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DEBLURRING AND DENOISING OF IMAGES WITH MINIMIZATION OF VARIATION AND NEGATIVE NORMS

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Abstract

A method based on the minimization of variation is presented for the identification of a completely unknown blur operator. We assume the knowledge of a blurred image and its original version. The class of blurring operators is identified in the class of compact operators. A variational method with negative norms is then used for the restoration of a blurred and noised image. The restoration method works for a wide class of blurring operators and we do not assume that the blur operator commutes with the Laplacian.

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1. Introduction

An image is considered to be a bounded and open set $\Omega \subset \mathbb{R}^2$ with Lipschitz continuous boundary. The model of image degradation commonly used in the literature is

$$f = Ru + n \tag{1.1}$$

where $f, u : \Omega \to \mathbb{R}$ are the degraded image and original image respectively, R is a linear operator, typically representing blur and n is white Gaussian noise with 0 mean and variance σ^2. An important problem in image processing is the one of restoring the original image u from the blurred and noised version f.

Blur can be introduced by an improperly focused lens, relative motion between the camera and the scene or atmospheric turbulence. The problem of deblurring with a

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known blur operator has been extensively addressed in the literature. See, for example, [9], [18] and the references therein. In many situations, the blur operator is partially known or completely unknown. Two approaches have been taken for restoring a blurred image with an unknown blur operator. In the first approach, restoration and simultaneous identification of the unknown blur operator is attempted, see for example [12], [15] and [20]. In the second approach, the identification of the blur operator (or point spread function PSF) from a blurred image is performed prior to the restoration, see for example [4], [8], [10], [17] and [19]. Some of these techniques assume a statistical model of the image, that is, the image is modelled by an autoregressive process and the blur as a moving average process. More recently, parametric methods have been used to identify PSF models. In the present case, we will take the second approach and model the unknown blur by a linear operator.

We employ here the well-known technique of the minimization of a regularized energy function (see [5]) for the identification of the blur operator as well as the image restoration. For the identification of the blur operator R we assume the knowledge of a blurred image f_0 and its original unblurred version u_0. The identification is done by minimizing the energy function

$$E(R) = \int_\Omega |f_0 - Ru_0|^2 \, d\Omega + \gamma \operatorname{trace}(R^* R),$$

where $\gamma > 0$ is a weight parameter and the trace$(\cdot)$ functional (see, for example, [7]) is used in order to introduce a Hilbert space structure on the linear manifold of "blurring operators" and to force the identification of a compact operator (see Section 2).

Once the blurring operator has been identified, the restoration of the original image from the noised and blurred one will be carried out. One of the earlier techniques of image restoration, proposed by Rudin, Osher, and Fatemi [16], involves the minimization of the energy functional

$$F(u) = J(u) + \lambda \int_\Omega |(f - Ru)|^2 \, d\Omega.$$

Here, $J(u)$ is a regularizing term, $\lambda > 0$ is a weight parameter and $\int_\Omega |(f - Ru)|^2 d\Omega$ is a fidelity term. The term $J(u)$ is the total variation (in the sense of measures; see Section 3) of the function u. This model allows for discontinuities along curves and therefore, edges are better restored. Its drawback, however, is that small details and oscillating patterns [13] in the image are mostly treated as noise and are thus lost in the restoration process. In this paper we use the energy functional

$$G(u) = J(u) + \lambda \int_\Omega \left|\nabla \Delta^{-1} (f - Ru)\right|^2 d\Omega,$$

where ∇ is the gradient operator and Δ^{-1} is the inverse of the Laplacian. This energy functional was used in [14] to handle images with oscillating patterns. Their results show that oscillating patterns are better separated from pure noise using this functional. We include in this paper a direct proof of the existence and "uniqueness" of a minimizer without assuming that the operator R commutes with the Laplacian. In fact, we do not even assume that R is a compact operator as far as image restoration is concerned, as we shall see in Section 3.

2. Identification of the blur operator

In this section we discuss the problem of identification of the operator R in (1.1) assuming the knowledge of an original image $u_0 \in L^2(\Omega)$ and its blurred version $f_0 = Ru_0 \in L^2(\Omega)$. Recovering R from this equation is an ill-posed problem [5]. To regularize it, we resort to the minimization of an energy functional such as

$$E(R) = \|f_0 - Ru_0\|^2_{L^2(\Omega)} + \gamma \|R\|^2_*$$

for some suitable operator norm $\|\cdot\|^2_*$. The operator R is sought in a space H of compact operators on $L^2(\Omega)$. The induced operator norm on $L^2(\Omega)$ could be used, however, to ensure recovery of a compact operator, the Hilbert norm is imposed on H. For a given $V \in H$, the Hilbert norm of V is defined by

$$\|V\|^2_H = \text{trace}(V^*V) = \sum_{j=1}^{\infty} \|Ve_j\|^2,$$

for some, and hence all, orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of $L^2(\Omega)$. An operator $V \in H$ with finite Hilbert norm is a Hilbert-Schmidt operator and thus is compact (see [7]). For ease of notation we will denote $L^2(\Omega)$ by W. The energy functional E can now be written as

$$E(R) = \|f_0 - Ru_0\|^2_W + \gamma \|R\|^2_H. \tag{2.1}$$

The existence of a unique minimizer R is ensured by the strict convexity of the E.

The following lemma checks the continuous embedding of H into $\mathscr{L}(W)$, the space of bounded operators on W, as well as the completeness of H.

LEMMA 2.1. *We have the following properties:*

(1) *There is a $c > 0$ such that $\|V\| \le c\|V\|_H$ for all $V \in H$ (here $\|\cdot\|$ is the induced operator norm on W).*

(2) *H is a Hilbert space with respect to the Hilbert norm.*

PROOF. To show (1), let $V \in H$ and let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for W consisting of eigenvectors of V^*V with corresponding eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$. Then

$$\|V\|_H^2 = \sum_{j=1}^{\infty} \|V e_j\|^2 = \sum_{j=1}^{\infty} \langle V^*Ve_j, e_j\rangle = \sum_{j=1}^{\infty} \lambda_j \geq r_\sigma(V^*V) = \|V^*V\|,$$

where V^* is the adjoint of V and $r_\sigma(V^*V)$ is the spectral radius of V^*V. We claim that there is a $C > 0$ such that $\|V^*V\| \geq C\|V\|^2$ for all $V \in H$. If not then there is a sequence $\{V_n\}$ in H such that $\|V_n\| = 1$ but $\|V_n^*V_n\| \to 0$. Let $\{u_n\}_{n=1}^{\infty}$ be a norm one sequence in W such that $\|V_n u_n\|_W \geq 1/2$. Then

$$\frac{1}{4} \leq \|V_n u_n\|_W^2 = \langle V_n^* V_n u_n, u_n\rangle \leq \|V_n^* V_n\| \to 0,$$

which is a contradiction. This proves (1) with $c = \sqrt{C^{-1}}$.

To show (2), let $\{V_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the norm $\|\cdot\|_H$. By Property (1) of this lemma, $\{V_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in the original norm of $\mathscr{L}(W)$. Then $V_n \to V$ and V is compact. Let $\{\lambda_j\}_{j=1}^{\infty}$ be the eigenvalues of V^*V listed in decreasing order. Since $V_n^*V_n \to V^*V$, for each λ_j there is a sequence $\lambda_j^{(k_n)}$ of eigenvalues of $V_{k_n}^* V_{k_n}$ such that $\lambda_j^{(k_n)} \to \lambda_j$ as $n \to \infty$. Thus, for any $N \in \mathbb{N}$, there is a subsequence $\{V_{k_n}\}_{n=1}^{\infty}$ such that $\lambda_j^{(k_n)} \to \lambda_j$ as $n \to \infty$, $j = 1, 2, \ldots, n$. Since $\{V_n\}_{n=1}^{\infty}$ is bounded in the norm $\|\cdot\|_H$, say by $M > 0$, $\sum_{j=1}^{\infty} \lambda_j^{(n)} \leq M$ for all n. Therefore,

$$\sum_{j=1}^{N} \lambda_j = \lim \sum_{j=1}^{N} \lambda_j^{(k_n)} \leq \overline{\lim} \sum_{j=1}^{\infty} \lambda_j^{(k_n)} \leq M.$$

Thus $\|V\|_H^2 = \sum_{j=1}^{\infty} \lambda_j \leq M < \infty$, that is, $V \in H$. □

Next we turn to the minimization of the energy functional (2.1). The Euler-Lagrange equation corresponding to (2.1) is

$$\langle f_0 - Ru_0, Vu_0\rangle_W = \gamma \langle R, V\rangle_H \quad \forall V \in H. \tag{2.2}$$

Define the operator $T_{u_0} : H \to W$ by $T_{u_0}V = Vu_0$ for all $V \in H$.

LEMMA 2.2. *The operator T_{u_0} is a bounded linear operator and there is a $c \in (0, 1)$ such that $\|u_0\|_W \leq \|T_{u_0}\| \leq c^{-1}\|u_0\|_W$.*

PROOF. By Lemma 2.1 we have

$$\|T_{u_0}V\| = \|Vu_0\|_W \leq \|V\| \|u_0\|_W \leq c^{-1} \|V\|_H \|u_0\|_W,$$

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Hence $\|T_{u_0}\| \le c^{-1}\|u_0\|_W$. On the other hand, if we take $P \in H$ as the projection onto the subspace $\operatorname{span}\{u_0\}$, then

$$\left\|T_{u_0}P\right\| = \|Pu_0\|_W = \|u_0\|_W .$$

It follows that $\|u_0\|_W \le \|T_{u_0}\| \le c^{-1}\|u_0\|_W$. □

In light of the above lemma, we can rewrite Equation (2.2) as

$$\left(\gamma I + T^*_{u_0}T_{u_0}\right) R = T^*_{u_0} f_0, \tag{2.3}$$

which must be satisfied by the minimizer R of the energy E. Since $T^*_{u_0}T_{u_0}$ is non-negative, Equation (2.3) is solvable for any positive γ. However, taking large values of γ tends to produce minimizers which are not in good agreement with the actual operator R. On the other hand, if large values of γ are allowed, then Equation (2.3) can be solved by fixed point iterations as the following lemma will show.

LEMMA 2.3. *Define the affine operator* $F : H \to H$ *by*

$$FV = \frac{1}{\gamma}T^*_{u_0}(f_0 - Vu_0) = \frac{1}{\gamma}T^*_{u_0}\left(f_0 - T_{u_0}V\right).$$

For $\|u_0\|^2_W/\gamma$ *sufficiently small,* F *is a contraction and therefore the iterations*

$$R^{p+1} = FR^p$$

converge to a fixed point of F *(the solution of* (2.3)*) for any choice of the initial guess* R^0.

PROOF. Let $U, V \in H$. Then, by Lemmas 2.1 and 2.2

$$\begin{aligned}\|FU - FV\|_H &= \frac{1}{\gamma}\left\|T^*_{u_0}(Vu_0 - Uu_0)\right\|_W \le \frac{c^{-1}}{\gamma}\|u_0\|^2_W\|V - U\| \\ &\le \frac{c^{-2}}{\gamma}\|u_0\|^2_W\|V - U\|_H .\end{aligned}$$

Therefore, the result follows for sufficiently small $\|u_0\|^2_W/\gamma$. □

2.1. The discretized problem Let $\{V_n\}_{n=1}^\infty$ be a complete set in H. That is, the set of finite linear combinations of the elements in $\{V_n\}_{n=1}^\infty$ is dense in H. Let H_n be the subspace of H given by

$$H_n = \operatorname{span}\{V_1, V_2, \dots, V_n\}$$

and let P_n be the orthogonal projection of H onto H_n with respect to the norm $\|\cdot\|_H$. Observe that $\bigcup_{n=1}^\infty H_n$ is dense in H.

LEMMA 2.4. *The sequence* $\{P_n\}_{n=1}^{\infty}$ *converges strongly to the identity operator on* H. *In other words, for every* $V \in H$, $P_n V \to V$ *as* $n \to \infty$.

PROOF. Let $V \in H$ and let $\varepsilon > 0$ be given. Since $\bigcup_{n=1}^{\infty} H_n$ is dense in H, there is an n_0 and a $\widetilde{V} \in H_{n_0}$ such that $\|V - \widetilde{V}\|_H < \varepsilon$. Now, for $n \geq n_0$, since $P_n V$ is the orthogonal projection of V in H_n and since $H_{n_0} \subset H_n$, we have

$$\|V - P_n V\|_H \leq \left\|V - \widehat{V}\right\|_H \quad \forall \widehat{V} \in H_n.$$

In particular,

$$\|V - P_n V\|_H \leq \left\|V - \widetilde{V}\right\|_H < \varepsilon. \qquad \square$$

Consider the discretized problem: Find $R_n \in H_n$ such that

$$\left\langle S R_n, \widehat{V}\right\rangle_H = \left\langle T_{u_0}^* f_0, \widehat{V}\right\rangle_H \quad \forall \widehat{V} \in H_n, \tag{2.4}$$

where

$$S = \left(\gamma I + T_{u_0}^* T_{u_0}\right),$$

or the fixed point version of it: Find $R_n \in H_n$ such that

$$\left\langle F R_n, \widehat{V}\right\rangle_H = \left\langle R_n, \widehat{V}\right\rangle_H \quad \forall \widehat{V} \in H_n, \tag{2.5}$$

with F defined as in Lemma 2.3. The operator equivalent of either (2.4) or (2.5) is

$$\left(\gamma I_n + P_n T_{u_0}^* T_{u_0}\right) R_n = P_n T_{u_0}^* f_0. \tag{2.6}$$

It is easy to see that this "matrix equation" is solvable for each $\gamma > 0$ and, for sufficiently large γ, independent of n, the fixed point formulation (2.6) converges for any choice of the initial guess R_n^0. We will proceed now to show that the sequence of solutions $\{R_n\}_{n=1}^{\infty}$ of (2.6) converges strongly to the solution of (2.3). For this purpose, it suffices to show that the operators

$$S_n := P_n S P_n, \quad n = 1, 2, \ldots \tag{2.7}$$

converge to the operator S in the *discrete-stable* sense (see [3]).

LEMMA 2.5. *The operators* S_n *defined by (2.7) converge to* S *in the discrete-stable sense.*

PROOF. To show this we have to establish two things: (1) $S_n \to S$ strongly, and (2) $\|S_n \widehat{V}\|_H \geq M \|\widehat{V}\|_H$ for all n sufficiently large and all $\widehat{V} \in H_n$. To establish (1), let $V \in H$. Since P_n converges strongly to I, $P_n V \to V$. Since S is continuous, $S P_n V \to SV$. Now

$$\begin{aligned}
\|S_n V - SV\|_H &= \|P_n S P_n V - SV\|_H \leq \|P_n S P_n V - P_n SV\|_H + \|P_n SV - SV\|_H \\
&\leq \|P_n\| \, \|S P_n V - SV\|_H + \|P_n SV - SV\|_H \\
&= \|S P_n V - SV\|_H + \|P_n SV - SV\|_H
\end{aligned}$$

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and the right-hand side goes to zero as n goes to ∞. To establish (2), let $\widehat{V} \in H_n$. Then

$$\left(S_n \widehat{V}, \widehat{V}\right)_H = \gamma \left\|\widehat{V}\right\|_H^2 + \left\|T_{u_0} \widehat{V}\right\|_H^2 \geq \gamma \left\|\widehat{V}\right\|_H^2 .$$

Therefore, (2) follows with $M := \gamma$ independent of n. □

COROLLARY 2.6. *The solutions* R_n, $n = 1, 2, \ldots$ *of either* (2.4) *or* (2.5) *converge to the solution* R *of* (2.3) *in the strong sense.*

PROOF. See [3]. □

In summary, we have shown that, for a fixed, sufficiently large n, the solution R_n of either (2.4) or (2.5) is a good approximation of the solution R of (2.3) which can be obtained by solving (2.4) or by fixed-point iteration. In the numerical experiments of Section 4 we used a Krylov Conjugate Gradient method (see [11]) to obtain a "approximate" solution of Equation (2.4).

3. The restoration problem

Before considering the problem of restoration of an image from its blurred and noised version, we need some preliminary results and notation. Denote the norm in $H^{-1}(\Omega)$ by $\|\cdot\|_{-1}$. Here $H^{-1}(\Omega)$ is a Hilbert space with the inner product defined by

$$\langle L, M \rangle = \left\langle \nabla \Delta^{-1} L, \nabla \Delta^{-1} M \right\rangle .$$

The underlying space of images is taken to be the space $BV(\Omega)$ of functions of bounded variation in the sense of measure. It is shown in [1] that $BV(\Omega)$ can be identified with those functions $u \in L^1(\Omega)$ such that u has a weak gradient, denoted Du, which extends to a continuous linear functional on $C_0(\Omega)^2$. If $u \in W^{1,1}(\Omega)$ then $Du = \nabla u$. It is known that $BV(\Omega)$ is compactly embedded in $L^1(\Omega)$ and continuously embedded in $L^2(\Omega)$. On $BV(\Omega)$ we define the functional

$$J(u) = \int_\Omega |Du| \, d\Omega ,$$

that is, the total variation of $u \in BV(\Omega)$. The problem of image restoration can be stated as

$$\text{Minimize } J(u) \tag{3.1}$$

subject to

$$\int_\Omega Ru \, d\Omega = \int f \, d\Omega, \quad \|Ru - f\|_{-1}^2 \leq \sigma^2$$

over all $u \in BV(\Omega)$ such that $Ru \in H^{-1}(\Omega)$.

It will be shown below that the above minimization problem is equivalent to

$$\text{Minimize } E(u) := J(u) + \frac{\lambda}{2} \|Ru - f\|_{-1}^2. \tag{3.2}$$

Proposition 3.1 and its corollary generalize the results in [2] where the $L^2(\Omega)$ norm was used. The function J can be regarded as a convex and lower semicontinuous function on $L^2(\Omega)$, ($J(u) = +\infty$ if $u \notin BV(\Omega)$).

The following assumptions are made:

A1. $R : H^{-1}(\Omega) \to H^{-1}(\Omega)$ is a continuous linear operator.

A2. $R1 = 1$.

A3. $\|f - \int f\|_{-1} \geq \sigma$. See the discussion in [2].

REMARK. Observe that A1 is satisfied, for example if $R : L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator since $L^2(\Omega)$ is densely and compactly embedded in $H^{-1}(\Omega)$. The (continuous) adjoint operator R^* is understood as a map from $H^{-1}(\Omega)$ into itself.

PROPOSITION 3.1. *Suppose* $f \in \overline{R(BV(\Omega))}$. *If* $u \in R(BV(\Omega))$ *is a solution of* (3.1) *then there exists a* $\lambda \geq 0$ *such that*

$$-\lambda R^* (Ru - f) \in \partial J(u).$$

Here $\partial J(u) \subset H^{-1}(\Omega)$ *is the subdifferential of* J *at* u.

PROOF. Set

$$G(u) = \chi_{\overline{B}(f,\sigma)}(u) = \begin{cases} 1, & u \in \overline{B}(f,\sigma) \\ \infty, & u \notin \overline{B}(f,\sigma) \end{cases},$$

where $\overline{B}(f,\sigma)$ is the closed ball in $H^{-1}(\Omega)$ centred at f with radius σ. Problem (3.1) is equivalent to

$$\min J(u) + G(Ru). \tag{3.3}$$

It can be shown (see [2] or [6]) that, under the assumption $f \in \overline{R(BV(\Omega))}$,

$$\partial (J + G \circ R)(u) = \partial J(u) + \partial (G \circ R)(u) \quad \text{and}$$
$$\partial (G \circ R)(u) = R^* \partial G(Ru)$$

with $\partial G(u) = \{0\}$ if $u \in B(f,\sigma)$ and, for $u \in \partial B(f,\sigma)$,

$$\partial G(u) = \{\lambda \mathscr{F}(u - f) : \lambda \geq 0\},$$

where $\mathscr{F} : H^{-1}(\Omega) \to \left(H^{-1}(\Omega)\right)^*$ is the duality mapping defined by

$$\mathscr{F}(u) = \left\{ z \in \left(H^{-1}(\Omega)\right)^* : \|z\|_{(H^{-1}(\Omega))^*} = \|u\|_{-1},\ \langle u, z\rangle = \|u\|_{-1}^2 \right\}.$$

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Such a mapping exists by the the Hahn-Banach Theorem (see [21]). Observe that $\mathscr{F}(u) = u$ if $(H^{-1}(\Omega))^*$ is identified with $H^{-1}(\Omega)$ itself or $\mathscr{F}(u) = \Delta^{-1}u$ if $(H^{-1}(\Omega))^*$ is identified with $H_0^1(\Omega)$. In our case $\mathscr{F}(u) = u$. Thus

$$\partial\,(J + G \circ R)\,(u) = \partial J\,(u) + R^*\partial G\,(Ru)\,.$$

If u is a solution of (3.3) then $0 \in \partial(J + G \circ R)(u)$. Since any solution of (3.1) satisfies $\|Ru - f\|_{-1} = \sigma$ (a slight modification of the argument in [2] p. 170), this shows that there exists a $\lambda \geq 0$ such that

$$0 \in \partial J\,(u) + \lambda R^*\,(Ru - f)\,. \qquad \square$$

COROLLARY 3.2. *The minimization problem* (3.1) *is equivalent to the minimization problem* (3.2) *for all* $\lambda \geq 0$.

PROOF. By Proposition 3.1, a minimizer u of (3.1) satisfies

$$0 \in \partial J(u) + \lambda R^*(Ru - f).$$

Therefore, u minimizes (3.2). On the other hand, a minimizer u of (3.2) also minimizes (3.1) with $\|Ru - f\|_{-1} = \sigma$ (see [2] p. 170). $\square$

We show next that problem (3.1) has a "unique" minimizer. Our proof is partly motivated by the argument in [14] p. 354.

THEOREM 3.3. *Assume that* $f \in \overline{R(BV(\Omega))}$. *Then* (3.1) *has a solution* $u \in BV(\Omega)$ *and* $Ru \in H^{-1}(\Omega)$ *is unique.*

PROOF. Let $\{u_n\}$ be a minimizing sequence for (3.1) satisfying the constraints. Then $J(u_n) \leq M$. From the generalized Poincaré inequality (see [22], Lemma 4.1.3)

$$\left\| u_n - \frac{\int_\Omega Ru_n d\Omega}{|\Omega|} \right\|_{L^1(\Omega)} \leq CJ\,(u_n)\,.$$

Since $\int Ru_n = \int f$,

$$\left\| u_n - \frac{\int_\Omega f d\Omega}{|\Omega|} \right\|_{L^1(\Omega)} \leq M.$$

We conclude that $\{u_n\}$ is bounded in $L^1(\Omega)$ and, consequently, in $L^2(\Omega)$. Thus $\{u_n\}$ is $BV(\Omega)$ bounded. Since $BV(\Omega)$ is compactly embedded in $L^1(\Omega)$, (a subsequence) $u_n \to u$ in $L^1(\Omega)$ and

$$J(u_n) \leq \liminf J(u_n).$$

Also

$$\int_\Omega f d\Omega = \int_\Omega Ru_n d\Omega = \int_\Omega Ru d\Omega.$$

On the other hand, since $BV(\Omega)$ is continuously embedded in $L^2(\Omega)$, (a subsequence) $u_n \rightharpoonup u$ in $L^2(\Omega)$. By the continuity and linearity of R we have $Ru_n \rightharpoonup Ru$ in $H^{-1}(\Omega)$. Therefore,

$$\|f - Ru\|_{-1} \leq \liminf \|f - Ru_n\|_{-1} \leq \sigma.$$

Hence, u is a solution of (3.1). It can also be shown that u satisfies $\|Ru - f\|_{-1}^2 = \sigma^2$ (see [2] p. 170).

To show uniqueness, assume that u and v are solutions of (3.1), then

$$J\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\left(J(u) + J(v)\right) = \min J$$

and

$$\int_\Omega R\frac{u+v}{2} d\Omega = \int_\Omega f d\Omega,$$

$$\left\| R\frac{u+v}{2} - f \right\|_{-1}^2 \leq \sigma^2.$$

Hence, $(u+v)/2$ is a solution of (3.1). Consequently,

$$\left\| R\frac{u+v}{2} - f \right\|_{-1}^2 = \sigma^2 = \|Ru - f\|_{-1}^2 = \|Rv - f\|_{-1}^2.$$

By the strict convexity of $\|\cdot\|_{-1}^2$ we conclude that $Ru = Rv$. □

Following [14], the formal computation of the Euler-Lagrange equation for (3.2) yields

$$\operatorname{div} \frac{Du}{|Du|} = \lambda \Delta^{-1} R^* (Ru - f), \tag{3.4}$$

$$\frac{\partial \operatorname{div} \frac{Du}{|Du|}}{\partial n} |_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial n} |_{\partial\Omega} = 0 \tag{3.5}$$

where $\partial u/\partial n$ is the outward normal derivative of u and, for $v \in \mathbb{R}^2$,

$$|v| = \sqrt{v_1^2 + v_2^2}.$$

To see how the derivative of $\|Ru - f\|_{-1}^2$ is computed, we proceed as follows. Let

$$K(u) = \|Ru - f\|_{-1}^2.$$

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Then, for $h \in C_0^\infty(\Omega)$,

$$K(u+th) = \|Ru - f\|_{-1}^2 + 2t\langle Ru - f, Rh\rangle + t^2\|Rh\|_{-1}^2 .$$

Therefore,

$$\begin{aligned}\frac{d}{dt}K(u+th)_{|t=0} &= 2\langle Ru - f, Rh\rangle = 2\langle R^*(Ru-f), h\rangle \\ &= 2\left(\nabla\Delta^{-1}R^*(Ru-f), \nabla\Delta^{-1}h\right) = -2\left(\Delta^{-1}R^*(Ru-f), h\right).\end{aligned}$$

This gives the right-hand side of (3.4).

Equation (3.4) is solved by driving to steady state

$$\begin{gathered} u_t = -\left(\operatorname{div}\frac{Du}{|Du|} - \lambda\Delta^{-1}R^*(Ru-f)\right), \\ \left.\frac{\partial \operatorname{div}\frac{Du}{|Du|}}{\partial n}\right|_{\partial\Omega} = 0, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \\ u(0) = f. \end{gathered} \tag{3.6}$$

To justify this procedure, we show that the energy $E(u)$ in (3.2) decreases with time. The PDE (3.6) can be written as

$$u_t = -E'(u).$$

Now,

$$E : BV(\Omega) \to \mathbb{R} \quad \Rightarrow \quad E'(u) \in (BV(\Omega))^* \subset H^{-1}(\Omega) \quad \forall u \in BV(\Omega).$$

Thus, considering $u_t \in H^{-1}(\Omega)$, we have

$$\frac{d}{dt}E(u) = \left(E'(u), u_t\right) = -\|E'(u)\|_{-1}^2 .$$

4. Numerical Experiments

Experiment results with simulated blurred and noised images are described in this section. The fixed-point iteration technique described in Sections 2 and 3 was used for estimating the unknown parameters of the blur operator R. Since the technique is globally convergent, we started with an initial guess of $R = 0$. We also assumed the *a priori* knowledge of a source image to compute the blur operator. The blur operator or PSF representation of the blurring operator R was assumed to have a support region of 30×11 pixels. The image in Figure 1(b) was obtained by adding Gaussian noise with zero mean and variance of 0.01. Figure 1(c) shows the noisy-blurred picture. Restoration was done using the technique described in Section 3. The

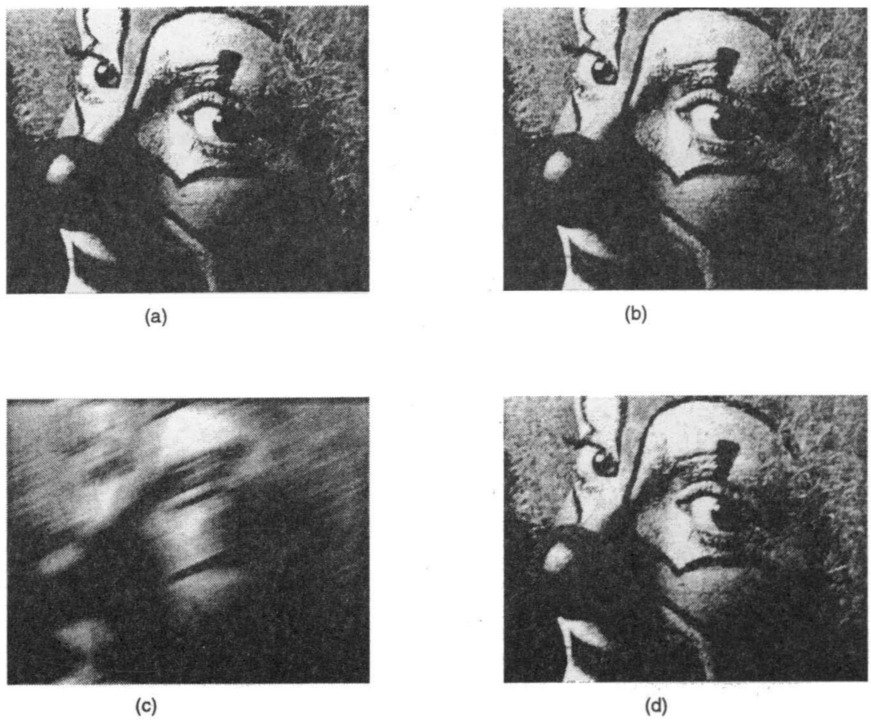

FIGURE 1. (a) Original picture, (b) Picture with added noise of mean zero and variance 0.01, (c) Blurred picture with added noise, (d) Restored picture.

initial estimation was selected to be the blurred image. The calculation was terminated when the difference between two consecutive iterations was less than 1.0E-4.

The restored image is shown in Figure 1(d) and for comparison the original image is included in Figure 1(a). A general observation with this approach is that increased sharpness in the restored image is traded with noise amplification. As shown in Figure 1(d), the technique enhances the quality of the picture by reducing the noise and removing the effect of the blur while preserving the texture of the image. This simulation case confirms the effectiveness of the technique in removing the blurring effect and reducing the noise.

The value of the weighting factor λ was traded between noise reduction and the sharpness of the picture. The value of λ that gives a good visual picture varies from one picture to another. For this particular case, the picture shown was obtained for $\lambda = 200$. See [14] for more discussion on the choice of λ.

Before performing the calculations, the values of the pixels of the pictures were normalized to the interval [0, 1]. This is why a large value weighting factor λ was used.

The technique was also tested for various type of images. The simulation results

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shown in Figure 2 are for a coloured image and in Figure 3 are fingerprints.

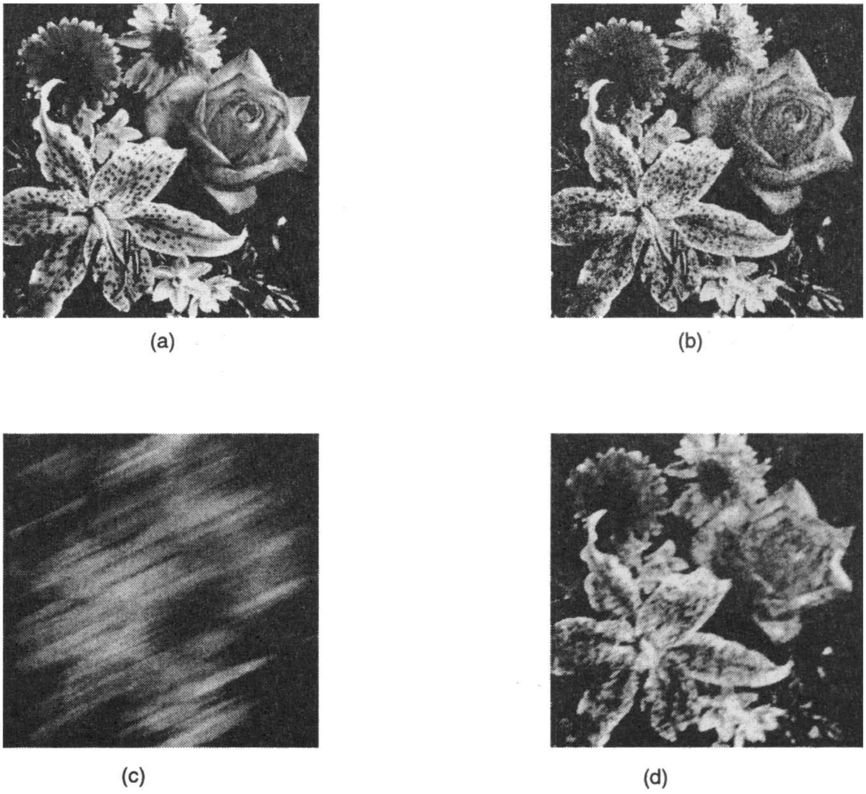

FIGURE 2. (a) Original picture, (b) Picture with added noise of mean zero and variance 0.01, (c) Blurred picture with added noise, (d) Restored picture.

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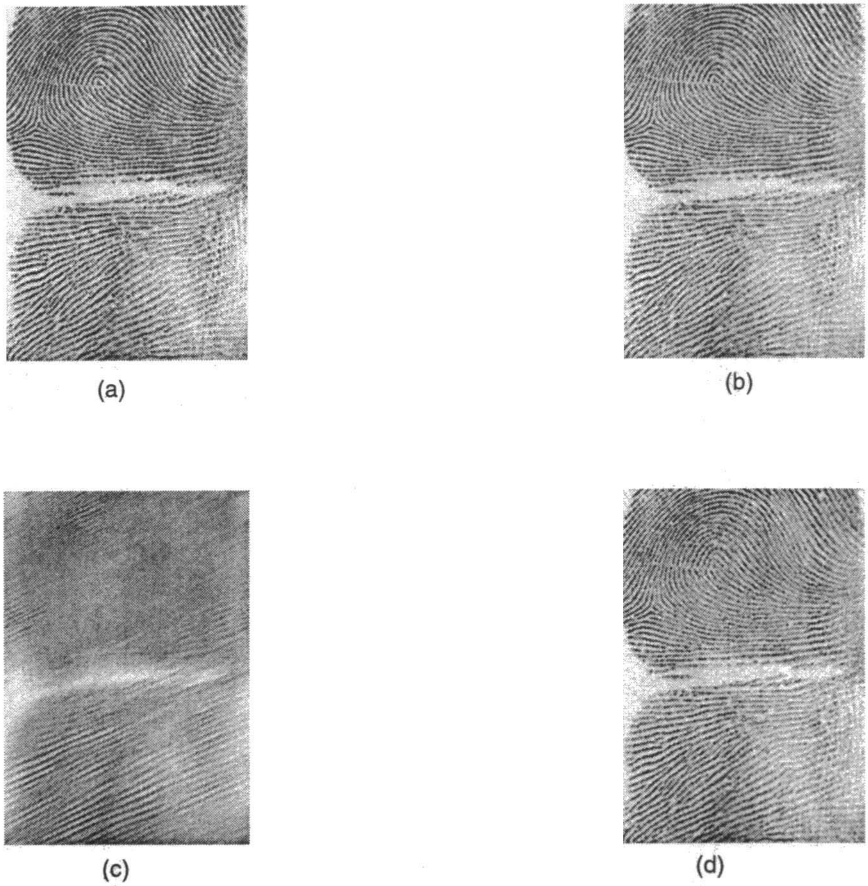

FIGURE 3. (a) Original picture, (b) Picture with added noise of mean zero and variance 0.01, (c) Blurred picture with added noise, (d) Restored picture.

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