_m\} \text{ and}$$
$$A_{j,k}: = \{\omega: \exists\ n \leqq k^3 \text{ such that } w(n, j, k) < v_j \text{ and}$$
$$(w(n+1, j, k) - w(n, j, k))(\omega) < \rho_{n,j,k}\}.$$

One has $P(C_m) \leqq 1/m^2$ (according to the definition of v_j) and $P(A_{j,k}) \leqq 1/j^2k^2$ (according to the definition of $\rho_{n,j,k}$). Now, if (f, ω) belongs to $\mathscr{K}_m'$ and f does not belong to $\mathscr{K}_m$, one has either $\omega \in C_m$ or

$$\omega \in \bigcup_{j \leqq m} \bigcup_{k \geqq m} A_{j,k}.$$

Thus, the probability of such a situation is less than

$$\frac{1}{m^2} + \sum_{j \leqq m} \sum_{k \geqq m} \frac{1}{j^2k^2} \leqq 3\epsilon_m$$

which ends the proof of the theorem.

Remark. Let Y be an $\mathbf{L}$-valued process, uniformly bounded in norm by α and $\mathbf{B}^I$-predictable; let X be the process defined by

$$X_t : = \int_{]0,t]}^{\lrcorner} Y\, dZ.$$

Then, it is easily seen that X belongs to $\mathscr{C}(Q')$ as defined in the previous theorem (see properties 2.1 and 2.2).

7. Hypotheses and notations on a**.** Let us recall that the aim of this paper is to prove the theorem of Section 8, i.e., to prove the existence of a "weak" solution X of the stochastic differential equation:

$$dX_t(\omega) = a(X, \omega, t)\, dZ_t(\omega).$$

The hypotheses on the functional a are as follows:

1 a is an $\mathbf{L}$-valued $\mathbf{B}^{\mathbf{H}}$-predictable process

2 for every element (f, ω, t) of $(D^{\mathbf{H}} \times \Omega \times T)$, $\|a(f, \omega, t)\| \leqq \alpha$

3 for every element (ω, t) of $(\Omega \times T)$, the mapping $f \to a(f, \omega, t)$ is τ_u-continuous.

Let us remark that property 2) implies the following property (see Proposition 6.4 in [**15**]):

4) let (ω, t) be an element of $(\Omega \times T)$ and let (f, f') be a pair of elements of $D^{\mathbf{H}}$ such that $f(s) = f'(s)$ for $s < t$; then, $a(f, \omega, t) = a(f', \omega, t]$: in other words a depends only on the strict past of the function f.

Then a can be approximated as follows:

PROPOSITION. *Let a be a process fulfilling hypotheses 1, 2 and 3 above. Let $w(n, j, k)$ be the family of stopping times as defined in Section 6. For every integer $k > 0$ and for every element (f, ω) of $(D^{\mathbf{H}} \times \Omega)$, let us define:*

$$f_k(\omega) := \sum_{n,j} f(w(n, j, k)(\omega)) 1_{[w(n,j,k), w(n+1,j,k)[}(\omega)$$

and

$$a_k(f, \omega, t) := a(f_k(\omega), \omega, t).$$

Then, the sequence of processes $(a_k)_{k>0}$ is a sequence of $\mathbf{B}^{\mathbf{H}}$-predictable processes, uniformly bounded by α, τ_u-continuous and "constant by pieces" with respect to the first variable; moreover, this sequence converges to a in the following sense:

(7.1) *for every element (m, ω, t) of $(\mathbf{N} \times \Omega \times T)$ with $t < v_m(\omega)$*

$$\lim_{k\to\infty} \sup_{f \in \mathscr{K}_m'(\omega)} \|a_k(f, \omega, t) - a(f, \omega, t)\|\} = 0$$

where v_m and $\mathscr{K}_m'$ are defined as in Section 6 and

$$\mathscr{K}_m'(\omega) := \{f : (f, \omega) \in \mathscr{K}_m'\}.$$

Proof. m, ω and t are fixed through this proof with $j < m$, $k \geqq m$ and $t < v_m(\omega)$. When (f, ω) belongs to $\mathscr{K}_m'$, we have

$$\|f(t) - f(w(n, j, k))\| \leqq \lambda_{j,k}$$

for $w(n, j, k) \leqq t < w(n + 1, j, k)$; this implies

$$\sup_t \|f_k(t) - f(t)\| \leqq \lambda_{m,k}.$$

Let us define:

$$S_m(\omega) := \{f : (f, \omega) \in \mathscr{K}_m' \text{ and } f = f\, 1_{[0, v_m(\omega)[}\}.$$

The boundness property above shows that $S_m(\omega)$ is a τ_u-compact subset of $D^{\mathbf{H}}$; thus, restricted to $S_m(\omega)$, the mapping $f \rightsquigarrow a(f, \omega, t)$ is uniformly continuous and this proves the proposition (indeed $\lim_{k\to\infty} \lambda_{m,k} = 0$ and (f_k, ω) belongs to $\mathscr{K}_m'$).

8. Main theorem.

THEOREM. *The hypotheses and the notation introduced in Sections 2 and*

7 are in force. Moreover, for every element (f, ω, t) *of* $(D^{\mathbf{H}} \times \Omega \times T)$, *let us define* $\bar{Z}_t(f, \omega) := Z_t(\omega)$, $\bar{X}_t(f, \omega) := f(t)$ *(canonical process) and* $\bar{Q}_t(f, \omega) := Q_t(\omega)$.

Then there exists a probability R on $(D^{\mathbf{H}} \times \Omega, \mathscr{D}^{\mathbf{H}} \otimes \mathscr{F})$ *such that:*

(i) *for every element A of* $\mathscr{F}$, $R(D^{\mathbf{H}} \times A) = P(A)$ *(i.e., R is a rule as defined in Section* 3)

(ii) *there exists a sequence* $(X^n)_{n>0}$ *of* **H**-*valued* $\mathbf{B}^I$ *adapted processes such that, if, for every integer n,* $R(n)$ *is the rule associated with* X^n *(as defined in Section* 3), *R is the limit in rule of the sequence* $(R(n))_{n>0}$

(iii) *for the probability R, Z is* **L**-**dominated and* $\mathbf{K}'$-**dominated by* $\bar{Q}$ *in the following sense: for every* $\mathbf{B}^I$-*stopping time u and for every* $\mathbf{B}^{\mathbf{H}}$-*predictable (uniformly) bounded process Y with values in* **L** *or in* $\mathbf{K}'$, *we have:*

$$E_R\left\{\sup_{t<u} \cdot \left\|\int_{]0,t]} Y_s\, d\bar{Z}_s\right\|^2\right\} \leqq E_R\left\{\bar{Q}_{u-} \int_{]0,u[} \|Y_s\|^2 d\bar{Q}_s\right\}$$

(iv) *for the probability R, we have:*

$$\bar{X}_t = \int_{]0,t]} a(\bar{X}, \omega, s) d\bar{Z}_s$$

this integral being a stochastic integral which is well defined according to property (iii) *(see* [**15**]).

In other words, $\bar{X}$ *is a "weak solution" for the stochastic differential equation* $dX = a(X)\, dZ$ *in a sense more precise than the one introduced by Strook and Varadhan (see* [**30**], [**31**] *and* [**23**]).

In general, such a probability R is not unique.

Proof. Let $(a_k)_{k>0}$ be the sequence of $\mathbf{B}^{\mathbf{H}}$-predictable processes defined in Section 7. For every element ω of Ω, $a_k(f, \omega, t)$ is "constant by pieces" and depends only on the strict past of f; thus, there is a (unique) process X^k which is a (strong) solution of the (stochastic) differential equation

$$dX_t^k = a(X^k, ., t)\, dZ_t;$$

this process X^k is **H**-valued, cadlag and $\mathbf{B}^I$-adapted.

For every integer k, let $R(k)$ be the rule associated with X^k (as defined in Section 3). The problem is to state that $(R(k))_{k>0}$ admits a subsequence which converges in rule to a rule R and to verify that this rule fulfills properties (i), (ii), (iii) and (iv).

According to its construction, X^k is of the following form:

$$X^k = \int Y\, dZ \quad \text{with} \quad \sup_{\omega, t} \cdot \|Y_t(\omega)\| \leqq \alpha;$$

thus X^k belongs to $\mathscr{C}(Q')$ with $Q' := \alpha Q$ (see the remark in Section 6); then the theorem of Section 6 is in force: in particular, for every $\epsilon > 0$, there exists a τ_s-compact subset $\mathscr{K}_m$ of $D^{\mathbf{H}}$ such that, for every integer $k > 0$,

$$P[(X^k)^{-1}\mathscr{K}_m)] \geqq 1 - \epsilon.$$

Now, according to the theorem in Section 4, there is a subsequence of the sequence $(R(k))_{k>0}$ which converges in rule to a rule R. Let us assume that the sequence $(R(k))_{k>0}$ itself is convergent in rule to the rule R.

Let b be an element of $\mathscr{E}^{\mathbf{L}}$ or of $\mathscr{E}^{\mathbf{K}'}$ and v be a $\mathbf{B}^I$-stopping time. For every integer k, one has:

$$(8.1)\quad E_{R(k)}\left\{\sup_{t<v} \cdot \left\|\int_{]0,t]} b(\bar{X}, ., t)d\bar{Z}_t\right\|^2\right\} \leqq E_{R(k)}\left\{\bar{Q}_{v-} \int_{]0,v[} \|b(\bar{X})\|^2 d\bar{Q}\right\}$$

(by property 2.1). On the one hand, the convergence in rule of the sequence $(R(k))_{k>0}$ to the rule R implies (remembering that b is a step process and the proposition in Section 5):

$$(8.2)\quad E_R\left\{\sup_{t<v} \cdot \left\|\int_{]0,t]} b(\bar{X})d\bar{Z}\right\|^2\right\} \leqq E_R\left\{\bar{Q}_{v-} \int_{]0,v[} \|b(\bar{X})\|^2 d\bar{Q}\right\}.$$

On the other hand, these same inequalities (8.1) and (8.2) are still valid for every uniformly bounded, $\mathbf{L}$-valued or $\mathbf{K}'$-valued, $\mathbf{B}^{\mathbf{H}}$-predictable process b (see [**15**]).

Through the sequel of the proof q is a positive number, v is a $\mathbf{B}^I$-stopping time such that $\sup_\omega Q_{v-} \leqq q$, $V =]0, v[$ and ϵ is a positive number. Let m be an integer such that $\epsilon_m \leqq \epsilon/\alpha$ where ϵ_m is defined as in Section 6. Let us recall that

$$\mathscr{K}_m'(\omega) := \{f : (f, \omega) \in \mathscr{K}_m'\}.$$

Let b be a $\mathbf{B}^{\mathbf{H}}$-predictable process bounded (in norm) by 2α. Let us define:

$$(8.3)\quad b_m(\omega, t) := \sup \|b(f, \omega, t)\|$$

this supremum being taken for $f \in \mathscr{K}_m'(\omega)$ and

$$(8.4)\quad \mu := \left[4q^2\alpha^2\epsilon_m + qE_P\left\{\int_V b_m{}^2(\omega, t)dQ_t(\omega)\right\}\right]^{1/2}.$$

Remembering that $R(k)(\mathscr{K}_m') \geqq 1 - \epsilon_m$ and $R(\mathscr{K}_m') \geqq 1 - \epsilon_m$, inequalities (8.1) and (8.2) imply:

$$(8.5)\quad E_{R(k)}\left\{\sup_{t<v} \cdot \left\|\int_{]0,t]} b(\bar{X})d\bar{Z}\right\|^2\right\} \leqq \mu^2$$

and

(8.6) $$E_R\left\{\sup_{t<v}\left\|\int_{]0,t]} b(\bar{X})d\bar{Z}\right\|^2\right\} \leqq \mu^2.$$

Let g be an element of $G^{\mathbf{H}}$ bounded by 1 in norm; the previous inequalities and the Cauchy-Schwarz inequality imply:

(8.7) $$\left|E_{R(k)}\left\{\left\langle g, \int_V b(\bar{X})d\bar{Z}\right\rangle\right\}\right| \leqq \mu$$

and

(8.8) $$\left|E_R\left\{\left\langle g, \int_V b(\bar{X})d\bar{Z}\right\rangle\right\}\right| \leqq \mu.$$

Let us define:

$$\beta(1, n, k) := E_R\left\{\left\langle g, \int_V [a(\bar{X}) - a_n(\bar{X})d\bar{Z}]\right\rangle\right\}$$

$$\beta(2, n, k) := [E_R - E_{R(n+k)}]\left\{\left\langle g, \int_V a_n(\bar{X})d\bar{Z}\right\rangle\right\}$$

$$\beta(3, n, k) := E_{R(n+k)}\left\{\left\langle g, \int_V (a_n - a_{n+k})(\bar{X})d\bar{Z}\right\rangle\right\}$$

$$\beta(4, n, k) := [E_{R(n+k)} - E_R]\{\langle g, \bar{X}_{v-}\rangle\}$$

$$\gamma(v, g) := E_R\left\{\left\langle g, \int_V a(\bar{X})d\bar{Z} - \bar{X}_{v-}\right\rangle\right\}.$$

According to the definition of X^{n+k}, one has:

$$E_{R(n+k)}\left\{\left\langle g, \bar{X}_{v-} - \int_V a_{n+k}(\bar{X})d\bar{Z}\right\rangle\right\} = 0$$

which implies:

$$\sum_{j=1}^{4} \beta(j, n, k) = \gamma(v, g).$$

Now, we have to prove that, for every integer j,

$$\lim_{n,k} \beta(j, n, k) \leqq 2\epsilon$$

which implies $\gamma(v, g) = 0$.

Property (7.1) and inequality (8.8) imply

$$\lim_n \beta(1, n, k) \leqq \epsilon;$$

in the same way, property (7.1) and inequality (8.7) imply that there is an integer n' such that, for every integer k, we have:

$$\beta(3, n', k) \leqq 2\epsilon.$$

Moreover, n' being fixed, the convergence in rule implies that

$$\lim_{k\to\infty} \beta(2, n', k) = 0$$

(let us recall that, for n' fixed, $a_{n'}$ is a "step" process and the theorem in Section 5 applies, $\mathscr{K}_m{}'(\omega)$ being a τ_u-compact subset of $D^{\mathbf{H}}$).

At last, the convergence in rule implies

$$\lim_{n,k} \beta(4, n, k) = 0$$

and we get $\gamma(v, g) = 0$.

The set $G^{\mathbf{H}}$ is dense in $L^{\mathbf{H}\infty}(D^{\mathbf{H}} \times \Omega, \mathscr{D}^{\mathbf{H}} \otimes \mathscr{F}, R)$ (for the topology $\sigma(L^\infty, L^1)$); thus we have

$$\bar{X}_{v-} = \int_V a(\bar{X})d\bar{Z} \quad R\text{-a.e.}$$

which implies

$$\bar{X} = \int a(\bar{X})d\bar{Z}$$

up to R-equivalence.

9. Comments. The notion of convergence in rule as defined in this paper was introduced in [**20**]. Some analoguous notions were studied before: See, for example, [**25**], [**26**], [**27**], [**3**], [**17**], etc.

The use of this notion in stochastic differential equations appears in [**20**] and [**21**].

The notion of weak solutions for stochastic differential equations is due to Strook and Varadhan and was studied by several authors: See, for example, [**30**], [**31**], [**12**], [**23**], [**13**], [**9**], etc.

The fundamental steps of our proof originate from [**22**], except the theorem in Section 5 which originates from [**10**]. Our proof is new inasmuch as it does not need to solve first a "martingale problem".

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