L log L CRITERION FOR A CLASS OF SUPERDIFFUSIONS

RONG-LI LIU * ** AND
YAN-XIA REN,* *** Perking University
RENMING SONG,** **** University of Illinois

Abstract

In Lyons, Pemantle and Peres (1995), a martingale change of measure method was developed in order to give an alternative proof of the Kesten–Stigum L log L theorem for single-type branching processes. Later, this method was extended to prove the L log L theorem for multiple- and general multiple-type branching processes in Biggins and Kyprianou (2004), Kurtz et al. (1997), and Lyons (1997). In this paper we extend this method to a class of superdiffusions and establish a Kesten–Stigum L log L type theorem for superdiffusions. One of our main tools is a spine decomposition of superdiffusions, which is a modification of the one in Englander and Kyprianou (2004).

Keywords: Diffusions; superdiffusions; Poisson point process; Kesten–Stigum theorem; martingale; martingale change of measure

2000 Mathematics Subject Classification: Primary 60J80; 60F15
Secondary 60J25

1. Introduction and main result

Suppose that \( \{Z_n, n \geq 1\} \) is a Galton–Watson branching process with each particle having probability \( p_n \) of giving birth to \( n \) children. Let \( L \) stand for a random variable with this offspring distribution. Let \( m := \sum_{n=1}^{\infty} n p_n \) be the mean number of children per particle. Then \( Z_n/m^n \) is a nonnegative martingale. Let \( W \) be the limit of \( Z_n/m^n \) as \( n \to \infty \). Kesten and Stigum [8] proved that if \( 1 < m < \infty \) (that is, in the supercritical case) then \( W \) is nondegenerate (i.e. not almost surely zero) if and only if

\[
E(L \log^+ L) = \sum_{n=1}^{\infty} p_n n \log n < \infty.
\]

This result is usually referred to as the Kesten–Stigum L log L theorem. In [1], Asmussen and Hering generalized this result to the case of branching Markov processes under some conditions.

Lyons et al. [14] developed a martingale change of measure method in order to give an alternative proof of the Kesten–Stigum L log L theorem for single-type branching processes.
Later, this method was extended to prove the $L \log L$ theorem for multiple- and general multiple-type branching processes in [2], [12], and [13].

In this paper we will extend this method to a class of superdiffusions and establish an $L \log L$ criterion for superdiffusions. To state our main result, we need to introduce the setup we are going to work with first.

Let $a_{ij}$, $i, j = 1, \ldots, d$, be bounded functions in $C^1(\mathbb{R}^d)$ such that all their first partial derivatives are bounded. We assume that the matrix $(a_{ij})$ is symmetric and satisfies

$$0 < a|\nu|^2 \leq \sum_{i,j} a_{ij} \nu_i \nu_j \quad \text{for all } x \in \mathbb{R}^d \text{ and } \nu \in \mathbb{R}^d$$

for some positive constant $a$. Let $b_i$, $i = 1, \ldots, d$, be bounded Borel functions on $\mathbb{R}^d$.

We will use $(Y, \Pi_{\lambda}, x \in \mathbb{R}^d)$ to denote a diffusion process on $\mathbb{R}^d$ corresponding to the operator

$$L = \frac{1}{2} \nabla \cdot a \nabla + b \cdot \nabla.$$

In this paper we will always assume that $\beta$ is a bounded Borel function on $\mathbb{R}^d$ and that $D$ is a bounded domain in $\mathbb{R}^d$. We will use $(Y^D, \Pi_{\lambda}, x \in D)$ to denote the process obtained by killing $Y$ upon exiting from $D$, that is,

$$Y^D_t = \begin{cases} Y_t & \text{if } t < \tau, \\ \varnothing & \text{if } t \geq \tau, \end{cases}$$

where $\tau = \inf\{ t \geq 0 : Y_t \notin D \}$ is the first exit time of $D$ and $\varnothing$ is a cemetery point. Any function $f$ on $D$ is automatically extended to $D \cup \{ \varnothing \}$ by setting $f(\varnothing) = 0$. For convenience, we use the following convention throughout this paper. For any probability measure $P$, we also use $P$ to denote the expectation with respect to $P$. When there is only one probability measure involved, we sometimes also use $E$ to denote the expectation with respect to that measure.

We will use $\{P_t\}_{t \geq 0}$ to denote the following Feynman–Kac semigroup:

$$P_t f(x) = \Pi_{\lambda} \left( \exp \left( \int_0^t \beta(Y^D_s) ds \right) f(Y^D_t) \right), \quad x \in D.$$ 

It is well known that the semigroup $\{P_t\}_{t \geq 0}$ is strongly continuous in $L^2(D)$ and, for any $t > 0$, $P_t$ has a bounded, continuous, and strictly positive density $p(t, x, y)$.

Let $\{\tilde{P}_t\}_{t \geq 0}$ be the dual semigroup of $\{P_t\}_{t \geq 0}$ defined by

$$\tilde{P}_t f(x) = \int_D p(t, y, x) f(y) dy, \quad x \in D.$$ 

It is well known that $\{\tilde{P}_t\}_{t \geq 0}$ is also strongly continuous on $L^2(D)$.

Let $A$ and $\tilde{A}$ be the generators of the semigroups $\{P_t\}_{t \geq 0}$ and $\{\tilde{P}_t\}_{t \geq 0}$ on $L^2(D)$, respectively. We can formally write $A$ as $\lambda p + \beta$, where $\lambda p$ is the restriction of $L$ to $D$ with Dirichlet boundary condition. Let $\sigma(A)$ and $\sigma(\tilde{A})$ respectively denote the spectrum of $A$ and $\tilde{A}$. It follows from Jentzsch’s theorem [16, Theorem V.6.6, p. 337] and the strong continuity of $\{P_t\}_{t \geq 0}$ and $\{\tilde{P}_t\}_{t \geq 0}$ that the common value $\lambda_1 := \sup \Re(\sigma(A)) = \sup \Re(\sigma(\tilde{A}))$ is an eigenvalue of multiplicity 1 for both $A$ and $\tilde{A}$, and that an eigenfunction $\phi$ of $A$ associated with $\lambda_1$ can be chosen to be strictly positive almost everywhere (a.e.) on $D$ and an eigenfunction $\tilde{\phi}$ of $\tilde{A}$ associated with $\lambda_1$ can be chosen to be strictly positive a.e. on $D$. We assume that $\phi$ and $\tilde{\phi}$
are strictly positive a.e. on $D$. By [9, Proposition 2.3] we know that $\phi$ and $\tilde{\phi}$ are bounded and continuous on $D$, and they are in fact strictly positive everywhere on $D$. We choose $\phi$ and $\tilde{\phi}$ so that $\int_D \phi(x)\tilde{\phi}(x)\,dx = 1$.

Throughout this paper, we make the following assumptions.

**Assumption 1.1.** $\lambda_1 > 0$.

**Assumption 1.2.** The semigroups $\{P_t\}_{t \geq 0}$ and $\{\widehat{P}_t\}_{t \geq 0}$ are intrinsic ultracontractive, that is, for any $t > 0$, there exists a constant $c_t > 0$ such that
\[
p(t, x, y) \leq c_t \phi(x)\tilde{\phi}(y) \quad \text{for all } (x, y) \in D \times D.
\]

Assumption 1.2 is a very weak regularity assumption on $D$. It follows from [9] and [10] that Assumption 1.2 is satisfied when $D$ is a bounded Lipshitz domain. For other, more general, examples of domain $D$ for which Assumption 1.2 is satisfied, we refer the reader to [10] and the references therein.

Let $E_t = \sigma(Y^{D}_s, s \leq t)$. For any $x \in D$, we define a probability measure $\Pi^\phi_x$ by the martingale change of measure:
\[
\frac{d\Pi^\phi_x}{d\Pi_x}\bigg|_{E_t} = \frac{\phi(Y^{D}_t)}{\phi(x)} \exp\left\{-\int_0^{t \wedge \tau} (\lambda_1 - \beta(Y_s)) \, ds\right\}.
\]

The process $(Y^{D}, \Pi^\phi_x)$ is an ergodic Markov process and its transition density is given by
\[
p^\phi(t, x, y) = \frac{\exp[-\lambda_1 t]}{\phi(x)} p(t, x, y)^\phi(y).
\]

The function $\phi\tilde{\phi}$ is the unique invariant density for the process $(Y^{D}, \Pi^\phi_x)$.

By our choices for $\phi$ and $\tilde{\phi}$, $\int_D \phi(x)\tilde{\phi}(x)\,dx = 1$. Thus, it follows from [9, Theorem 2.8] that
\[
\left|\frac{\exp[-\lambda_1 t]p(t, x, y)}{\phi(x)\tilde{\phi}(y)} - 1\right| \leq ce^{-vt}, \quad x \in D,
\]
for some positive constants $c$ and $v$, which is equivalent to
\[
\sup_{x \in D} \left|\frac{p^\phi(t, x, y)}{\phi(y)\tilde{\phi}(y)} - 1\right| \leq ce^{-vt}.
\]

Thus, for any $f \in L^\infty(D)$, we have
\[
\sup_{x \in D} \left|\int_D p^\phi(t, x, y)f(y)\,dy - \int_D \phi(y)\tilde{\phi}(y)f(y)\,dy\right| \leq ce^{-vt} \int_D \phi(y)\tilde{\phi}(y)f(y)\,dy.
\]

Consequently, we have
\[
\lim_{t \to \infty} \sup_{x \in D} \sup_{f \in L^\infty(D)} \left(\int_D \phi(y)\tilde{\phi}(y)f(y)\,dy\right)^{-1} \times \left|\int_D p^\phi(t, x, y)f(y)\,dy - \int_D \phi(y)\tilde{\phi}(y)f(y)\,dy\right| = 0.
\]
Theorem 1.1. Suppose that $\mu$ with initial value $\mu$ is a $\sigma$-finite kernel. For any finite Borel measure $\mu$ on $D$, we define a probability measure $\Pi^\phi_\mu$ as follows:

$$\Pi^\phi_\mu = \int_D \mu(dx) \frac{\phi(x)}{\langle \phi, \mu \rangle} \pi^\phi.$$

Note that, for any $A \in \mathcal{E}_t$, $\Pi^\phi_{\phi\mu}(A) = \frac{1}{\langle \phi, \mu \rangle} \Pi_\mu \left( \phi(Y_t^D) \exp \left\{ -\int_0^{t\wedge \tau} (\lambda_1 - \beta(Y_s)) ds \right\} \right).$

The superdiffusion $X$ we are going to study is a $(Y, \psi(\lambda) - \beta\lambda)$-superprocess, which is a measure-valued Markov process with underlying spatial motion $Y$, branching rate $dr$, and branching mechanism $\psi(\lambda) - \beta\lambda$, where

$$\psi(x, \lambda) = \int_0^\infty (e^{-\lambda r} - 1 + \lambda r)n(x, dr)$$

for some $\sigma$-finite kernel $n$ from $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, that is, $n(x, dr)$ is a $\sigma$-finite measure on $\mathbb{R}_+$ for each fixed $x$, and $n(\cdot, A)$ is a measurable function for each Borel set $A \subset \mathbb{R}_+$. In this paper we will always assume that $\sup_{x \in D} \int_0^\infty (r \wedge r^2)n(x, dr) < \infty$. Note that this assumption implies that, for fixed $\lambda > 0$, $\psi(x, \lambda)$ is bounded on $D$.

Let $(Y, \Pi_{r,x})$ denote a diffusion with generator $L$, birth time $r$, and starting point $x$. For any $\mu \in M_F(D)$, the family of all finite Borel measures on $D$, we will use $(X, P_{r,\mu})$ to denote a $(Y, \psi(\lambda) - \beta\lambda)$-superprocess with starting time $r$ such that $P_{r,\mu}(X_r = \mu) = 1$. We will simply denote $(X, P_{0,\mu})$ as $(X, \mu)$. Let $X_{t,D}$ be the exit measure from $[0, t) \times D$, and let $\partial^D_t$ be the union of $(0, t) \times \partial D$ and $[t \times D$.

Define $\phi^t : [0, t) \times \partial D \to [0, \infty)$ for each fixed $t \geq 0$, such that $\phi^t(u, x) = \phi(x)$ for $(u, x) \in [0, t) \times D$ and $\phi^t(u, x) = 0$ for $(u, x) \in [0, t) \times \partial D$. In particular, we extend $\phi^t$ to $\partial D$ by setting it to be 0 on the boundary. Then

$$\{M_t(\phi) := \exp[-\lambda_t t]\langle \phi^t, X_{t,D} \rangle, t \geq 0\}$$

is a $P_\mu$-martingale with respect to $\mathcal{F}_t := \sigma(X_{s,D}, s \leq t)$ (see Lemma 2.1, below) and $P_\mu(M_t(\phi)) = \langle \phi, \mu \rangle, t \geq 0$. It is easy to check that $\{M_t(\phi), t \geq 0\}$ is a multiplicative functional of $X_{t,D}$.

To state our main result, we first define a new kernel $n^\phi(x, dr)$ from $(D, \mathcal{B}(D))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that, for any nonnegative measurable function $f$ on $\mathbb{R}_+$,

$$\int_0^\infty f(r)n^\phi(x, dr) = \int_0^\infty f(r\phi(x))n(x, dr), x \in D.$$

The following theorem is the main result of the paper.

**Theorem 1.1.** Suppose that $(X_t)$ is a $(Y, \psi(\lambda) - \beta\lambda)$-superdiffusion starting from time 0 and with initial value $\mu$. Set

$$l(y) := \int_1^\infty r \log rn^\phi(y, dr).$$
The purpose of this section is to give a spine decomposition of $L \log L$ criterion for a class of superdiffusions.

1. If $\int_D \tilde{\phi}(y) l(y) \, dy < \infty$ then $M_\infty(\phi)$ is nondegenerate under $P_\mu$ for any $\mu \in M_F(D)$.

2. If $\int_D \tilde{\phi}(y) l(y) \, dy = \infty$ then $M_\infty(\phi)$ is degenerate for any $\mu \in M_F(D)$.

The proof of this theorem is accomplished by combining the ideas from [14] with the ‘spine decomposition’ of [5]. The new feature here is that we consider a different branching mechanism. The new branching mechanism considered here is essential. With this branching mechanism, we can establish a strong (that is, almost-sure) version of the spine decomposition, as opposed to the weak (that is, in distribution) version in [5]. The reason is that the branching mechanism we consider here results in discrete immigration points, as opposed to the quadratic branching case where immigration is continuous in time.

In the next section we first give a spine decomposition of the superdiffusion $X$ under a martingale change of measure with the help of Poisson point processes. Then, in Section 3 we use this decomposition to give a proof of Theorem 1.1.

2. Decomposition of superdiffusions under the martingale change of measure

Let $\mathcal{F}_t = \sigma(X_t, S, s \leq t)$. We define a probability measure $\tilde{P}_\mu$ by the martingale change of measure:

$$
\frac{d\tilde{P}_\mu}{dP_\mu} \mid \mathcal{F}_t = \frac{1}{\langle \phi, \mu \rangle} M_t(\phi).
$$

The purpose of this section is to give a spine decomposition of $X$ under $\tilde{P}_\mu$.

The most important step in proving Theorem 1.1 is a decomposition of $X$ under $\tilde{P}_\mu$. We could decompose $X$ under $P_\mu$ as the sum of two independent measure-valued processes. The first process is a copy of $X$ under $P_\mu$. The second process is, roughly speaking, obtained by taking an ‘immortal particle’ that moves according to the law of $Y$ under $\Pi_0^{0,\mu}$ and spins off pieces of mass that continue to evolve according to the dynamics of $X$.

To give a rigorous description of this decomposition of $X$ under $\tilde{P}_\mu$, let us first recall some results on Poisson point processes. Let $(S, \delta)$ be a measurable space. We will use $\mathcal{M}$ to denote the family of $\sigma$-finite counting measures on $(S, \delta)$ and $\mathcal{B}(\mathcal{M})$ to denote the smallest $\sigma$-field on $\mathcal{M}$ with respect to which all $\nu \in \mathcal{M} \mapsto \nu(B) \in \mathbb{Z}^+ \cup \{\infty\}$, $B \in \delta$, are measurable. For any $\sigma$-finite measure $\tilde{N}$ on $\delta$, we call an $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$-valued random variable $\xi$ a Poisson random measure with intensity $\tilde{N}$

(a) for each $B \in \delta$ with $\tilde{N}(B) < \infty$, $\xi(B)$ has a Poisson distribution with parameter $\tilde{N}(B)$;

(b) for $B_1, \ldots, B_n$ disjoint, the variables $\xi(B_1), \ldots, \xi(B_n)$ are independent.

Suppose that $\tilde{N}$ is a $\sigma$-finite measure on $(0, \infty) \times S$, if $e = (e(t), t \geq 0)$ is a process taking values in $S \cup \{\Upsilon\}$, where $\Upsilon$ is an isolated additional point and $e(0) = \Upsilon$, such that the random counting measure $\xi = \sum_{t \geq 0} \delta_{(e(t))}$ is a Poisson random measure on $(0, \infty) \times S$ with intensity $\tilde{N}$, then $e$ is called a Poisson point process with compensator $\tilde{N}$. If, for every $t > 0$, $\tilde{N}((0, t] \times S) < \infty$ then $e$ can also be expressed as $e = (((\sigma_i, e_i), i = 1, \ldots, N_t), t \geq 0)$, where $e_i = e(\sigma_i)$ and $N_t$ is a Poisson process with instant intensity $\tilde{N}(dt \times S)$. The following proposition follows easily from [15, Proposition 19.5].

**Proposition 2.1.** Suppose that $e = (e(t), t \geq 0)$ is a Poisson point process with compensator $\tilde{N}$. Let $f$ be a nonnegative Borel function on $(S \cup \{\Upsilon\}) \times [0, \infty)$ with $f(\Upsilon, t) = 0$ for
all $t > 0$. If $\int_{0}^{t} \int_{S} |1 - e^{-\theta(s)}| \tilde{N}(ds, dx) < \infty$ for all $t > 0$ then

$$
E(\exp\left\{ - \sum_{0 \leq s \leq t} f(\theta(s), s) \right\}) = \exp\left\{ - \int_{0}^{t} \int_{S} (1 - e^{-\theta(s)}) \tilde{N}(ds, dx) \right\}.
$$

Moreover, if $\int_{0}^{\infty} \int_{S} f(s, x) \tilde{N}(ds, dx) < \infty$ then

$$
E\left( \int_{0}^{\infty} \int_{S} f(s, x) \tilde{N}(ds, dx) \right) = \int_{0}^{\infty} \int_{S} f(s, x) \tilde{N}(ds, dx).
$$

To give a formula for the one-dimensional distribution of the exit measure process under $\tilde{P}_{\mu}$, we recall some results from [4] first.

According to [4], for any nonnegative bounded continuous function $f : \partial \Omega \rightarrow \mathbb{R}$, we have

$$
P_{r, \mu}(\exp\{ - f, X_{t}^{r,D} \}) = \exp\{ - U'(f)(r, \cdot, \mu) \},
$$

where $U'(f)$ denotes the unique nonnegative solution to

$$
- \frac{\partial U(s, x)}{\partial s} = LU + \beta U(s, x) - \psi(U(s, x)), \quad x \in D, \quad s \in (0, t),
$$

$$
U = f \quad \text{on} \quad \partial \Omega.
$$

More precisely, $U'(f)$ satisfies the following integral equation:

$$
U'(f)(r, x) + \Pi_{r,x} \int_{r}^{t \wedge \tau_{r}} [\psi(U'(f))(s, Y_{s}) - \beta(Y_{s})U'(f)(s, Y_{s})] ds
$$

$$
= \Pi_{r,x} f(t \wedge \tau_{r}, Y_{t \wedge \tau_{r}}), \quad r \leq t, \quad x \in D,
$$

where $\tau_{r} = \inf\{ t \geq r : X_{t} \notin D \}$. Since $Y$ is a time-homogeneous process, we find that $X_{t}^{r,D}$ under $P_{r, \mu}$ has the same distribution as $X_{t-r}^{r,D}$ under $P_{\mu}$. The first moment of $\langle f, X_{t}^{r,D} \rangle$ is given by

$$
P_{r,x}(f, X_{t}^{r,D}) = \Pi_{r,x} \left( f(t \wedge \tau_{r}, Y_{t \wedge \tau_{r}}) \exp\left\{ \int_{r}^{t \wedge \tau_{r}} \beta(Y_{s}) ds \right\} \right).
$$

**Lemma 2.1.** $\{M_{t}(\phi), \ t \geq 0\}$ is a $P_{\mu}$-martingale with respect to $\mathcal{F}_{t}$.

**Proof.** It follows from the first moment formula (2.5) that

$$
P_{r,x}(\phi'(t), X_{t}^{r,D}) = \Pi_{r,x} \left( \phi(Y_{t}) \exp\left\{ \int_{r}^{t} \beta(Y_{s}) ds \right\}, \ t < \tau_{r} \right)
$$

$$
= P_{t-r}(\phi(x)) \quad \text{for} \ r \leq t, \ x \in D.
$$

It is obvious that $P_{r,x}(\phi'(t), X_{t}^{r,D}) = 0$ for $x \in \partial D$. By the special Markov property of $X$ and the invariance of $\phi$ under $\exp[-\lambda_{t}] P_{t}$,

$$
P_{\mu}(M_{t}(\phi) | \mathcal{F}_{s}) = \exp[-\lambda_{s}] P_{X_{t}^{s,D}}(\exp[-\lambda_{1}(t-s)]\phi', X_{t}^{s,D}))
$$

$$
= \exp[-\lambda_{s}] \exp[-\lambda_{1}(t-s)] P_{t-s} \phi, X_{t}^{s,D}|_{D}
$$

$$
= \exp[-\lambda_{s}] \phi', X_{s}^{D}
$$

$$
= M_{s}(\phi) \quad \text{for} \ s \leq t,
$$

where $X_{s}^{D}|_{D}$ is the restriction of the measure $X_{s}^{D}$ on $\{s\} \times D$. 

Downloaded from https://www.cambridge.org/core. IP address: 54.191.40.80, on 20 Aug 2017 at 02:44:30, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0021900200005593
Now we give a formula for the one-dimensional distribution of \( X \) under \( \tilde{P} \).

**Theorem 2.1.** Suppose that \( \mu \) is a finite measure on \( D \) and that \( g \in C_b^+ (\partial_t, D) \). Then

\[
\tilde{P}(\exp(-g, X^{t,D})) = P(\exp(-g, X^{t,D})) \\
\times \Pi_\phi^\mu \left( \exp \left\{ - \int_0^{t \wedge \tau_0} \psi'(Y_s, U^t(g)(Y_s)) \, ds \right\} \right),
\]

(2.6)

where \( U^t(g) \) is the unique solution of (2.3) or, equivalently, (2.4) with \( f \) replaced by \( g \).

**Proof.** This theorem can be proved using the same argument as that given in [5] to obtain Theorem 5 therein, with some obvious modifications. We omit the details.

From (2.6) we can see that the superprocess \((X^{t,D}, \tilde{P})\) can be decomposed into two independent parts in the sense of distributions. The first part is a copy of the original superprocess and the second part is an immigration process. To explain the second part more precisely, we need to introduce another measure-valued process \((\hat{X}^t)\).

(a) Suppose that \( \tilde{Y} = (\tilde{Y}_t, t \geq 0) \) is defined on some probability space \((\Omega, P, \phi)\) and that \( \tilde{Y} = (\tilde{Y}_t, t \geq 0) \) has the same law as \((Y, \Pi_\phi^\mu)\). Here \( \tilde{Y} \) serves as the spine or the immortal particle, which visits every part of \( D \) for large times since it is an ergodic diffusion.

(b) Suppose that \( m = \{m_t, t \geq 0\} \) is a point process taking values in \((0, \infty) \cup \{\Upsilon\}\) such that, conditional on \( \sigma(\tilde{Y}_t, t \geq 0) \), \( m \) is a Poisson point process with intensity \( r n(\tilde{Y}_t, dr) \). Now \((0, \infty)\) is the ‘space of mass’ and \( m_1 = \Upsilon \) simply means that there is no immigration at \( t \). We suppose that \( \{m_t, t \geq 0\} \) is also defined on \((\Omega, P, \phi)\). Set \( D_m = \{t : m_t(\omega) \neq \Upsilon\} \). Note that \( D_m \) is almost surely (a.s.) countable. The process \( m \) describes the immigration mechanism: along the path of \( \tilde{Y} \), at the moment \( t \in D_m \), a particle with mass \( m_t \) is immigrated into the system at the position \( \tilde{Y}_t \).

(c) Once the particles are in the system, they begin to move and branch according to a \((Y, \psi(\lambda) - \beta \lambda)\)-superprocess independently.

We use \((X^\sigma, t \geq \sigma)\) to denote the measure-valued process generated by the mass immigrated at time \( \sigma \) and position \( \tilde{Y}_\sigma \). Conditional on \( \{\tilde{Y}_t, m_t, t \geq 0\}, \{X^\sigma, \sigma \in D_m\} \) are independent \((Y, \psi - \beta \lambda)\)-superprocesses. The birth time of \( X^\sigma \) is \( \sigma \) and the initial value of \( X^\sigma \) is \( m_\sigma \delta_{\tilde{Y}_\sigma} \). Set

\[
\hat{X}^{\sigma, t,D} = \sum_{\sigma \in (0,t] \cap D_m} X^{\sigma, (t,D)},
\]

where, for each \( \sigma \in D_m \), \( X^{\sigma, (t,D)} \) is the exit measure of the superprocess \( X^\sigma \) from \((0,t) \times D\). The Laplace functional of \( \hat{X}^{\sigma, t,D} \) is described in the following proposition.

**Proposition 2.2.** The Laplace functional of \( \hat{X}^{\sigma, t,D} \) under \( P_{\mu, \phi} \) is

\[
\Pi_\phi^\mu \left( \exp \left\{ - \int_0^t \psi'(Y_s, U^t(g)(Y_s, s)) \, ds \right\} \right).
\]
Proof. For any \( g \in C^+_b(\partial^i.D) \), using (2.2), we have
\[
P_{\mu,\phi}(\exp(-\langle g, \tilde{X}_{t,D}^i \rangle)) = P_{\mu,\phi}\left(\exp\left(-\sum_{\sigma \in (0,t] \cap D_m} \langle g, X^\sigma_{t,D} \rangle \right) \Bigg| \tilde{Y}, \sigma, m \right) = P_{\mu,\phi}\left(\prod_{\sigma \in (0,t] \cap D_m} \exp(-m_\sigma U^i(g)(\tilde{Y}_\sigma, \sigma)) \right) = P_{\mu,\phi}\left(\prod_{\sigma \in (0,t] \cap D_m} m_\sigma U^i(g)(\tilde{Y}_\sigma, \sigma) \right).
\]

Using Proposition 2.1, we obtain
\[
P_{\mu,\phi}(\exp(-\langle g, \tilde{X}_{t,D}^i \rangle)) = P_{\mu,\phi}\left(\exp\left(-\int_0^t \int_0^\infty (1 - \exp(-rU^i(g)(Y_s,s))) r n(Y_s, dr) ds \right) \right) = P_{\mu,\phi}\left(\exp\left(-\int_0^t \psi'(Y_s, U^i(g)(s, Y_s)) ds \right) \right).
\]

Without loss of generality, we suppose that \((X_t, t \geq 0; \tilde{P}_\mu)\) is a superdiffusion defined on \((\Omega, \tilde{P}_\mu, \phi)\), equivalent to \((X_t, t \geq 0; P_{\mu})\) and independent of \(\tilde{X}\). Proposition 2.2 says that we have the following decomposition of \(X^i.D\) under \(\tilde{P}_\mu\): for any \(t > 0\),
\[
(X^i.D, \tilde{P}_\mu) = (X^i.D + \tilde{X}^{i,D}, P_{\mu,\phi}) \quad \text{in distribution}, \quad \text{(2.7)}
\]
where \(X^i.D\) is the exit measure of \(X\) from \([0, t) \times D\). Since \((X_t, t \geq 0; \tilde{P}_\mu)\) is generated from the time-homogeneous Markov process \((X_t, t \geq 0; P_{\mu})\) via a nonnegative martingale multiplicative functional, \((X_t, t \geq 0; \tilde{P}_\mu)\) is also a time-homogeneous Markov process (see [17, Section 62]). From the construction of \((\tilde{X}^{i,D}, t \geq 0; P_{\mu,\phi})\) we see that \((\tilde{X}^{i,D}, t \geq 0; P_{\mu,\phi})\) is a time-homogeneous Markov process. For a rigorous proof of \((\tilde{X}^{i,D}, t \geq 0; P_{\mu,\phi})\) being a time-homogeneous Markov process, we refer the reader to [6]. Although [6] dealt with the representation of the superprocess conditioned to stay alive forever, we can check that the arguments there work in our case. Therefore, (2.7) implies that
\[
(X^i.D, t \geq 0; \tilde{P}_\mu) = (X^i.D + \tilde{X}^{i,D}, P_{\mu,\phi}) \quad \text{in distribution}.
\]

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need some preparations. The following elementary result is taken from [3].

Lemma 3.1. ([3, Exercise 1.3.8].) Let \(Y \geq 0\) with \(E(Y) < \infty\), and let \(0 \leq a < E(Y)\). Then
\[
P(Y > a) \geq \frac{(E(Y) - a)^2}{E(Y^2)}.
\]

Proposition 3.1. Set \(h(x) = P_{\delta_x}(M_\infty(\phi))/\phi(x)\).

1. \(h\) is nonnegative and invariant for the process \((Y^D, \Pi^\phi)\).

2. Either \(M_\infty\) is nondegenerate under \(P_\mu\) for all \(\mu \in M_F(D)\) or \(M_\infty\) is degenerate under \(P_\mu\) for all \(\mu \in M_F(D)\).
Proof. 1. Since $\phi' (\cdot, u) = \phi(\cdot)$ for each $u \in [0, t]$ and $\phi$ is identically 0 on $\partial D$, we have, by the special Markov property of $X$,

$$h(x) = \frac{1}{\phi(x)} P_{\lambda t} \left( \lim_{s \to \infty} \exp \left[ -\lambda_1 (t + s) \right] \phi^{t + s}, X^{t + s}, D \right)$$

$$= \frac{\exp \left[ -\lambda_1 t \right]}{\phi(x)} P_{\lambda t} \left( P_{X^{t}, D} \left( \lim_{s \to \infty} \exp \left[ -\lambda_1 s \right] \phi^s, X^s, D \right) \right)$$

$$= \frac{\exp \left[ -\lambda_1 t \right]}{\phi(x)} P_{\lambda t} \left( P_{X^{t}, D} (M_{\infty}) \right)$$

$$= \frac{\exp \left[ -\lambda_1 t \right]}{\phi(x)} P_{\lambda t} \left( (h \phi)^t, X^t, D \right)$$

$$= \frac{1}{\phi(x)} \Pi_t \left( \exp \left[ \int_0^{t \wedge \tau} (\beta - \lambda_1)(Y_s) \, ds \right] (h \phi)(Y_D) \right), \quad x \in D.$$

By the definition of $\Pi_T$ we obtain $h(x) = \Pi_T (h(Y_D))$. So, $h$ is an invariant function of the process $(Y^D, \Pi_T)$. The nonnegativity of $h$ is obvious.

2. Since $h$ is nonnegative and invariant, if there exists an $x_0 \in D$ such that $h(x_0) = 0$, then $h \equiv 0$ on $D$. Since $P_{\mu} (M_{\infty} (\phi)) = (h \phi, \mu)$, we then have $P_{\mu} (M_{\infty} (\phi)) = 0$ for any $\mu \in M_F (D)$. If $h > 0$ on $D$ then $P_{\mu} (M_{\infty} (\phi)) > 0$ for any $\mu \in M_F (D)$.

Using Proposition 3.1, we see that, to prove Theorem 1.1, we only need to consider the case $\mu(dx) = \tilde{\phi}(x) \, dx$. So, in the remaining part of this paper we will always suppose that $\mu(dx) = \tilde{\phi}(x) \, dx$.

Lemma 3.2. Let $(m_t, t \geq 0)$ be the Poisson point process constructed in Section 2. Define

$$\sigma_0 = 0, \quad \sigma_i = \inf \{ s \in D_m : s > \sigma_{i-1}, m_s \phi(Y_s) > 1 \}, \quad \eta_i = m_{\sigma_i}, \quad i = 1, 2, \ldots.$$

If $\int_D \tilde{\phi}(y) l(y) \, dy < \infty$ then

$$\sum_{s \in D_m} \exp \left[ -\lambda_1 s \right] m_s \phi(Y_s) < \infty \quad P_{\mu, \phi} \text{-a.s.}$$

If $\int_D \tilde{\phi}(y) l(y) \, dy = \infty$ then

$$\limsup_{i \to \infty} \exp \left[ -\lambda_1 \sigma_i \right] \eta_i \phi(Y_{\sigma_i}) = \infty \quad P_{\mu, \phi} \text{-a.s.}$$

Proof. Since $\tilde{\phi}$ is bounded from above, $\sigma_i$ is strictly increasing with respect to $i$. We first prove that if $\int_D \tilde{\phi}(y) l(y) \, dy < \infty$ then

$$\sum_{s \in D_m} \exp \left[ -\lambda_1 s \right] m_s \phi(Y_s) < \infty \quad P_{\mu, \phi} \text{-a.s.}$$
For any \( \varepsilon > 0 \), we write the sum above as

\[
\sum_{s \in D_m} \exp(-\lambda_1 s) m_s \phi(\tilde{Y}_s)
= \sum_{s \in D_m} \exp(-\lambda_1 s) m_s \phi(\tilde{Y}_s) \mathbf{1}_{\{\phi(\tilde{Y}_s) m_s \leq e^{\varepsilon s}\}} + \sum_{s \in D_m} \exp(-\lambda_1 s) m_s \phi(\tilde{Y}_s) \mathbf{1}_{\{m_s \phi(\tilde{Y}_s) > e^{\varepsilon s}\}}
= \sum_{s \in D_m} \exp(-\lambda_1 s) m_s \phi(\tilde{Y}_s) \mathbf{1}_{\{\phi(\tilde{Y}_s) m_s \leq e^{\varepsilon s}\}} + \sum_{i=1}^{\infty} \exp(-\lambda_1 s_i) \eta_i \phi(\tilde{Y}_{s_i}) \mathbf{1}_{\{\eta_i \phi(\tilde{Y}_{s_i}) > \exp(\varepsilon s_i)\}}
= I + II.
\]  

(3.1)

By (2.1) we have

\[
\sum_{i=1}^{\infty} P_{\mu,\phi}(\eta_i \phi(\tilde{Y}_{s_i}) > \exp(\varepsilon s_i)) = \sum_{i=1}^{\infty} P_{\mu,\phi}(P_{\mu,\phi}(\eta_i \phi(\tilde{Y}_{s_i}) > \exp(\varepsilon s_i) | \sigma(\tilde{Y})))
= P_{\mu,\phi} \left( \int_{\tilde{Y}} \exp \left( \int_0^{\infty} \right) \right)
= \Pi_{\phi\mu} \left( \int_0^{\infty} \left( \int_{\phi(\tilde{Y})} \right) \right)
\]

Recall that, under \( \Pi_{\phi\mu} \), \( Y \) starts at the invariant measure \( \phi(x) \mu(dx) = \phi(x) \tilde{\phi}(x) dx \). So we have

\[
\sum_{i=1}^{\infty} P_{\mu,\phi}(\eta_i \phi(\tilde{Y}_{s_i}) > \exp(\varepsilon s_i)) = \int_0^{\infty} \int_{\tilde{Y}} dy \phi(y) \tilde{\phi}(y) \int_{\phi(\tilde{Y})}^{\infty} r n(y, \sigma) ds
= \int_{\tilde{Y}} \phi(y) \tilde{\phi}(y) dy \int_{\phi(\tilde{Y})}^{\infty} r n(y, \sigma) ds
= \varepsilon^{-1} \int_{\tilde{Y}} \phi(y) (\phi(y))^{\frac{1}{\varepsilon}} dy.
\]

By the assumption that \( \int_{\tilde{Y}} \phi(y) (\phi(y))^{\frac{1}{\varepsilon}} dy < \infty \) and the Borel–Cantelli lemma, we obtain

\[
P_{\mu,\phi}(\eta_i \phi(\tilde{Y}_{s_i}) > \exp(\varepsilon s_i) \text{ infinitely often}) = 0 \text{ for all } \varepsilon > 0,
\]

which implies that

\[
II < \infty \text{ } P_{\mu,\phi} \text{-a.s.} \quad (3.2)
\]

Meanwhile, for \( \varepsilon < \lambda_1 \),

\[
P_{\mu,\phi} I = P_{\mu,\phi} \left( \sum_{s \in D_m} \exp(-\lambda_1 s) m_s \phi(\tilde{Y}_s) \mathbf{1}_{\{m_s \leq e^{\varepsilon s}\}} \right)
= \Pi_{\phi\mu} \left( \int_0^{\infty} dt \exp(-\lambda_1 t) \int_0^{\phi(\tilde{Y})} r n(Y_t, \sigma) \right)
\leq \Pi_{\phi\mu} \left( \int_0^{\infty} dt \exp(-\lambda_1 t) \int_0^{1} r^2 n(Y_t, \sigma) \right)
+ \Pi_{\phi\mu} \left( \int_0^{\infty} dt \exp(-t) \int_1^{\infty} r n(Y_t, \sigma) \right).
\]
where for the second term of the last inequality we used the fact that \( r \leq \phi(Y_t)^{-1}e^{r t} \) implies that \( r \phi(Y_t) \leq e^{r t} \).
By the assumption that \( \sup_{x \in D} \int_0^\infty (r \wedge r^2)m_x(r\nu) < \infty \) we have \( P_{\mu,\phi} I < \infty \), which implies that

\[
I < \infty \quad P_{\mu,\phi} \text{-a.s.} \tag{3.3}
\]

Combining (3.1), (3.2), and (3.3), we see that

\[
\limsup_{i \to \infty} \exp\{ -\lambda_1 \sigma_i \} \eta_i \phi(\tilde{Y}_{\sigma_i}) < \infty \quad P_{\mu,\phi} \text{-a.s.} \tag{3.4}
\]

It suffices to prove that, for any \( K > 0 \),

\[
\limsup_{i \to \infty} \exp\{ -\lambda_1 \sigma_i \} \eta_i \phi(\tilde{Y}_{\sigma_i}) > K \quad P_{\mu,\phi} \text{-a.s.} \tag{3.4}
\]

Set \( K_0 := 1 \vee \left( \max_{x \in D} \phi(x) \right) \). Then, for \( K \geq K_0 \),

\[
K \inf_{x \in D} \phi(x) \geq 1.
\]

Note that, for any \( T \in (0, \infty) \), conditional on \( \sigma(\tilde{Y}) \),

\[
\sharp\{ i : \sigma_i \in (0, T); \eta_i > K \phi(\tilde{Y}_{\sigma_i})^{-1} \exp\{\lambda_1 \sigma_i\} \}
\]

is a Poisson random variable with parameter \( \int_0^T dt \int_{K \phi(Y_t)^{-1} \exp\{\lambda_1 t\}} r_n(\tilde{Y}, dr) \) a.s. Since \((\tilde{Y}, P_{\mu,\phi})\) has the same distribution as \((Y, \Pi_{\mu,\phi})\), we have

\[
P_{\mu,\phi} \int_0^T dt \int_{K \phi(\tilde{Y}_t)^{-1} \exp\{\lambda_1 t\}} r_n(\tilde{Y}_t, dr) = \int_0^T dt \int_D d\phi(y) \phi(y) \int_{K \phi(y)^{-1} \exp\{\lambda_1 t\}} r_n(y, dr) = \infty;
\]

thus,

\[
\int_0^T dt \int_{K \phi(\tilde{Y}_t)^{-1} \exp\{\lambda_1 t\}} r_n(\tilde{Y}_t, dr) < \infty \quad P_{\mu,\phi} \text{-a.s.}
\]

Consequently, we have

\[
\sharp\{ i : \sigma_i \in (0, T); \eta_i > K \phi(\tilde{Y}_{\sigma_i})^{-1} \exp\{\lambda_1 \sigma_i\} \} < \infty \quad P_{\mu,\phi} \text{-a.s.}
\]

So, to prove (3.4), we need to prove that

\[
\int_0^\infty dt \int_{K \phi(\tilde{Y}_t)^{-1} \exp\{\lambda_1 t\}} r_n(\tilde{Y}_t, dr) = \infty \quad P_{\mu,\phi} \text{-a.s.},
\]

which is equivalent to

\[
\int_0^\infty dt \int_{K \phi(Y_t)^{-1} \exp\{\lambda_1 t\}} r_n(Y_t, dr) = \infty \quad \Pi_{\mu,\phi} \text{-a.s.} \tag{3.5}
\]
For this purpose, we first prove that
\[
\Pi_\phi^\mu \left( \int_0^\infty dt \int_{K\phi(Y_t)} r_n(y, dr) \right) = \infty. \tag{3.6}
\]
Applying Fubini’s theorem, we obtain
\[
\Pi_\phi^\mu \left( \int_0^\infty dt \int_{K\phi(Y_t)} r_n(y, dr) \right) = \int_D \phi(y) \tilde{\phi}(y) dy \int_0^\infty \int_{K\phi(y)} r_n(y, dr) \int_0^{(1/\lambda_1) \ln (r\phi(y)/K)} dt
\]
for some positive constant A, where in the inequality we used the facts that \(K\phi(y)^{-1} > 1\) for any \(y \in D\) and \(\sup_{y \in D} \int_1^\infty r_n(y, dr) < \infty\). Since
\[
\int_D \tilde{\phi}(y) dy \int_1^\infty r \ln r \phi^\theta(y, dr) = \infty,
\]
and
\[
\int_D \tilde{\phi}(y) dy \int_1^K r \ln r \phi^\theta(y, dr) \leq K \log K \int_D \tilde{\phi}(y) n(y, [\|\phi\|^{-1}_\infty, \infty)) dy < \infty,
\]
we obtain
\[
\int_D \tilde{\phi}(y) dy \int_K^\infty r \ln r \phi^\theta(y, dr) = \infty,
\]
and, therefore, (3.6) holds.

By (1.1), there exists a constant \(c > 0\) such that, for any \(t > c\) and any \(f \in L^\infty_\phi(D)\),
\[
\frac{1}{2} \int_D \phi(y) \tilde{\phi}(y) f(y) dy \leq \int_D p^\theta(t, x, y) f(y) dy \leq 2 \int_D \phi(y) \tilde{\phi}(y) f(y) dy, \quad x \in D.
\tag{3.7}
\]
For \(T > c\), we define
\[
\xi_T = \int_0^T dt \int_{K\phi(Y_t)} r_n(Y_t, dr), \quad A_T = \int_T^\infty dt \int_D \tilde{\phi}(y) dy \int_{K\phi(Y_t)} r_n(y, dr).
\]
Our goal is to prove (3.5), which is equivalent to
\[
\xi_\infty := \int_0^\infty dt \int_{K\phi(Y_t)} r_n(Y_t, dr) = \infty \quad \Pi_\phi^\mu - a.s.
\]
Since \( \{ \xi_\infty = \infty \} \) is an invariant event, by the ergodic property of \( Y \) under \( \Pi_{\phi\mu}^\phi \), it is enough to prove that
\[
\Pi_{\phi\mu}^\phi (\xi_\infty = \infty) > 0.
\] (3.8)

Note that
\[
\Pi_{\phi\mu}^\phi \xi_T = \int_0^T dt \int_D \tilde{\phi}(y) dy \int_0^\infty r n^\phi(y, dr) \geq A_T
\] (3.9)
and
\[
\lim_{T \to \infty} \Pi_{\phi\mu}^\phi \xi_T \geq \Lambda_{\infty}
\] (3.10)
where \( C \) is a positive constant. By Lemma 3.1,
\[
\Pi_{\phi\mu}^\phi (\xi_T \geq \frac{1}{2} \Pi_{\phi\mu}^\phi \xi_T) \geq \frac{(\Pi_{\phi\mu}^\phi \xi_T)^2}{4 \Pi_{\phi\mu}^\phi (\xi_T^2)}.
\] (3.11)

If we can prove that there exists a constant \( \hat{C} > 0 \) such that, for all \( T > c \),
\[
\frac{(\Pi_{\phi\mu}^\phi \xi_T)^2}{4 \Pi_{\phi\mu}^\phi (\xi_T^2)} \geq \hat{C},
\] (3.12)
then by (3.11) we would obtain
\[
\Pi_{\phi\mu}^\phi (\xi_T \geq \frac{1}{2} \Pi_{\phi\mu}^\phi \xi_T) \geq \hat{C},
\]
and therefore,
\[
\Pi_{\phi\mu}^\phi (\xi_\infty \geq \frac{1}{2} \Pi_{\phi\mu}^\phi \xi_T) \geq \Pi_{\phi\mu}^\phi (\xi_T \geq \frac{1}{2} \Pi_{\phi\mu}^\phi \xi_T) \geq \hat{C} > 0.
\]
Since \( \lim_{T \to \infty} \Pi_{\phi\mu}^\phi \xi_T = \infty \) (see (3.10)), the above inequality implies (3.8). Now we only need to prove (3.12). For this purpose, we first estimate \( \Pi_{\phi\mu}^\phi (\xi_T^2) \):
\[
\Pi_{\phi\mu}^\phi \xi_T^2 = \Pi_{\phi\mu}^\phi \int_0^T dt \int_D K_{\phi(Y_t)}^{-1} \exp[\lambda_1 t] r n(Y_t, dr) \int_0^T ds \int_D K_{\phi(Y_s)}^{-1} \exp[\lambda_1 s] un(Y_s, du)
\]
\[
= 2 \Pi_{\phi\mu}^\phi \int_0^T dt \int_D K_{\phi(Y_t)}^{-1} \exp[\lambda_1 t] r n(Y_t, dr) \int_0^T ds \int_D K_{\phi(Y_s)}^{-1} \exp[\lambda_1 s] un(Y_s, du)
\]
\[
= 2 \Pi_{\phi\mu}^\phi \int_0^T dt \int_D K_{\phi(Y_t)}^{-1} \exp[\lambda_1 t] r n(Y_t, dr) \int_t^{T} ds \int_D K_{\phi(Y_s)}^{-1} \exp[\lambda_1 s] un(Y_s, du)
\]
\[
+ 2 \Pi_{\phi\mu}^\phi \int_0^T dt \int_D K_{\phi(Y_t)}^{-1} \exp[\lambda_1 t] r n(Y_t, dr) \int_0^{(t+c) \wedge T} ds \int_D K_{\phi(Y_s)}^{-1} \exp[\lambda_1 s] un(Y_s, du)
\]
\[
= \text{III} + \text{IV},
\]
where

\[ III = 2 \Pi_{\phi(t)}^\phi \int_0^T dt \int_{\Omega \times [0,1]} \varphi^\phi(Y_t, dr) \int_t^{(t+c) \wedge T} ds \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \]

and

\[ IV = 2 \Pi_{\phi(t)}^\phi \int_0^T dt \int_{\Omega \times [0,1]} \varphi^\phi(Y_t, dr) \int_t^{T} ds \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \]

\[ \leq 2 \int_0^T dt \int_{\Omega} \int_{\Omega} \varphi^\phi(y, \tilde{\varphi}(y)) \int_{\Omega \times [0,1]} \varphi^\phi(Y_t, dr) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \]

\[ \times \int_{(t+c) \wedge T} ds \int_{\Omega} \int_{\Omega} \varphi^\phi(s - t, y, z) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \]

By our assumption on the kernel \( n \) we have \( \| \int_{0}^{\infty} \varphi^\phi(\cdot, dr) \|_\infty < \infty \). Since \( K \inf_{x \in B} \varphi(x) - 1 \geq 1 \), we have

\[ III \leq C_1 \Pi_{\phi(t)}^\phi \xi_T \]

for some positive constant \( C_1 \) which does not depend on \( T \). Using (3.7) and the definition of \( n^\phi \), we obtain

\[ \int_{(t+c) \wedge T} ds \int_{\Omega} \varphi^\phi(s - t, y, z) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \]

\[ \leq 2 \int_{(t+c) \wedge T} ds \int_{\Omega} \varphi^\phi(\tilde{\varphi}(y), \tilde{\varphi}(z)) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \]

\[ \leq 2 \int_{t}^{T} ds \int_{\Omega} \varphi^\phi(\tilde{\varphi}(y), \tilde{\varphi}(z)) \int_{0}^{\infty} \varphi^\phi(z, du) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \]

\[ = 2 \int_{t}^{T} ds \int_{\Omega} \varphi^\phi(\tilde{\varphi}(y), \tilde{\varphi}(z)) \int_{0}^{\infty} \varphi^\phi(z, du) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \int_{\Omega \times [0,1]} \varphi^\phi(Y_s, du) \]

\[ = 2 A_T. \]

Then, using (3.9), we have

\[ IV \leq 4 A_T \Pi_{\phi(t)}^\phi \xi_T \leq 4 (\Pi_{\phi(t)}^\phi \xi_T)^2. \]

Combining the above estimates for \( III \) and \( IV \), we find that there exists a \( C_2 > 0 \) independent of \( T \) such that, for \( T > c \),

\[ \Pi_{\phi(t)}^\phi (\xi_T^2) \leq 4 (\Pi_{\phi(t)}^\phi \xi_T)^2 + C_1 \Pi_{\phi(t)}^\phi \xi_T \leq C_2 (\Pi_{\phi(t)}^\phi \xi_T)^2. \]

Then we have (3.12) with \( \hat{C} = 1/C_2 \), and the proof of the theorem is now complete.

**Definition 3.1.** Suppose that \( (\Omega, \mathcal{F}, P) \) is a probability space, that \( \{ \mathcal{F}_t, \ t \geq 0 \} \) is a filtration on \( (\Omega, \mathcal{F}) \), and that \( \mathcal{G} \) is a sub-\( \sigma \)-field of \( \mathcal{F} \). A real-valued process \( U_t \) on \( (\Omega, \mathcal{F}, P) \) is called a \( P(\cdot \mid \mathcal{G}) \)-martingale with respect to \( \{ \mathcal{F}_t, \ t \geq 0 \} \) if

(i) it is adapted to \( \{ \mathcal{F}_t \cup \mathcal{G}, \ t \geq 0 \} \);

(ii) for any \( t \geq 0 \), \( E(|U_t| \mid \mathcal{G}) < \infty \); and
(iii) for any \( t > s \),
\[
E(U_t \mid \mathcal{F}_s \vee \mathcal{G}) = U_s \quad \text{a.s.}
\]

We say that \( U_t \) on \((\Omega, \mathcal{F}, P)\) is a \( P(\cdot \mid \mathcal{G})\)-submartingale or a \( P(\cdot \mid \mathcal{G})\)-supermartingale with respect to \( \mathcal{F}_t, t \geq 0 \) if, in addition to (i) and (ii), for any \( t > s \),
\[
E(U_t \mid \mathcal{F}_s \vee \mathcal{G}) \geq U_s \quad \text{a.s.}
\]
or, respectively,
\[
E(U_t \mid \mathcal{F}_s \vee \mathcal{G}) \leq U_s \quad \text{a.s.}
\]

The following result is a folklore. Since we could not find a reference for this result, we provide a proof for completeness.

**Lemma 3.3.** Suppose that \((\Omega, \mathcal{F}, P)\) is a probability space, that \( \{\mathcal{F}_t, t \geq 0\} \) is a filtration on \((\Omega, \mathcal{F})\), and that \( \mathcal{G} \) is a \( \sigma \)-field of \( \mathcal{F} \). If \( U_t \) is a \( P(\cdot \mid \mathcal{G})\)-submartingale with respect to \( \{\mathcal{F}_t, t \geq 0\} \) satisfying
\[
\sup_{t \geq 0} E(\|U_t\| \mid \mathcal{G}) < \infty \quad \text{a.s.},
\]
then there exists a finite random variable \( U_\infty \) such that \( U_t \) converges a.s. to \( U_\infty \).

**Proof.** By Definition 3.1, \( U_t \) is a submartingale with respect to \( \{\mathcal{F}_t \vee \mathcal{G}, t \geq 0\} \). Let \( \Omega_n = \{\sup_{t \geq 0} E(\|U_t\| \mid \mathcal{G}) \leq n\} \). Assumption (3.13) implies that \( P(\Omega_n) \uparrow 1 \). Note that, for each fixed \( n \), \( 1_{\Omega_n} U_t \) is a submartingale with respect to \( \{\mathcal{F}_t \vee \mathcal{G}, t \geq 0\} \) with
\[
\sup_{t \geq 0} E(1_{\Omega_n} U_t) = \sup_{t \geq 0} E(E(1_{\Omega_n} U_t \mid \mathcal{G}))
\]
\[
= \sup_{t \geq 0} E(1_{\Omega_n} E(U_t \mid \mathcal{G}))
\]
\[
\leq E\left( \sup_{t \geq 0} E(U_t \mid \mathcal{G}) \mid \Omega_n \right)
\]
\[
< \infty.
\]
The martingale convergence theorem says that there exists a finite random variable \( U_\infty \) defined on \( \Omega_n \) such that \( U_t \) converges to \( U_\infty \) on \( \Omega_n \) as \( t \to \infty \). Therefore, there exists a finite \( U_\infty \) on the whole space \( \Omega \) such that \( U_t \) converges to \( U_\infty \) a.s.

The next result is basically [3, Theorem 4.3.3].

**Lemma 3.4.** Suppose that \((\Omega, \mathcal{F})\) is a measurable space and that \( \{\mathcal{F}_t, t \geq 0\} \) is a filtration on \((\Omega, \mathcal{F})\) with \( \mathcal{F}_t \uparrow \mathcal{F} \). If \( P \) and \( Q \) are two probability measures on \((\Omega, \mathcal{F})\) such that, for some nonnegative \( P \)-martingale \( Z_t \) with respect to \( \{\mathcal{F}_t, t \geq 0\} \),
\[
\frac{dQ}{dP} \bigg|_{\mathcal{F}_t} = Z_t.
\]
Then the limit \( Z_\infty := \limsup_{t \to \infty} Z_t \) exists and is finite a.s. under \( P \). Furthermore, for any \( F \in \mathcal{F} \),
\[
Q(F) = \int_F Z_\infty \, dP + Q(F \cap \{Z_\infty = \infty\}).
\]
and, consequently,
\[ P(Z_\infty = 0) = 1 \iff Q(Z_\infty = \infty) = 1, \]
\[ \int_\Omega Z_\infty \, dP = \int_\Omega Z_0 \, dP \iff Q(Z_\infty < \infty) = 1. \]

**Proof of Theorem 1.1.** We first prove that if \( \int_D \widetilde{h}(y) l(y) \, dy < \infty \) then \( M_\infty \) is nondegenerate under \( P_\mu ). \) Since \( M_t^{-1}(\phi) \) is a positive supermartingale under \( P_\mu ), \) \( M_t(\phi) \) converges to some nonnegative random variable \( M_\infty(\phi) \in (0, \infty) \) under \( P_\mu \). By Lemma 3.4, we only need to prove that
\[ \widetilde{P}_\mu (M_\infty(\phi) < \infty) = 1. \]

By (2.7), \((X^{t, D}, \widetilde{P}_\mu )\) has the same law as \((X^{t, D} + \tilde{X}^{t, D}, P_\mu , \phi)\), where \( X^{t, D} \) is the first exit measure of the superprocess \( X \) from \((0, t) \times D\) and \( \tilde{X}^{t, D} = \sum_{\sigma \in (0, t) \cap D_m} X^{\sigma, (t, D)} \). Define
\[ W_t(\phi) := \sum_{\sigma \in (0, t) \cap D_m} \langle \phi', X^{\sigma, (t, D)} \rangle \exp\{-\lambda_1 t\}. \]

Then,
\[ (M_t(\phi), t \geq 0; \widetilde{P}_\mu) = (M_t(\phi) + W_t(\phi), t \geq 0; P_\mu , \phi) \quad \text{in distribution}, \] (3.15)
where \( \{M_t(\phi), t \geq 0\} \) is copy of the martingale defined in (1.2) and is independent of \( W_t(\phi)\). Let \( \mathcal{G} \) be the \( \sigma \)-field generated by \( \{\bar{Y}_s, m_s, t \geq 0\}. \) Then, conditional on \( \mathcal{G}, (X^T \sigma , t \geq \sigma; \mathcal{P}_\mu , \phi) \)
has the same distribution as \((X^{t-\sigma}, t \geq \sigma; \mathcal{F}_{m_s}, \lambda_1 )\) and the \((X^T , t \geq \sigma; \mathcal{P}_\mu , \phi)\) are independent for \( \sigma \in \mathcal{D}_m \). Then we have
\[ W_t(\phi) \overset{D}{=} \sum_{\sigma \in (0, t) \cap \mathcal{D}_m} \exp\{-\lambda_1 \sigma\} M^\mu_{t-\sigma}(\phi), \] (3.16)
where, for each \( \sigma \in \mathcal{D}_m, M^\mu_{t}(\phi) \) is a copy of the martingale defined by (1.2) with \( \mu = m_\sigma \delta_{\mathcal{G}} \) and, conditional on \( \mathcal{G}, \{M^\mu_{t}(\phi), \sigma \in \mathcal{D}_m\} \) are independent. Here \( \overset{D}{=} \) denotes equality in distribution. To prove (3.14), by (3.15), it suffices to show that
\[ P_{\mu , \phi} \left( \lim_{t \to \infty} [M_t(\phi) + W_t(\phi)] < \infty \right) = 1. \]

Since \( \{M_t(\phi), t \geq 0\} \) is a nonnegative martingale under the probability \( P_{\mu , \phi} \), it converges \( P_{\mu , \phi} \)-a.s. to a finite random variable \( M_\infty(\phi) \) as \( t \to \infty \). So we only need to prove that
\[ P_{\mu , \phi} \left( \lim_{t \to \infty} W_t(\phi) < \infty \right) = 1. \] (3.17)

Define \( \mathcal{H}_t := \mathcal{G} \vee \sigma(X^{\sigma, (\sigma, t)}; \sigma \in [0, t] \cap \mathcal{D}_m, s \in [\sigma, t]) \). Then \( (W_t(\phi)) \) is a \( P_{\mu , \phi} (\cdot \mid \mathcal{G}) \)-nonnegative submartingale with respect to \( (\mathcal{H}_t) \). By (3.16) and Lemma 3.2,
\[ \sup_{t \geq 0} P_{\mu , \phi} (W_t(\phi) \mid \mathcal{G}) = \sup_{t \geq 0} \sum_{s \in (0, t) \cap \mathcal{D}_m} \exp\{-\lambda_1 s\} \mathcal{F}_{m_s}(\bar{Y}_s) \]
\[ \leq \sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} \mathcal{F}_{m_s}(\bar{Y}_s) \]
\[ < \infty \quad P_{\mu , \phi} \text{-a.s.} \]
Then, by Lemma 3.3, $W_t(\phi)$ converges $P_{\mu,\phi}$-a.s. to $W_\infty(\phi)$ as $t \to \infty$ and $P_{\mu,\phi}(W_\infty(\phi) < \infty) = 1$; therefore, (3.17) holds.

Now we turn to the proof of the second part of the theorem. Assume that $\int_D \tilde{\phi}(y) l(y) \, dy = \infty$. We are going to prove that $M_\infty(\phi) := \lim_{t \to \infty} M_t(\phi)$ is degenerate with respect to $P_{\mu,\phi}$.

By [7, Proposition 2], $1/M_t(\phi)$ is a supermartingale under $\tilde{P}_{\mu}$, and, thus, $1/(M_t(\phi) + W_t(\phi))$ is a nonnegative supermartingale under $P_{\mu,\phi}$. Recall that $M_t(\phi)$ is a nonnegative martingale under $P_{\mu,\phi}$. Then the limits $\lim_{t \to \infty} M_t(\phi)$ and $1/\lim_{t \to \infty} (M_t(\phi) + W_t(\phi))$ exist and are finite $P_{\mu,\phi}$-a.s. Therefore, $\lim_{i \to \infty} W_t(\phi)$ exists in $[0, \infty)$ $P_{\mu,\phi}$-a.s. Recall the definition of $(\eta_i, \sigma_i, i = 1, 2, \ldots)$ in Lemma 3.2, and note that $\lim_{i \to \infty} \sigma_i = \infty$. By Lemma 3.2,

$$\limsup_{t \to \infty} W_t(\phi) \geq \limsup_{i \to \infty} W_{\sigma_i}(\phi) \geq \limsup_{i \to \infty} \exp\{-\lambda_1 \sigma_i \eta_i \phi(\tilde{Y}_{\sigma_i})\} = \infty \quad P_{\mu,\phi}\text{-a.s.}$$

So we have

$$\lim_{t \to \infty} W_t(\phi) = \infty \quad P_{\mu,\phi}\text{-a.s.}$$

By (3.15),

$$\tilde{P}_{\mu}(M_\infty(\phi) = \infty) = 1.$$ 

It follows from Lemma 3.4 that $P_{\mu}(M_\infty = 0) = 1$.

**Remark 3.1.** The argument of this paper actually works for general superprocesses. Our main result remains valid for any general $(Y, \psi(\lambda) - \beta \lambda)$-superprocess with $Y$ being a reasonable Markov process such that Assumptions 1.1 and 1.2 are satisfied. For examples of discontinuous Markov processes satisfying Assumption 1.2, we refer the reader to [11] and the references therein.

**Acknowledgements**

We thank Andreas Kyprianou for helpful discussions. We also thank the anonymous referee for helpful comments.

**References**


