THE \( \lambda \)-PROPERTY IN SCHREIER’S SPACE \( S \) AND
THE LORENTZ SPACE \( d(a, 1) \)

by THADDEUS J. SHURA and DAVID TRAUTMAN

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0. Abstract. We add Schreier’s space \( S \) and the Lorentz space \( d(a, 1) \) to the list of
classical Banach spaces which enjoy the \( \lambda \)-property, investigate the extreme point
structure of \( S \), and show that \( d(a, 1) \) has a \( \lambda \)-function which is continuous on \( S \),
though not even uniformly so.

1. Introduction. Let \( X \) be a Banach space, \( B_X \) the unit ball of \( X \), \( S_X \) the surface of
\( B_X \), and \( \text{ext } B_X \) the set of extreme points of \( B_X \). For points \( x, y \in X \), we write \([x, y)\) for
\( \{\lambda x + (1 - \lambda)y : 0 < \lambda \leq 1\} \).

DEFINITION 1.1. (a) \( X \) has the \( k \)-property, if for each \( x \in B_X \), there exists \( e \in \text{ext } B_X \),
\( y \in B_X \), \( 0 < \lambda \leq 1 \) such that
\[ x = \lambda e + (1 - \lambda)y. \]
In this case we say that the triple \((e, y, \lambda)\) is amenable to \( x \), and write \((e, y, \lambda) \sim x\).

(b) If \( X \) has the \( \lambda \)-property, for each \( x \in B_X \), we define
\[ \lambda(x) := \sup\{\lambda : (e, y, \lambda) \sim x\}. \]

(c) If there exists \( \lambda_0 > 0 \) such that \( \lambda(x) \geq \lambda_0 \), for all \( x \in B_X \), we say that \( X \) has the
uniform \( \lambda \)-property.

(d) Finally, we say that \( X \) has the convex series representation property (C.S.R.P.), if
for each \( x \in B_X \), there exist \( \lambda_n \geq 0 \), \( e_n \in \text{ext } B_X \), \( (n = 1, 2, \ldots ) \), such that
\[ x = \sum_n \lambda_n e_n \]
and \( \sum_n \lambda_n = 1 \).

These notions were developed by R. Aron and R. H. Lohman in [1], where (among
other results) they proved: the uniform \( \lambda \)-property implies C.S.R.P. An easy exercise
shows that C.S.R.P. implies the \( \lambda \)-property. Spaces that enjoy either the \( \lambda \)-property or
the uniform \( \lambda \)-property are not rare [1], [3], [4], [8], and it is our belief that some strong
theorems are lurking behind these concepts. In an attempt to better understand these
properties we decided to investigate a couple of “exotic” sequence spaces. We begin with
Schreier’s space \( S \).

2. Schreier’s space \( S \).

DEFINITION 2.1. (a) Let \( R^N \) denote the (vector) space of real sequences \( x = (x(1), x(2), \ldots ) \) which are finitely-non-zero (i.e., have “finite support”). A subset \( E \) of the
natural numbers \( N \) is admissible, if \( E = \{n_1, n_2, \ldots , n_k\} \), with \( k \leq n_1 < n_2 \ldots < n_k \). We
denote by \( \mathcal{A} \) the collection of all admissible subsets of \( N \).

(b) For \( x \in R^N \), we define
\[ \|x\|_S := \sup_{E \in \mathcal{A}} \sum_{j \in E} |x(j)|. \]
(Routine calculations show that \( \|\cdot\|_S \) is a norm on \( R^N \).)
The space $S$ has been studied extensively in [2], where it is shown that $S$ is hereditarily-$c_0$; (hence $l_1$ does not embed in it). In this section we shall show that $S$ has enough extreme points to enjoy C.S.R.P., even though we fall short of a useful characterization of $\text{ext } B_S$. First we note that $S$ is not $c_0$ in disguise.

**Proposition 2.2.** $S$ is not isomorphic to $c_0$.

**Proof.** If we denote by $\{s_n\}_{n=1}^\infty$ the canonical unit vector basis for $S$, then for each $n$, $l_1^n$ is isometric to the norm-one complemented subspace of $S$ spanned by $\{s_i : n + 1 \leq i \leq 2n\}$. Thus (see [6, p. 74]) $S^*$ fails to have finite co-type. Hence $S^*$ is not isomorphic to $c_0^*$, and so $S$ is not isomorphic to $c_0$.

For each $n$, let $S_n := \text{span} \{s_i : i \leq n\}$. Since $S_n$ is finite-dimensional, $\text{ext } B_n \neq \emptyset$ (where $B_n := B_{S_n}$). In fact we shall show that $\text{ext } B_n \cap \text{ext } B_S \neq \emptyset$.

The reader can easily show (by using 1-sets introduced below) that the vectors $(1, 1)$, $(1, 1, 1, \frac{1}{2})$, and $(1, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ are all in $\text{ext } B_S$ (when we write $x = (x(1), x(2), \ldots, x(n))$, we mean $x(j) = 0$ when $j > n$).

**Definition 2.3.** Let $x \in B_S$.

(a) If $E \in \mathcal{A}$, and $\sum_{j \in E} |x(j)| = 1$, we say that $E$ is a 1-set for $x$.

(b) If (in addition) $E = \{n_1 < n_2 < \ldots < n_k\}$ and $k < n_1$, we say $E$ is a non-maximal 1-set, for $x$.

Since for $E \in \mathcal{A}$, $x \rightarrow \sum_{j \in E} |x(j)|$ is a semi-norm, we clearly have the following result.

**Lemma 2.4.** Let $x, b, c \in B_S$ with $x = \lambda b + (1 - \lambda)c$ for some $0 < \lambda < 1$. Then any 1-set for $x$ is a 1-set for $b$ and $c$.

A slight modification of the above shows that for vectors $x, b_1, b_2, \ldots$ in $B_S$ and scalars $\lambda_1, \lambda_2, \ldots$ each $> 0$ with $\sum_n \lambda_n = 1$ and $x = \sum_n \lambda_nb_n$, every 1-set for $x$ is a 1-set for each $b_n$.

**Lemma 2.5.** Let $n \geq 1$ and $x \in \text{ext } B_n$. If $x$ has a non-maximal 1-set $E$, then $x \in \text{ext } B_S$.

**Proof.** Clearly we may assume that $\text{max } E \leq n$. Suppose $x = \lambda b + (1 - \lambda)c$, for some $0 < \lambda < 1$ and some $b, c \in B_S$. If $E$ is a non-maximal 1-set for $x$, then, by Lemma 2.4, $E$ is a non-maximal 1-set for $b$ and $c$. So $b(j) = 0 = c(j)$, for $j > n$, since $E \cup \{j\} \in \mathcal{A}$ for every $j > n$. But $x(j) = b(j) = c(j)$ for $j \leq n$, since $x \in \text{ext } B_n$. Thus $x = b = c$, and $x \in \text{ext } B_S$.

We note that $\text{ext } B_n \notin \text{ext } B_S$. Some calculations show that $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \text{ext } B_5 \sim \text{ext } B_S$, for instance. To show that $S$ has C.S.R.P. we need some lemmata about certain representations.

**Lemma 2.6.** Let $x \in B_S$. Then for all $\epsilon > 0$, there exists $N$ such that for $E \in \mathcal{A}$ with $N < \text{min } E$, we have $\sum_{j \in E} |x(j)| < \epsilon$.

**Proof.** If $x \in B_S$, then $\|x - y\| < \epsilon$ for some finite vector $y \in B_S$. 

(c) Schreier's space $S$ is the $\|\cdot\|_S$-completion of $R^{(N)}$. (From here on, we will write $"||\cdot||"$, for $\|\cdot\|_S$.)
LEMMA 2.7. Let $x \in B_s$ have infinite support. Then there exist vectors $b, c \in B_s$ such that $x = \frac{1}{2}b + \frac{1}{3}c$, and $b$ has finite support.

Proof. Without loss of generality we may assume $\|x\| = 1$. Let

$$\alpha := \min \{ |x(j)| : j \in E \in \mathcal{A}, E \text{ is a 1-set for } x, x(j) \neq 0 \},$$

and

$$\epsilon := \min \left\{ 1 - \sum_{j \in E} |x(j)| : E \in \mathcal{A}, \sum_{j \in E} |x(j)| < 1 \right\}.$$

Lemma 2.6 implies that $\epsilon > 0$. Choose an integer $M$ larger than any element in any 1-set for $x$ and larger than any element in any $E \in \mathcal{A}$ which determines $\epsilon$. Finally, enlarge $M$ (if needed) so that for $E \in \mathcal{A}$, $\min E \geq M$ implies

$$\sum_{j \in E} |x(j)| < \frac{1}{2} \min \{ \alpha, \epsilon \}.$$

Now define vectors $b$ and $c$ by

$$\begin{cases} 
  b(j) = x(j) = c(j), & \text{for } 1 \leq j < M, \\
  b(j) = 0, & \text{for } j \geq M, \\
  c(j) = 2x(j), & \text{for } j \geq M.
\end{cases}$$

The only thing left to show is that $c \in B_s$. Towards this end, let $E \in \mathcal{A}$. If $\max E < M$, then $\sum_{j \in E} |c(j)| \leq \|x\| = 1$. If $\max E \geq M$, then

$$\sum_{j \in E} |c(j)| = \sum_{j < M} |c(j)| + \sum_{j \geq M} |c(j)| < 1 - \epsilon + 2 \cdot \frac{\epsilon}{2} = 1.$$

Note that for $x \in B_s$, applying the above Lemma recursively we obtain a representation $x = \sum_{i} 2^{-i}b_i$, where each $b_i \in B_s$ and each $b_i$ has finite support. Also note that we immediately obtain the following result.

COROLLARY 2.8. If $x \in \text{ext} B_s$, then $x$ has finite support.

In fact, we can show more.

LEMMA 2.9. Let $x \in B_s$ have finite support. Then $x$ can be represented as $x = \frac{1}{2}b + \frac{1}{3}c$, for two vectors $b, c \in B_s$ each of finite support, and each having a non-maximal 1-set.

Proof. Without loss of generality, we may assume $x \neq 0$. Let $N = \max(\text{support } x) + 1$, and let $\epsilon = \min \left\{ 1 - \sum_{j \in E} |x(j)| : E \in \{n_1, \ldots, n_k\}, k < n_1 < N \right\}$. (If $\epsilon = 0$, then $x$ already has a non-maximal 1-set, and we can choose $b = c = x$.) Choose $M > N$ such that

$$\frac{N - 2}{M} < \epsilon.$$ (The case where $N \leq 2$ is trivial.)
Define vectors $b$ and $c$ via

$$
\begin{align*}
    b(j) &= x(j) = c(j), & \text{for } 1 \leq j \leq M, \\
    b(j) &= 0 = c(j), & \text{for } j > 2M, \\
    b(j) &= \frac{1}{M} = -c(j), & \text{for } M + 1 \leq j \leq 2M.
\end{align*}
$$

Clearly, $x = \frac{1}{2}b + \frac{1}{2}c$, and $\|b\| = \|c\|$.

To show that $\|b\| \leq 1$, let $E \in \mathcal{A}$. If $\min E \geq N$, then $\sum_{j \in E} |b(j)| \leq M \cdot \frac{1}{M} = 1$. If $\max E < N$, then $\sum_{j \in E} |b(j)| = \sum_{j \in E} |x(j)| \leq 1$. In the only remaining case

$$
\sum_{j \in E} |b(j)| = \left(\sum_{j \in E} + \sum_{j \neq N} \right) |b(j)| \\
\leq 1 - \varepsilon + \frac{N - 2}{M} < 1.
$$

So $\|b\| = \|c\| = 1$, and each has $\{M + 1, M + 2, \ldots, 2M\}$ for a non-maximal 1-set.

**Theorem 2.10.** Schreier’s space $S$ has C.S.R.P.

**Proof.** Let $x \in B_S$. By the remark following Lemma 2.7, we may write $x = \sum_n 2^{-n} b_n$, where $\|b_n\| = 1$ and support $b_n$ is finite, $(n = 1, 2, \ldots)$. Using Lemma 2.9 on each $b_n$, we can write $x = \sum \lambda_j c_j$, for some choices of $\lambda_j$ and $c_j$ such that $\sum \lambda_j = 1$, $\|c_j\| = 1$, and each vector $c_j$ has finite support and a non-maximal 1-set. Now each vector $c_j$ belongs to some $S_n$, where $n := n(j)$. Since $S_n$ has C.S.R.P. [2], for each $j$ we can write $c_j = \sum_i \lambda_{j,i} e_{j,i}$, a convex series where the $e_{j,i} \in \text{ext} B_n$. Finally $x = \sum_{i,j} \lambda_{j,i} e_{j,i}$, and the vectors $e_{j,i}$ all belong to $\text{ext} B_S$, by Lemmas 2.4 and 2.5.

This of course implies that $S$ has the $\lambda$-property although we do not know whether it has the uniform $\lambda$-property. We mention here that the extreme points of $B_S$ all have supports with even cardinality (we omit the proof). It is of interest to note the following result.

**Proposition 2.11.** $\text{ext} B_S$ is countable.

**Proof.** The earlier lemmas show that $\text{ext} B_S \subseteq \bigcup_n \text{ext} B_n$. We now show that each $\text{ext} B_n$ is finite. Since $B_n$ is compact, it suffices to show that for each $x \in B_n$, there is a ball (in the $B_n$ topology) of radius $\varepsilon = \varepsilon(x)$ such that this ball meets $\text{ext} B_n$ (at most) at the point $x$. Let $x \in B_n$, and assume $\|x\| = 1$. Define

$$
\delta_1 = \min\{|x(j)| : x(j) \neq 0\},
$$

$$
\delta_2 = \min\left\{1 - \sum_{j \in E} |x(j)| : E \in \mathcal{A} \text{ and } \sum_{j \in E} |x(j)| < 1 \right\}.
$$

Let $\delta = \frac{1}{2} \min\{\delta_1, \delta_2\}$, and choose $\varepsilon > 0$ so that $2\varepsilon < \delta$.

Suppose $y \in B_n$ with $\|x - y\| < \varepsilon$. Note that by choice of $\varepsilon$, whenever $x(j) \neq 0$, $x(j)$
and $y(j)$ have the same sign. Now define $z$ by

$$z(j) = \begin{cases} 0, & \text{if } j > n, \\ 2y(j) - x(j), & \text{if } j \leq n. \end{cases}$$

Clearly $z \in S_n$ and $y = \frac{1}{2}x + \frac{1}{2}z$. If we can show $z \in B_n$, then $y \notin \text{ext } B_n$, unless $y = x$ and $x \in \text{ext } B_n$. Note that $\varepsilon$ was chosen small enough so that $x(j)$, $y(j)$, and $z(j)$ have the same sign as $j$ ranges over the support of $x$. So for all $j$, we have

$$x(j) - z(j) = 2(x(j) - y(j)),$$

$$|x(j)| - |z(j)| = 2(|x(j)| - |y(j)|).$$

Letting $E \in A$, we may assume $E \subset \{1, \ldots, n\}$. If $E$ is not a 1-set for $x$, then

$$\sum_{j \in E} |z(j)| \leq \sum_{j \in E} |x(j)| + 2n \varepsilon < 1.$$ 

If $E$ is a 1-set for $x$, then letting $E_0 = \{j \in E : x(j) = 0\}$, and $E_1 = E \setminus E_0$, we have

$$\sum_{j \in E} |z(j)| = \sum_{j \in E_1} |z(j)| + \sum_{j \in E_0} |z(j)|$$

$$= \sum_{j \in E_1} |x(j)| - \sum_{j \in E_1} (|x(j)| - |z(j)|) + \sum_{j \in E_0} |z(j)|$$

$$= \sum_{j \in E_1} |x(j)| - 2 \sum_{j \in E_1} (|x(j)| - |y(j)|) + 2 \sum_{j \in E_0} |y(j)|$$

$$= 2 \sum_{j \in E} |y(j)| - \sum_{j \in E_1} |x(j)| \leq 1.$$ 

Thus $\|z\| \leq 1$.

3. The Lorentz sequence space $d(a, 1)$. We consider here Lorentz sequence spaces of type $d(a, 1)$. These “weighted” versions of $l_1$ turn out to have the $\lambda$-property, while failing the uniform $\lambda$-property. This was demonstrated in Theorems 5 and 6 in [8], both of which we improve here by producing the exact form of the $\lambda$-function for norm-one vectors. This is then used to prove a continuity result. First we establish some definitions and notation.

**Definition 3.1.** Let $a = (a_n) \in c_0 \setminus l_1$ be a positive strictly decreasing sequence with $a_1 = 1$. The space $d(a, 1)$ consists of all real sequences $x = (x(n)) \in c_0$ such that $\sup_{\pi} \sum |x(\pi(n))| a_n < \infty$, where the supremum is taken over all permutations $\pi$ of the natural numbers. (If $\|x\|$ is taken to be this supremum, then $d(a, 1)$ is a Banach space.)

If $x = (x(n)) \in d(a, 1)$, and $x \neq 0$, we write

$$M_1(x) = \|x\|_\infty, \quad \text{and} \quad F_1(x) = \{n : |x(n)| = M_1(x)\},$$

$$M_2(x) = \|x - xc_{F_1(x)}\|_\infty, \quad F_2(x) = \{n : |x(n)| = M_2(x)\},$$

where $c_{F_1(x)}$ is the characteristic function of $F_1(x)$, etc. Then $M_k(x) \downarrow 0$, and if $M_k(x) > 0$, then $M_k(x) > M_{k+1}(x)$. Also $F_k(x)$ and $F_j(x)$ are disjoint if $M_k(x), M_j(x) > 0$ and $k \neq j$. Let

$$N(x) = \{k : M_k(x) - M_{k+1}(x) > 0\}, \quad \text{and for } k \in N(x), \quad n_k(x) = \text{card} \left( \bigcup_{i=1}^{k} F_i(x) \right),$$

$$s_k(x) = \sum_{n=1}^{n_k(x)} a_n.$$
If we let \( n_0(x) = 0 \), then we can write \( \|x\| \) as

\[
\|x\| = \sum_{k \in N(x)} M_k(x) \cdot (s_k(x) - s_{k-1}(x)).
\]

Importantly, for \( x \in d(a, 1) \), \( \|x\| \) can also be realized in another way.

**Proposition 3.2.** For any \( x \in d(a, 1) \),

\[
\|x\| = \sum_k M_k(x) \cdot [s_k(x) - s_{k-1}(x)] = \sum_n [M_n(x) - M_{n+1}(x)] \cdot s_n(x).
\]

**Proof.** It suffices to note that either sum is equal to

\[
\sum_k \sum_{j=k}^\infty (M_k(x) - M_{k+1}(x))(s_j(x) - s_{j-1}(x))
\]

The extreme points of \( B_{d(a,1)} \) were characterized by W. J. Davis [10].

**Proposition 3.3.** \( e \in \text{ext } B_{d(a,1)} \) if and only if \( e \) has the form

\[
e = \left( \sum_{n=1}^{k} a_n \right)^{-1} \left( \sum_{n=1}^{k} \varepsilon_n x_{a_n} \right),
\]

for some integer \( k \), \( i_1 < i_2 < \ldots < i_k \), and signs \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \), (where \( (x_i) \) is the canonical unit vector basis of \( d(a, 1) \).)

Using this characterization, we can establish the following result.

**Proposition 3.4.** The space \( d(a, 1) \) has C.S.R.P.

**Proof.** Assume first that \( \|x\| = 1 \), and that \( x \) has the form

\[
x = (x(1) \geq x(2) \geq \ldots \geq x(k) > 0).
\]

For any \( j \), define \( s_j = \sum_{i=1}^{j} a_i \), and denote by \( e_m \) that extreme point with non-negative coefficients and support = \( \{1, 2, \ldots, m\} \). Further denote by \( \nu^n \) that vector defined by \( \nu^n(i) = 1 \), if \( 1 \leq n \), and \( 0 \), otherwise. Then

\[
x = (x(1), x(2), \ldots, x(k), 0, \ldots)
= x(k) \cdot \nu^k + (x(1) - x(k)), \ldots, x(k-1) - x(k), 0, \ldots)
= \ldots = x(k)\nu^k + (x(k-1) - x(k))\nu^{k-1}
+ (x(k-2) - x(k-1))\nu^{k-2} + \ldots (x(2) - x(3))\nu^2 + (x(1) + x(2))\nu^1
= [x(k)s_k]e_k + [(x(k-1) - x(k))s_{k-1}]e_{k-1}
+ [(x(k-2) - x(k-1))s_{k-2}]e_{k-2} + \ldots
+ [(x(2) - x(3))s_2]e_2 + [(x(1) - x(2))s_1]e_1.
\]

Let \( \alpha_l = (x(l) - x(l+1))s_l \), \( (l = k, k-1, \ldots, 2, 1) \) and note that

\[
\alpha_k + \alpha_{k-1} + \ldots + \alpha_1 = x(k)(s_k - s_{k-2}) + x(k-1)(s_{k-1} - s_{k-2})
+ \ldots + x(2)(s_2 - s_1) + x(1)s_1
= x(k)a_k + x(k-1)a_{k-1} + \ldots + x(2)a_2 + x(1)a_1
= \|x\| = 1.
\]
THE $\lambda$-PROPERTY

Now assume that $||x|| = 1$ and that $x$ has the form $x = (x(1) \geq x(2) \geq \ldots > 0)$. Then (using the notation above) $1 = ||x|| = \lim \sum _{k \geq 1} a_x(i) = \lim \sum _{k \geq 1} \alpha_i$. Arbitrary vectors $x$ with $||x|| = 1$ are an isometry away from the two cases already considered, and if $||x|| < 1$,

$$x = ||x|| \cdot \frac{x}{||x||} + \frac{1 - ||x||}{2} \cdot e + \frac{1 - ||x||}{2} (-e)$$

(where $e$ is any extreme point), leads to a convex series representation.

Proposition 3.4 implies that $d(a, 1)$ has the $\lambda$-property, but we can say more. In [8] a lower bound is proven for the $\lambda$-function.

If $x \in B_{d(a, 1)}$, $x \neq 0$, then $\lambda(x) \geq \sup _{k \in \mathbb{N}(x)} [M_k(x) - M_{k+1}(x)] s_k(x)$. (*)

In the same paper an exact formula is given for unit vectors of finite support.

If $x \in d(a, 1)$ with $||x|| = 1$, and support $x$ is finite, then

$$\lambda(x) = \max _{k \in \mathbb{N}(x)} [M_k(x) - M_{k+1}(x)] s_k(x).$$ (**)

Proposition 3.2 allows us to replace the “sup” in (*) by a “max”, and we can now remove the hypothesis about support $x$ in (**).

Using the results above, we can also establish the following theorems.

**THEOREM 3.5.** Assume $x \in d(a, 1)$, $||x|| = 1$. Then

$$\lambda(x) = \max _{n} [M_n(x) - M_{n+1}(x)] s_n(x)$$

**THEOREM 3.6.** The $\lambda$-function for $d(a, 1)$ is continuous on $\{x : ||x|| = 1\}$, Lipschitz-continuous on $\{x : ||x|| \leq r\}$, $(0 < r < 1)$, though not even uniformly continuous on $\{x : ||x|| = 1\}$.

Consideration of space forces us to omit proofs of these last two results, which will appear in [9].

**REMARK.** R. H. Lohman [7] has recently shown that for Banach spaces the $\lambda$-property is equivalent to the C.S.R.P.

**REFERENCES**

4. A. S. Granero, The $\lambda$-function in the spaces $(\oplus \sum X)_p$ and $L_p(\mu, X)$ (preprint).


Kent State University at Salem
Salem, Ohio 44460
U.S.A.

The Citadel
Charleston
South Carolina 29409
U.S.A.