The Minimal Resolution Conjecture for Points on the Cubic Surface

M. Casanellas

Abstract. In this paper we prove that a generalized version of the Minimal Resolution Conjecture given by Mustață holds for certain general sets of points on a smooth cubic surface $X \subset \mathbb{P}^3$. The main tool used is Gorenstein liaison theory and, more precisely, the relationship between the free resolutions of two linked schemes.

1 Introduction

The Minimal Resolution Conjecture for points in projective spaces was first stated by Lorenzini in [12]. Roughly speaking it says that the graded minimal free resolution of a general set of t points in $\mathbb{P}^n$ has no *ghost* terms. This conjecture is known to hold for $n \leq 4$ ([1, 16]) and for large values of t for any n (cf. [11]), but it does not hold in general. Eisenbud, Popescu, Schreyer, and Walter [6] proved that it fails for every $n \geq 6$, $n \neq 9$. Farkas, Mustață, and Popa introduced a generalized version of this conjecture for points in arbitrary projective varieties (see [15] and [7]). Namely, if $X \subset \mathbb{P}^n$ is any projective variety, the Minimal Resolution Conjecture for a general set of t points Z on X predicts that the Betti numbers of the ideal of Z are completely determined by those of the ideal of X (see [7] and Section 2 for a precise statement of the conjecture). When $X = \mathbb{P}^n$, this formulation of the conjecture coincides with the conjecture formulated by Lorenzini.

This generalization of the Minimal Resolution Conjecture has been studied in [7] and has been proven to hold for a general set of points of any sufficiently large degree when $X \subset \mathbb{P}^n$ is a canonical curve. However, it always fails for sets of points on curves of large degree (see [7]). Guiffrida, Maggioni, and Ragusa [9] proved that this conjecture also holds for any general set of points when X is a smooth quadric surface in $\mathbb{P}^3$. In this paper we study the Minimal Resolution Conjecture for general sets of points on a smooth cubic surface in $\mathbb{P}^3$.

The goal of this paper is to prove that t general points on a smooth cubic surface $X \subset \mathbb{P}^3$ satisfy the Minimal Resolution Conjecture when t is equal to $\frac{3}{2}a(a-1) + a$, $\frac{3}{2}a(a-1) + 2a$, $\frac{3}{2}a(a-1) + a + 1$ or $\frac{3}{2}a(a-1) + 2a + 1$ for some $a \in \mathbb{Z}$ (see Theorem 3.2). We also give the precise minimal free resolutions of these sets of points. All these sets of points happen to be *level* so, in particular, this result provides proofs to some unjustified examples in [8] Appendix C (namely that 35, 36, 39, 41, and 42 general points on the cubic surface are level). It also provides a partial answer to Question 7.5

of [8] "*If X is an integral arithmetically Cohen–Macaulay variety of dimension ≤ 5, and Z is a general set of points on X, then does Z have the expected resolution? In particular, is it true that Z is level if it is numerically possible?*" and gives hope for an affirmative answer to this question.

Whereas [15] and [7] used Koszul cohomology to study the Minimal Resolution Conjecture for points on curves, the main tool used in this paper is Gorenstein liaison theory. R. Hartshorne proved in [10] that a general set of points in the cubic surface X can be Gorenstein linked in X to one point by a finite sequence of links. We make use of the G-links he describes to track the minimal free resolutions of the sets of points we are considering. It is a well known result in liaison theory that there is a relationship between the free resolutions of two linked schemes, but one needs to be careful because the minimality of the resolutions is not preserved in general.

Our initial motivation for studying the resolution of sets of points on the cubic surface was the connection between level sets of points and Ulrich bundles in the sense of [3], but this will be explained in a forthcoming paper.

The paper is organized as follows. In the next section we state the Minimal Resolution Conjecture and provide the results of Gorenstein liaison theory needed for the sequel. Section 3 is devoted to proving the main result of the paper.

2 Preliminaries

Throughout this section R denotes the polynomial ring in $n + 1$ variables over an algebraically closed field k. Throughout the paper, for a subscheme $Z \subset \mathbb{P}^n_k$ we denote by $I_Z \subset R$ its saturated ideal, *i.e.,* $I_Z = \oplus_{t \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{J}_Z(t))$. For any coherent sheaf $\mathcal{F}$ over a projective scheme $X \subset \mathbb{P}^n$, $H^i_*(\mathcal{F})$ denotes the sum $\oplus_{t \in \mathbb{Z}} H^i(X, \mathcal{F}(t))$. The regularity reg$(Z)$ of Z is defined to be the regularity of I_Z, if $Z \neq \mathbb{P}^n$ and 1 otherwise. The Hilbert polynomial of Z will be denoted as P_Z.

The results of this paper deal with minimal free resolutions of graded R-modules. We will always talk about *graded* minimal free resolutions over R.

Definition 2.1   If $Z \subset \mathbb{P}^n$ is a subscheme and

$$d_\bullet \quad 0 \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \to F_1 \xrightarrow{d_1} R \xrightarrow{d_0} R/I_Z \to 0$$

is the minimal free resolution of R/I_Z, the *Betti numbers* $b_{i,j}(Z)$ are defined as

$$F_i = \oplus_{j \in \mathbb{Z}} R(-i - j)^{b_{i,j}(Z)}.$$

The *Betti diagram* of Z has in the (j, i)-th position the Betti number $b_{i,j}(Z)$. The last nontrivial row of the Betti diagram of Z is indexed by reg$(Z) - 1$ (see [5]).

Minimality criterion   We recall that a free resolution

$$d_\bullet \quad 0 \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

is minimal if, after choosing a basis of the modules F_i, the matrices representing the maps d_i do not have any non-zero scalar entry.

Minimal Resolution Conjecture for points on embedded varieties (See [7].) Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension ≥ 1 and denote by P_X its Hilbert polynomial. Let Z be a *general* set of z points in X. In this paper we say that a statement holds for a *general Z* if there is a nonempty open subset of the family of all Z's for which the statement holds. We choose r such that $P_X(r-1) \leq z < P_X(r)$ and assume that $r \geq \mathrm{reg}(X) + 1$. Then the *Minimal Resolution Conjecture* formulated in [7] holds for the value z if for every set Z of z general points on X,

$$b_{i+1,r-1}(Z)b_{i,r}(Z) = 0 \text{ for all } i.$$

Mustață [15] proved that the first rows of the Betti diagram of a general set of points Z on X coincide with the Betti diagram of X and that there are two extra nontrivial rows at the bottom. He also gives lower bounds for the Betti numbers in these last two rows, and the Minimal Resolution Conjecture states that these lower bounds are attained for a general set of points. More precisely, his results can be summarized as follows.

Theorem 2.2 ([7, Theorem 1.2]) *Assume that $Z \subseteq X$ is a general set of z points, with $P_X(r-1) \leq z < P_X(r)$ for some $r \geq m+1$, where $m = \mathrm{reg}\, X$.*

(i)   *For every i and $j \leq r-2$, we have $b_{i,j}(Z) = b_{i,j}(X)$.*
(ii) $b_{i,j}(Z) = 0$, *for $j \geq r+1$ and there is an i such that $b_{i,r-1}(Z) \neq 0$.*
(iii) *If $d = \dim X$, then for every $i \geq 0$, we have $b_{i+1,r-1}(Z) - b_{i,r}(Z) = Q_{i,r}(z)$, where*

$$Q_{i,r}(z) = \sum_{l=0}^{d-1} (-1)^l \binom{n-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{n}{i}(z - P_X(r-1)).$$

In particular, $b_{i+1,r-1}(Z) \geq \max\{Q_{i,r}(z), 0\}$ and $b_{i,r}(Z) \geq \max\{-Q_{i,r}(z), 0\}$.

In other words, for $z \geq P_X(r-1)$ and $r \geq m+1 = \mathrm{reg}(X)+1$, the Minimal Resolution Conjecture is satisfied if and only if the minimal free resolution of Z has no *ghost* terms (*i.e.*, there are no identical free summands in two consecutive steps of the resolution). Note that the Betti numbers of X do not overlap with those of Z because we are requiring $r \geq m+1$. Indeed, the last row in the Betti diagram of X is indexed by $m-1$ while the part of the Betti diagram of Z that is not determined by $b_{i,j}(X)$ occurs in the rows $r-1$ and r. Here is an example of how the Betti numbers of X determine those of Z if the Minimal Resolution Conjecture holds.

Example 2.3 Let Z be a set of $z = 22$ general points on a smooth cubic surface $X \subset \mathbb{P}^3$. As $P_X(t) = \frac{3}{2}t(t+1) + 1$ and $\mathrm{reg}(X) = 3$, we take $r = 4$ so that $P_X(3) = 19 \leq 22 < P_X(4) = 31$. According to Theorem 2.2, the first 2 rows of the Betti diagram of Z coincide with the Betti diagram of X, and in rows 3 and 4 we have $b_{1,3} - b_{0,4} = 9$, $b_{2,3} - b_{1,4} = 12$, $b_{3,3} - b_{2,4} = 9$, $b_{4,3} - b_{3,4} = -3$. If the Minimal Resolution Conjecture holds for Z, one of the terms in each of these differences is 0. Therefore, as $b_{i,j}(Z)$ is always positive and $b_{i,j}(Z) = 0$ for $i \geq 4$, the Betti diagram should be as follows

	0	1	2	3
0	1	–	–	–
1	–	–	–	–
2	–	1	–	–
3	–	9	12	–
4	–	–	–	3

or, equivalently, the minimal free resolution should be

$$0 \to R(-7)^3 \to R(-5)^{12} \to R(-4)^9 \oplus R(-3) \to I_Z \to 0.$$

In Theorem 3.2 we will prove that, indeed, 22 general points on a smooth cubic surface satisfy the Minimal Resolution Conjecture.

Example 2.4 Mustaţă proved in [15] that when $z = P_X(r-1)$ or $z = P_X(r) - 1$, the Minimal Resolution Conjecture holds for every X.

When $X = \mathbb{P}^n$ the Minimal Resolution Conjecture has been deeply studied and is known to hold for $n \leq 4$ (see [1, 16]). However, it fails in the middle of the resolution for every $n \geq 6$, $n \neq 9$. The conjecture is also known to hold for any set of points on a smooth quadric surface in $\mathbb{P}^3$. (See [9]; although this paper precedes [15], they precisely prove that the minimal free resolution of any set of general points on a smooth quadric X is completely determined by the Betti numbers of X.) It is also known to hold, among other cases, for any set of general points on a canonical curve X and to fail when $X \subset \mathbb{P}^n$ is a curve of large degree (cf. [7]).

The main tool we will use to study the Minimal Resolution Conjecture on the cubic surface is Gorenstein liaison theory. A good reference for liaison theory is [14]. Here we just recall the definitions we need for this paper and some tools related to Gorenstein liaison.

We recall that a scheme $V \subset \mathbb{P}^N$ is an *arithmetically Cohen–Macaulay* scheme if its homogeneous coordinate ring is a Cohen–Macaulay ring (*i.e.,* $\dim R/I_V = \operatorname{depth} R/I_V$). If V is of dimension $d \geq 1$, V is arithmetically Cohen–Macaulay if and only if $H^i_*(\mathbb{P}^n, \mathcal{J}_V) = 0$ for $i = 1, \ldots, d$. Any 0-dimensional scheme is arithmetically Cohen–Macaulay. An obvious fact that we will use throughout the paper is the following: if $C \subset \mathbb{P}^n$ is arithmetically Cohen–Macaulay of dimension ≥ 1 and $Z \subset C$ is any subscheme, then $I_{Z,C} = H^0_*(\mathcal{J}_{Z,C}) = I_Z/I_C$.

A closed subscheme $V \subset \mathbb{P}^n$ of codimension c is *arithmetically Gorenstein* if its coordinate ring R/I_V is a Gorenstein *local graded ring (in the sense of [2]). This is equivalent to saying that its saturated homogeneous ideal I_V has a graded minimal free R-resolution of the following type:

$$0 \to R(-t) \to F_{c-1} \to \cdots \to F_1 \to F_0 \to I_V \to 0.$$

In other words, $V \subset \mathbb{P}^n$ is arithmetically Gorenstein if and only if V is arithmetically Cohen–Macaulay and the last module in the minimal free resolution of its saturated ideal has rank one.

Definition 2.5 ([14]) We say that two subschemes V_1 and V_2 of $\mathbb{P}^n$ are *directly Gorenstein linked*, or simply *directly G-linked*, by an arithmetically Gorenstein scheme $G \subset \mathbb{P}^n$ if $I_G \subset I_{V_1} \cap I_{V_2}$ and we have $I_G : I_{V_1} = I_{V_2}$ and $I_G : I_{V_2} = I_{V_1}$. We say that V_2 is the *residual* to V_1 in G. When G is a complete intersection we talk about a *CI-link*.

Whenever V_1 and V_2 do not share any common component, the fact of being directly *G*-linked by a scheme G is equivalent to saying that $G = V_1 \cup V_2$. *Gorenstein liaison* studies the equivalence relation generated by *G*-links.

One of the most interesting properties preserved by a *G*-link is the free resolution. To explain how it works we first recall the mapping cone procedure.

Lemma 2.6 (**Mapping cone procedure** [13]) *Given a short exact sequence of finitely generated R-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0,$$

and free resolutions

$$e_\bullet \quad 0 \to G_{n+1} \xrightarrow{e_{n+1}} G_n \xrightarrow{e_n} \cdots \xrightarrow{e_1} G_0 \xrightarrow{e_0} A \to 0$$

and

$$d_\bullet \quad 0 \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} B \to 0,$$

then the map α lifts to a map between the resolutions $\xi_\bullet : e_\bullet \longrightarrow d_\bullet$, and a free resolution for C is

$$0 \to G_{n+1} \xrightarrow{c_{n+2}} G_n \oplus F_{n+1} \xrightarrow{c_{n+1}} \cdots \xrightarrow{c_3} G_1 \oplus F_2 \xrightarrow{c_2} G_0 \oplus F_1 \xrightarrow{c_1} F_0 \xrightarrow{c_0} C \to 0$$

where

$$c_{i+1} = \begin{pmatrix} -e_i & 0 \\ \xi_i & d_{i+1} \end{pmatrix}, \quad 1 \le i \le n.$$

The resolution of C produced in Lemma 2.6 above is not necessarily minimal even if those of A and B are.

In the following lemma we recall how to pass from the free resolution of a scheme to the free resolution of its residual in an arithmetically Gorenstein scheme.

Lemma 2.7 (see [14]) *Let $Z, Z' \subset \mathbb{P}^n$ be two arithmetically Cohen–Macaulay subschemes of codimension c directly G-linked by an arithmetically Gorenstein scheme G. Let the minimal free resolutions of I_Z and I_G be*

$$0 \to F_c \xrightarrow{d_c} F_{c-1} \xrightarrow{d_{c-1}} \cdots \to F_1 \xrightarrow{d_1} I_Z \to 0$$

and

$$0 \to R(-t) \xrightarrow{e_c} G_{c-1} \xrightarrow{e_{c-1}} \cdots \to G_1 \xrightarrow{e_1} I_G \to 0$$

respectively. Then the functor $\mathrm{Hom}_R(\cdot, R(-t))$ applied to a free resolution of I_Z/I_G gives a free resolution of Z'. In particular,

$$0 \to F_1^\vee(-t) \to F_2^\vee(-t) \oplus G_1^\vee(-t) \to \cdots \to F_c^\vee(-t) \oplus G_{c-1}^\vee(-t) \to I_{Z'} \to 0$$

is a free resolution of $I_{Z'}$ (not necessarily minimal) obtained by mapping cone from the resolutions of I_G and I_Z.

3   Minimal Resolution Conjecture on the Cubic Surface

Let X be a smooth cubic surface in $\mathbb{P}^3$ defined by a homogeneous cubic form f. From now on R denotes the polynomial ring $k[x_0, \ldots, x_3]$ over an algebraically closed field k and R_X denotes the ring $R_X = R/(f)$. When Z is a subscheme of X, I_Z denotes its saturated ideal in R and $I_{Z,X} \subset R_X$ the ideal $I_Z/(f) = H^0_*(\mathcal{J}_{Z,X})$.

As the Hilbert polynomial of X is $P_X(t) = \frac{3}{2}t(t+1) + 1$, $P_X(t)$ general points on X satisfy the Minimal Resolution Conjecture and so do $P_X(t) - 1$ general points on X (see Example 2.4). We are going to prove that other families of general points in X satisfy the Minimal Resolution Conjecture.

Notation 3.1   We set the following notation.

(i)   If t is a positive integer, Z_t denotes a set of t general points in X.
(ii) $m(a) = \frac{3}{2}a(a-1) + a$, $n(a) = \frac{3}{2}a(a-1) + 2a$, $o(a) = \frac{3}{2}a(a-1) + a + 1$, $p(a) = \frac{3}{2}a(a-1) + 2a + 1$ for any $a \in \mathbb{Z}$.
(iii) C_0 is any smooth conic on X, Γ any twisted cubic on X and L any of the 27 lines in X.
(iv) H denotes a general hyperplane section of X.
(v)   If C is a curve on X, H_C will be a general hyperplane section of C and K_C a canonical divisor on C.

The curves C_0, Γ and L are arithmetically Cohen–Macaulay curves and, according to the proof of [10, Proposition 2.4], any arithmetically Cohen–Macaulay curve on X is linearly equivalent to $C_0 + aH_X$, $\Gamma + aH$, $L + aH$ or aH, for some $a \in \mathbb{N}$. The modules $H^0_*(\mathbb{P}^3, \mathcal{J}_{C_0,X})$, $H^0_*(\mathbb{P}^3, \mathcal{J}_{\Gamma,X})$ and $H^0_*(\mathbb{P}^3, \mathcal{J}_{L,X})$ are maximal Cohen–Macaulay R_X-modules and their minimal free R-resolutions are:

$$0 \longrightarrow R(-3)^2 \xrightarrow{\varphi_1} R(-1) \oplus R(-2) \longrightarrow I_{C_0,X} \longrightarrow 0,$$

$$0 \longrightarrow R(-3)^3 \xrightarrow{\varphi_2} R(-2)^3 \longrightarrow I_{\Gamma,X} \longrightarrow 0,$$

$$0 \longrightarrow R(-2) \oplus R(-3) \xrightarrow{\varphi_3} R(-1)^2 \longrightarrow I_{L,X} \longrightarrow 0.$$

For certain matrices ψ_j, the pairs (φ_j, ψ_j) are matrix factorizations of f (see [4]): $\varphi_j \cdot \psi_j = f \cdot Id$, $\psi_j \varphi_j = f \cdot Id$, $f = \det(\varphi_j) = \det(\psi_j)$, $j = 1, 2, 3$.

The degree and genus of smooth arithmetically Cohen–Macaulay curves C on X are:

(1) $d = 3a - 2$, $g = \frac{1}{2}(3a^2 - 7a + 4)$ if $C \sim L + (a-1)H$.
(2) $d = 3a - 1$, $g = \frac{1}{2}(3a^2 - 5a + 2)$ if $C \sim C_0 + (a-1)H$.
(3) $d = 3a$, $g = \frac{1}{2}(3a^2 - 3a)$ if $C \sim \Gamma + (a-1)H$.
(4) $d = 3a$, $g = \frac{1}{2}(3a^2 - 3a + 2)$ if $C \sim aH$.

En each case, one of these curves moves on a linear system of dimension $d + g - 1$. Note that the degree and genus of an arithmetically Cohen–Macaulay curve C determine the linear system in which C belongs. Indeed, the four possibilities above imply that if two arithmetically Cohen–Macaulay curves on X have the same degree and genus, they belong to the same linear system.

The goal of this paper is to prove the following result.

Theorem 3.2 *Let $X \subset \mathbb{P}^3$ be a smooth cubic surface and consider the above notation. Then, for $a \geq 3$, the graded minimal free resolutions of $m(a)$, $n(a)$, $o(a)$, and $p(a)$ general points on X are:*

$$0 \to R(-a-3)^{a-1} \to R(-a-1)^{3a} \to R(-a)^{2a+1} \oplus R(-3) \to I_{Z_{m(a)}} \to 0,$$

$$0 \to R(-a-3)^{2a-1} \to R(-a-2)^{3a} \to R(-a)^{a+1} \oplus R(-3) \to I_{Z_{n(a)}} \to 0,$$

$$0 \to R(-a-3)^a \to R(-a-1)^{3a-3} \oplus R(-a-2)^3 \to R(-a)^{2a} \oplus R(-3)$$
$$\to I_{Z_{o(a)}} \to 0,$$

$$0 \to R(-a-3)^{2a} \to R(-a-2)^{3a+3} \to R(-a)^a \oplus R(-a-1)^3 \oplus R(-3)$$
$$\to I_{Z_{p(a)}} \to 0.$$

As a consequence, the Minimal Resolution Conjecture on X holds for $m(a)$, $n(a)$, $o(a)$, and $p(a)$ whenever $a \geq 1$.

Our results are based on the proof of a Proposition of Hartshorne ([10, Proposition 2.4]), where it is proved that any set of general points on X can be G-linked to a point in X by a finite number of G-links. The main idea used to prove that result was the following lemma.

Lemma 3.3 *Let d, g be the degree and genus of a smooth arithmetically Cohen–Macaulay curve D on X and let $n, n' \in \mathbb{N}$ satisfy $g \leq n, n' \leq d + g - 1$ and $n + n' = \deg(mH_D - K_D)$ for some m. Then n (respectively n') general points on X lie on a smooth arithmetically Cohen–Macaulay curve C in the linear system $|D|$ and there is a divisor $G \subset C$ linearly equivalent to $mH_C - K_C$ which links n general points to n' general points.*

Proof See the proof of [10, Proposition 2.4]. ∎

Remark 3.4. (a) In applying Lemma 3.3 it will be important to keep in mind that, for an arithmetically Cohen–Macaulay curve C on X, the couple (d, g) of degree and genus of C determines the linear system in which C belongs (see the paragraph preceding Theorem 3.2).

(b) In the hypothesis of Lemma 3.3, n (respectively n') general points on X form a general divisor of degree n on C. However, the results of [15] for general points on curves do not apply to prove the Minimal Resolution Conjecture for $m(a)$, $n(a)$, $o(a)$ or $p(a)$.

We follow the proof of [10, Proposition 2.4] in reverse order: we perform links to obtain larger sets of points. In this way and according to [10, Proposition 2.4], we can pass from $Z_{m(a-1)}$ general points on X to $Z_{m(a+1)}$ general points by the following sequence of G-links in X:

- First link: $Z_{m(a-1)}$ is G-linked to $Z_{n(a-1)}$ by an arithmetically Gorenstein set of points G linearly equivalent to $(a-1)H_C$ on a smooth curve C linearly equivalent to $(a-1)H$ on X.

- Second link: $Z_{n(a-1)}$ is G-linked to $Z_{o(a)}$ by an arithmetically Gorenstein set of points G linearly equivalent to $(2a-2)H_C - K_C$ on a smooth curve C linearly equivalent to $C_0 + (a-1)H$ on X.
- Third link: $Z_{o(a)}$ is G-linked to $Z_{p(a)}$ by an arithmetically Gorenstein set of points G linearly equivalent to $(2a-1)H_C - K_C$ on a smooth curve C linearly equivalent to $\Gamma + (a-1)H$ on X.
- Fourth link: $Z_{p(a)}$ is G-linked to $Z_{m(a+1)}$ by an arithmetically Gorenstein set of points G linearly equivalent to $2aH_C - K_C$ on a smooth curve C linearly equivalent to $C_0 + aH$ on X.

This sequence of linkages will allow us to deduce the minimal free resolution of $Z_{m(a+1)}$ from that of $Z_{m(a-1)}$. To do this we need the following four results corresponding to the four links mentioned above:

Proposition 3.5 (First link) *Assume that the ideal $I_{Z_{m(a)},X} \subset R_X$ of $m(a) = \frac{3}{2}a(a-1) + a$ general points on X has the following graded minimal free resolution over R*

$$0 \to R(-a-3)^{a-1} \to R(-a-1)^{3a} \to R(-a)^{2a+1} \to I_{Z_{m(a)},X} \to 0.$$

Then the ideal $I_{Z_{n(a)},X} \subset S$ of $n(a) = \frac{3}{2}a(a-1)+2a$ general points on X has the following minimal free resolution over R:

$$0 \to R(-a-3)^{2a-1} \to R(-a-2)^{3a} \to R(-a)^{a+1} \to I_{Z_{n(a)},X} \to 0.$$

Proposition 3.6 (Second link) *Assume that the ideal $I_{Z_{n(a-1)},X} \subset R_X$ of $n(a-1) = \frac{3}{2}(a-2)(a-1) + 2a - 2$ general points on X has the following minimal free resolution over R*

$$0 \to R(-a-2)^{2a-3} \to R(-a-1)^{3a-3} \to R(-a+1)^{a} \to I_{Z_{n(a-1)},X} \to 0.$$

Then the ideal $I_{Z_{o(a)},X}$ of $o(a) = \frac{3}{2}a(a-1) + a + 1$ general points on X has the following minimal free resolution over R :

$$0 \to R(-a-3)^{a} \to R(-a-1)^{3a-3} \oplus R(-a-2)^{3} \to R(-a)^{2a} \to I_{Z_{o(a)},X} \to 0.$$

Proposition 3.7 (Third link) *Assume that the ideal $I_{Z_{o(a)},X}$ of $o(a) = \frac{3}{2}a(a-1)+a+1$ general points on X has the following minimal free resolution over R :*

$$0 \to R(-a-3)^{a} \to R(-a-1)^{3a-3} \oplus R(-a-2)^{3} \to R(-a)^{2a} \to I_{Z_{o(a)},X} \to 0.$$

Then the ideal $I_{Z_{p(a)},X}$ of $p(a) = \frac{3}{2}a(a-1) + 2a + 1$ general points on X has a minimal free resolution over R:

$$0 \to R(-a-3)^{2a} \to R(-a-2)^{3a+3} \to R(-a)^{a} \oplus R(-a-1)^{3} \to I_{Z_{p(a)},X} \to 0.$$

Proposition 3.8 (**Fourth link**)   *Assume that the ideal $I_{Z_{p(a)},X}$ of $p(a) = \frac{3}{2}a(a-1) + 2a + 1$ general points on X has the following minimal free R-resolution:*

$$0 \to R(-a-3)^{2a} \to R(-a-2)^{3a+3} \to R(-a)^a \oplus R(-a-1)^3 \to I_{Z_{p(a)},X} \to 0.$$

Then the ideal $I_{Z_{m(a+1)},X}$ of $Z_{m(a+1)}$ general points on X has a minimal free resolution over R:

$$0 \to R(-a-4)^a \to R(-a-2)^{3a+3} \to R(-a-1)^{2a+3} \to I_{Z_{m(a+1)},X} \to 0.$$

Assuming the four propositions, we prove the theorem as follows.

Proof of Theorem 3.2   For $a = 1, 2$ and 3 we are considering the following number of general points on X: $m(1) = 1, n(1) = o(1) = 2, p(1) = 3, m(2) = 5, n(2) = 7, o(2) = 6, p(2) = 8, m(3) = 12, n(3) = 15, o(3) = 13, p(3) = 16$. These are also general points in $\mathbb{P}^3$ and therefore the Minimal Resolution Conjecture is known to hold. Moreover, for $a = 3$ their minimal free resolutions correspond to the ones given in the statement of the theorem.

For $a \geq 4$ we shall first prove the following claim:

Claim.   For $a \geq 2$ the resolution of $Z_{m(a)}$ in X is the one given in Proposition 3.5, *i.e.,*

$$(1) \qquad 0 \to R(-a-3)^{a-1} \to R(-a-1)^{3a} \to R(-a)^{2a+1} \to I_{Z_{m(a)},X} \to 0.$$

Proof of Claim   We proceed by induction on a and we start by proving that it is true for $a = 2$ and $a = 3$.

$a = 2$: $m(2)$ general points in X are 5 general points in $\mathbb{P}^3$. Therefore $Z_{m(2)}$ has the following graded minimal free resolution:

$$0 \to R(-5) \to R(-3)^5 \to R(-2)^5 \to I_{Z_{m(2)}} \to 0.$$

Using the mapping cone procedure applied to the free resolutions of the ideals in the exact sequence

$$0 \to I_X \to I_{Z_{m(2)}} \to I_{Z_{m(2)},X} \to 0,$$

we obtain that the minimal free resolution of $I_{Z_{m(2)},X}$ coincides with (1) for $a = 2$.

$a = 3$: $m(3) = 12$ general points in $\mathbb{P}^3$ lie on a smooth cubic surface. Their minimal free resolution is
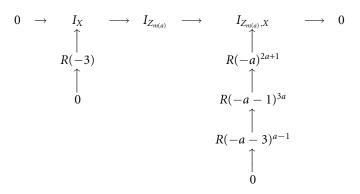
$$0 \to R(-6)^2 \to R(-4)^9 \to R(-3)^8 \to I_{Z_{m(3)}} \to 0.$$

Then we apply the mapping cone construction to the following commutative dia-

gram:

$$
\begin{array}{ccccccccc}
0 & \to & I_X & \longrightarrow & I_{Z_{m(3)}} & \to & I_{Z_{m(3)},X} & \to & 0 \\
& & \uparrow & & \uparrow & & & & \\
& & R(-3) & \xrightarrow{\xi_1} & R(-3)^8 & & & & \\
& & \uparrow & & \uparrow & & & & \\
& & 0 & & R(-4)^9 & & & & \\
& & & & \uparrow & & & & \\
& & & & R(-6)^2 & & & & \\
& & & & \uparrow & & & & \\
& & & & 0 & & & &
\end{array}
$$

As f is one of the generators of $I_{Z_{m(3)}}$, when we write the map ξ_1 as a matrix we see that it contains a non-zero scalar entry. Therefore, in the resolution of $I_{Z_{m(3)},X}$ obtained by mapping cone, there is one term $R(-3)$ that can be split off, giving the desired minimal free resolution.

$a \geq 4$: We assume that the resolution of $Z_{m(a-2)}$ in X is (1) for $a - 2$. Then Proposition 3.5 holds for $a - 2$, and so $n(a-2)$ satisfies the hypothesis of Proposition 3.6. This in turn implies that the hypothesis of Proposition 3.7 is satisfied for $a - 1$ and that of Proposition 3.8 also holds for $a - 1$. Therefore we obtain the desired minimal free resolution for $I_{Z_{m(a)},X}$ and hence the claim is proved. ∎

Now the Theorem immediately follows for $m(a)$ by applying the horseshoe lemma (cf. [17] 2.2.8) to the following diagram

$$
\begin{array}{ccccccccc}
0 & \to & I_X & \longrightarrow & I_{Z_{m(a)}} & \longrightarrow & I_{Z_{m(a)},X} & \longrightarrow & 0 \\
& & \uparrow & & & & \uparrow & & \\
& & R(-3) & & & & R(-a)^{2a+1} & & \\
& & \uparrow & & & & \uparrow & & \\
& & 0 & & & & R(-a-1)^{3a} & & \\
& & & & & & \uparrow & & \\
& & & & & & R(-a-3)^{a-1} & & \\
& & & & & & \uparrow & & \\
& & & & & & 0 & &
\end{array}
$$

The resolution obtained is minimal for $a \geq 3$ (but not for $a = 2$).

We now prove the theorem for $n(a)$, $o(a)$ and $p(a)$. The claim above and Propositions 3.5–3.8 provide us minimal free resolutions for $I_{Z_{n(a)},X}$ when $a \geq 2$ and for $I_{Z_{o(a)},X}$, $I_{Z_{p(a)},X}$ when $a \geq 3$. The horseshoe lemma gives us the desired minimal free resolutions for $n(a)$, $o(a)$ and $p(a)$ when $a \geq 3$. ∎

The hard work is proving the four propositions.

Proof of Proposition 3.5 $n = m(a)$ and $n' = n(a)$ satisfy the hypotheses of Lemma 3.3 with $d = 3a$ and $g = 1/2(3a^2 - 3a + 2)$ (which are the degree and genus of a

curve $C \sim aH$) because $n + n' = deg(aH_C)$. Therefore $Z_{m(a)}$ and $Z_{n(a)}$ are directly *CI*-linked on a smooth curve $C \sim aH$ by $G \sim aH_C$ and $I_G \subset R$ has the following minimal free resolution

$$0 \to R(-2a-3) \to R(-2a) \oplus R(-a-3)^2 \to R(-a)^2 \oplus R(-3) \to I_G \to 0.$$

By the mapping cone construction applied to the commutative diagram

$$
\begin{array}{ccccccc}
0 & \to & I_G & \xrightarrow{i} & I_{Z_{m(a)}} & \longrightarrow & I_{Z_{m(a)}}/I_G \to 0 \\
& & {\scriptstyle e_1}\uparrow & & {\scriptstyle d_1}\uparrow & & \\
& & R(-a)^2 \oplus R(-3) & \xrightarrow{\xi_1} & R(-a)^{2a+1} \oplus R(-3) & & \\
& & {\scriptstyle e_2}\uparrow & & {\scriptstyle d_2}\uparrow & & \\
& & R(-2a) \oplus R(-a-3)^2 & \xrightarrow{\xi_2} & R(-a-1)^{3a} & & \\
& & {\scriptstyle e_3}\uparrow & & {\scriptstyle d_3}\uparrow & & \\
& & R(-2a-3) & \xrightarrow{\xi_3} & R(-a-3)^{a-1} & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & & 0 & &
\end{array}
$$

we obtain the following free resolution of $I_{Z_{n(a)}}$ (see Lemma 2.7):

$$0 \to R(-a-3)^{2a+1} \oplus R(-2a) \to R(-a-2)^{3a} \oplus R(-a-3)^2 \oplus R(-2a)$$
$$\to R(-a)^{a+1} \oplus R(-3) \to I_{Z_{n(a)}} \to 0.$$

The generators of I_G can be taken in the set of minimal generators of $I_{Z_{m(a)}}$, so the map ξ_1 has non-zero scalar entries. This implies that the summand $R(-a-3)^2 \oplus R(-2a)$ in the free resolution above can be split off. Therefore, a minimal free resolution for $I_{Z_{n(a)}}$ is

$$0 \to R(-a-3)^{2a-1} \to R(-a-2)^{3a} \to R(-a)^{a+1} \oplus R(-3) \to I_{Z_{n(a)}} \to 0.$$

The mapping cone construction applied now to the exact sequence $0 \to I_X \to I_{Z_{n(a)}} \to I_{Z_{n(a)},X} \to 0$ gives the desired minimal free resolution for $I_{Z_{n(a)},X}$. ∎

Before proving Proposition 3.6 we need the following lemma.

Lemma 3.9 *Let C be a curve in X linearly equivalent to $C_0 + aH$ and let G be an effective divisor on C linearly equivalent to $2aH_C - K_C$. Then the minimal free R-resolution of $I_{G,C}$ is*

$$0 \to R(-2a-4) \to R(-a-3) \oplus R(-a-2) \oplus R(-2a-1) \to R(-a-1)^2 \to I_{G,C} \to 0.$$

Proof As $C \sim C_0 + aH$, we have that $I_{C,X} \cong H^0_*(X, \mathcal{J}_{C_0}(-a))$ and therefore we know the R- resolution of $I_{C,X}$. Thus we can apply the horseshoe lemma to the following diagram

$$
\begin{array}{ccccccccc}
0 & \to & I_X & \longrightarrow & I_C & \longrightarrow & I_{C,X} & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & R(-3) & & & & R(-a-1) \oplus R(-a-2) & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & R(-a-3)^2 & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$

and we obtain the following minimal free R-resolution for R/I_C:

$$0 \to R(-a-3)^2 \to R(-a-1) \oplus R(-a-2) \oplus R(-3) \to R \to R/I_C \to 0.$$

As C is an arithmetically Cohen–Macaulay curve, applying $\mathrm{Hom}(\,\cdot\,, R(-4))$ to this exact sequence we obtain a minimal free resolution for the canonical module $\omega_C = \mathrm{Ext}^2_R(R/I_C, R(-4))$:

$$0 \to R(-4) \to R(a-3) \oplus R(a-2) \oplus R(-1) \to R(a-1)^2 \to \omega_C \to 0.$$

When we shift this resolution by $-2a$ we obtain the desired minimal free resolution for $H^0_*(\mathcal{O}_C(K_C - 2aH_C)) \cong I_{G,C}$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2a-4) \to R(-a-3) \oplus R(-a-2) \oplus R(-2a-1) \to R(-a-1)^2 \to I_{G,C} \to 0.$$

∎

Proof of Proposition 3.6 $n = n(a-1)$ and $n' = o(a)$ satisfy the hypotheses of Lemma 3.3 with $d = 3a - 1$, $g = 1/2(3a^2 - 5a + 2)$ because $n + n' = \deg((2a-2)H_C - K_C)$ for a curve $C \sim C_0 + (a-1)H$. Therefore there is a smooth curve $C \sim C_0 + (a-1)H$ so that $n(a-1)$ general points on X and $o(a)$ general points on X are G-linked by $G \sim (2a-2)H_C - K_C$ on C.

According to Lemma 3.9, the minimal free resolution of $I_{G,C}$ is

$$0 \to R(-2a-2) \to R(-a-2) \oplus R(-a-1) \oplus R(-2a+1) \to R(-a)^2 \to I_{G,C} \to 0.$$

On the other hand, the mapping cone procedure applied to the following exact sequence

$$0 \to I_{C,X} \to I_{Z_{n(a-1)},X} \to I_{Z_{n(a-1)},C} \to 0$$

gives us the minimal free R-resolution of $I_{Z_{n(a-1)},C}$:

$$0 \to R(-a-2) \to R(-a) \oplus R(-a-1)^{3a-2} \to R(-a+1)^a \to I_{Z_{n(a-1)},C} \to 0.$$

Now we apply the mapping cone construction to the following commutative diagram:

$$
\begin{array}{ccccccc}
0 \longrightarrow & I_{G,C} & \xrightarrow{\;i\;} & I_{Z_{n(a-1)},C} & \longrightarrow & I_{Z_{n(a-1)},C}/I_{G,C} & \longrightarrow 0 \\
& \Big\uparrow{\scriptstyle e_1} & & \Big\uparrow{\scriptstyle d_1} & & & \\
& R(-a)^2 & \xrightarrow{\;\xi_1\;} & R(-a+1)^a & & & \\
& \Big\uparrow{\scriptstyle e_2} & & \Big\uparrow{\scriptstyle d_2} & & & \\
R(-a-1)\oplus R(-a-2) & \xrightarrow{\;\xi_2\;} & R(-a-1)^{3a-2}\oplus R(-a) & & & \\
\oplus R(-2a+1) & & & & & \\
& \Big\uparrow{\scriptstyle e_3} & & \Big\uparrow{\scriptstyle d_3} & & & \\
& R(-2a-2) & \xrightarrow{\;\xi_3\;} & R(-a-2)^{2a-1} & & & \\
& \Big\uparrow & & \Big\uparrow & & & \\
& 0 & & 0 & & &
\end{array}
$$

Writing ξ_2 in matrix form we shall prove that the first column of ξ_2 has a non-zero scalar entry. We assume that the first column of ξ_2 is $\begin{pmatrix} l \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for some linear form l, and we will arrive at a contradiction. Choosing a basis, we may denote the first column of e_2 by $\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$, the first column of d_2 by $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_a \end{pmatrix}$, and the entries of ξ_1 by $\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ \vdots & \vdots \\ p_a & q_a \end{pmatrix}$.

Then the equality $d_2\xi_2 = \xi_1 e_2$ implies that

$$
\text{(2)} \qquad\qquad x_i l = p_i L_1 + q_i L_2
$$

for all $i \in \{1,\ldots,a\}$. Note that, according to the proof of Lemma 3.9, L_1, L_2 are given by the curve C.

We now consider two different possibilities: either l, L_1, L_2 form a complete intersection in R or $l = \alpha L_1 + \beta L_2$ for some $\alpha, \beta \in k$.

If l, L_1, L_2 form a complete intersection, then (2) implies that $p_i \in (l, L_2)$ and $q_i \in (l, L_1)$ for all $i \in \{1,\ldots,a\}$. Thus, if $(\overline{f_1},\ldots,\overline{f_a}) \subset R/I_C$ are the generators of $I_{Z_{n(a-1)},C}$, we obtain that both generators of $I_{G,C}$ belong to $(\overline{l},\overline{L_1},\overline{L_2}) \subset R/I_C$ because they are defined by the product

$$
(\overline{f_1},\ldots,\overline{f_a}) \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ \vdots & \vdots \\ p_a & q_a \end{pmatrix}.
$$

But this implies that $I_{G,C} \subseteq (\bar{l}, \overline{L_1}, \overline{L_2})$, which is impossible because G is formed by two sets of general points in C.

If $l = \alpha L_1 + \beta L_2$, then we can change the generators of $I_{G,C} = (\bar{x}, \bar{y})$ in order to make L_1 or L_2 equal to l. Let us assume $L_2 = l$ and the other case can be treated analogously. From equation (2) we obtain in this case that $p_i \in (L_2)$ for all $i \in \{1, \ldots, a\}$ and in particular $I_{G,C} \subset (\overline{L_2}, \bar{y})$. As $\overline{L_2}, \bar{y}$ do not form a complete intersection in R/I_C, there is an associated prime of $(\overline{L_2}, \bar{y})$ of height 1. But this associated prime would contain a prime appearing in the primary decomposition of $I_{G,C}$ and this is impossible because all of them have height 1 in R/I_C.

In both cases we obtain a contradiction. Therefore there is a non-zero scalar entry in ξ_2 and, in particular, in the resolution of $I_{Z_{n(a-1)},C}/I_{G,C} \cong I_{Z_{n(a-1)}}/I_G$ obtained by the mapping cone, one term $R(-a-1)$ can be split off. Lemma 2.7 then gives the desired minimal free resolution for $I_{Z_{o(a)}}$. ∎

Before proving Proposition 3.7 we need the following result.

Lemma 3.10 *Assume that the ideal $I_{Z_{o(a)},X}$ of $o(a) = \frac{3}{2}a(a-1) + a + 1$ general points $Z_{o(a)}$ on X has the following minimal free resolution over R :*

$$0 \rightarrow R(-a-3)^a \rightarrow R(-a-1)^{3a-3} \oplus R(-a-2)^3 \rightarrow R(-a)^{2a} \rightarrow I_{Z_{o(a)},X} \rightarrow 0.$$

Then $Z_{o(a)}$ lies on a smooth curve $C \sim \Gamma + (a-1)H$ on X and $H^0_*(\mathcal{J}_{Z_{o(a)},C})$ has the following minimal free resolution:

$$0 \rightarrow R(-a-3)^a \rightarrow R(-a-1)^{3a} \rightarrow R(-a)^{2a} \rightarrow H^0_*(\mathcal{J}_{Z_{o(a)},C}) \rightarrow 0.$$

Proof As $\dim |\Gamma + (a-1)H| = 3a + 1/2(3a^2 - 3a) - 1 \geq o(a)$, any set of $o(a)$ general points lies on a smooth curve $C \sim \Gamma + (a-1)H$. As C is an arithmetically Cohen–Macaulay curve, we have the following exact sequence

$$0 \rightarrow H^0_*(\mathcal{J}_{C,X}) \xrightarrow{i} H^0_*(\mathcal{J}_{Z_{o(a)},X}) \rightarrow H^0_*(\mathcal{J}_{Z_{o(a)},C}) \rightarrow 0.$$

We can lift the map i to the minimal free resolutions of $I_{Z_{o(a)},X}$ and $I_{C,X}$ to get a commutative diagram:

$$
\begin{array}{ccccccc}
0 & \rightarrow & H^0_*(\mathcal{J}_{C,X}) & \xrightarrow{i} & H^0_*(\mathcal{J}_{Z_{o(a)},X}) & \rightarrow & H^0_*(\mathcal{J}_{Z_{o(a)},C}) & \rightarrow & 0 \\
& & \uparrow & & d_1 \uparrow & & & & \\
& & R(-a-1)^3 & \xrightarrow{\xi_1} & R(-a)^{2a} & & & & \\
& & \Phi \uparrow & & d_2 \uparrow & & & & \\
& & R(-a-2)^3 & \xrightarrow{\xi_2} & R(-a-2)^3 \oplus R(-a-1)^{3a-3} & & & & \\
& & \uparrow & & d_3 \uparrow & & & & \\
& & 0 & & R(-a-3)^a & & & & \\
& & & & \uparrow & & & & \\
& & & & 0 & & & &
\end{array}
$$

and the mapping cone construction gives a free resolution

$$0 \to R(-a-3)^a \oplus R(-a-2)^3 \to R(-a-1)^{3a} \oplus R(-a-2)^3 \to R(-a)^{2a}$$
$$\to H^0_*(\mathcal{J}_{Z_{o(a)},C}) \to 0,$$

that is not necessarily minimal. In order to prove that the terms $R(-a-2)^3$ can be split off we need to check that if, after choosing a basis, we write $\xi_2 = \left(\begin{smallmatrix} \sigma \\ \Lambda \end{smallmatrix} \right)$, where σ is a 3×3 matrix of scalars and Λ is a $(3a-3) \times 3$ matrix of linear homogeneous entries, then σ has rank 3.

Let us assume that σ has rank ≤ 2 instead. Then there is a combination of its columns that gives the 0 column. We can assume (after changing basis if necessary) that the first column of σ is zero. We write the matrix corresponding to d_2 as $\begin{pmatrix} \delta & D \end{pmatrix}$ (where δ is a $2a \times 3$ matrix and D a $2a \times (3a-3)$ matrix),

the first column of Λ as $\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_{3a-3} \end{pmatrix}$ and the first column of Φ as $\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$

Then from the equality

$$\begin{pmatrix} \delta & D \end{pmatrix} \begin{pmatrix} \sigma \\ \Lambda \end{pmatrix} = \xi_1 \Phi$$

applied to the first column of ξ_2 and Φ we obtain that

(3)
$$\overline{D} \cdot \begin{pmatrix} \overline{l_1} \\ \overline{l_2} \\ \vdots \\ \overline{l_{3a-3}} \end{pmatrix} = 0 \text{ in } R/(\varphi_1, \varphi_2, \varphi_3).$$

As the matrix Φ has determinant f, the ideal $(\varphi_1, \varphi_2, \varphi_3)$ might have either height 2 or 3 in R. We consider each case separately:

Case 1: height $(\varphi_1, \varphi_2, \varphi_3) = 3$.

In this case, in the ring $R/(\varphi_1, \varphi_2, \varphi_3)$ each $\overline{l_i}$ is equal to some linear form $\overline{l}$ multiplied by some constant c_i. Therefore (3) becomes

$$\overline{l}\,\overline{D} \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{3a-3} \end{pmatrix} = 0 \text{ in } R/(\varphi_1, \varphi_2, \varphi_3).$$

As $\overline{l}$ is a non-zero divisor in $R/(\varphi_1, \varphi_2, \varphi_3)$, we have that

$$\overline{D} \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{3a-3} \end{pmatrix} = 0 \text{ in } R/(\varphi_1, \varphi_2, \varphi_3).$$

This means that there exists a k-linear combination of the columns of $\bar{D}$ that gives the 0 column in $R/(\varphi_1, \varphi_2, \varphi_3)$. Changing basis if necessary, we can assume that there is a column in the matrix D whose entries belong to the ideal $(\varphi_1, \varphi_2, \varphi_3)$.

Let p be the point corresponding to the ideal $\mathfrak{p} = (\varphi_1, \varphi_2, \varphi_3)$ (note that this is a point in X because f is the determinant of Φ, so $f \in \mathfrak{p}$). We split the resolution of $I_{Z_{o(a)},X}$ into short exact sequences:

$$(4) \qquad 0 \to R(-a-3)^a \to R(-a-1)^{3a-3} \oplus R(-a-2)^3 \xrightarrow{d_2} E \to 0$$

$$(5) \qquad 0 \to E \to R(-a)^{2a} \to I_{Z_{o(a)},X} \to 0$$

and we are going to check how these sequences behave when we tensor by $R_\mathfrak{p}$. First, we need to prove the following claim.

Claim. p is not a point in $Z_{o(a)}$

Proof of claim If p were a point in $Z_{o(a)}$, then it would be a point in the curve C. Then, tensoring the minimal free resolution of $H^0_*(\mathcal{J}_{C,X})$ by $R_\mathfrak{p}$, we obtain that $\Phi_\mathfrak{p}$ is an isomorphism. But this is a contradiction since the entries of a column of Φ belong to $\mathfrak{p}$, so the claim is proved. ∎

As $p \notin Z_{o(a)}$, we have that $R_X/I_{Z_{o(a)},X} \otimes_R R_\mathfrak{p} = 0$ and in particular

$$Tor_i^R(R_X/I_{Z_{o(a)},X}, R_\mathfrak{p}) = 0$$

for $i \geq 0$. From this we also obtain that $I_{Z_{o(a)},X} \otimes_R R_\mathfrak{p}$ is isomorphic to $R_X \otimes_R R_\mathfrak{p}$.

From the exact sequence $0 \to I_X = (f) \to R \to R_X \to 0$ we obtain that $Tor_1^R(R_X, R_\mathfrak{p}) = 0$.

Now localizing the exact sequence $0 \to I_{Z_{o(a)},X} \to R_X \to R_X/I_{Z_{o(a)},X} \to 0$ and using the vanishing of the Tor groups above, we get that $Tor_i(I_{Z_{o(a)},X}, R_\mathfrak{p}) = 0$ for $i = 1, 2$.

Thus, after tensoring sequence (5) by $\otimes_R R_\mathfrak{p}$ we obtain the following short exact sequence:

$$(6) \qquad 0 \to E \otimes_R R_\mathfrak{p} \to R_\mathfrak{p}^{2a} \to I_{Z_{o(a)},X} \otimes_R R_\mathfrak{p} \to 0$$

Moreover, since $Tor_2(I_{Z_{o(a)},X}, R_\mathfrak{p}) = 0$ and $Tor_1^R(R, R_\mathfrak{p}) = 0$, we obtain from sequence (5) that $Tor_1(E, R_\mathfrak{p}) = 0$. Therefore, we have the following short exact sequence after localizing sequence (4):

$$(7) \qquad 0 \to R_\mathfrak{p}^a \to R_\mathfrak{p}^{3a} \xrightarrow{d_2 \otimes_R R_\mathfrak{p}} E \otimes_R R_\mathfrak{p} \to 0.$$

Now we are going to prove that $E \otimes_R R_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module. To prove this we need to check that the local cohomology modules $H^i_{\mathfrak{m}_\mathfrak{p}}(E \otimes_R R_\mathfrak{p})$ vanish for $i \leq 2$, where $\mathfrak{m}_\mathfrak{p}$ is the maximal ideal of the ring $R_\mathfrak{p}$. It follows immediately from sequence (7) that these modules are zero for $i = 0, 1$. Moreover, from the exact sequence (6), taking into account the isomorphism $I_{Z_{o(a)},X} \otimes_R R_\mathfrak{p} \cong R_X \otimes_R R_\mathfrak{p}$, we obtain that $H^2_{\mathfrak{m}_\mathfrak{p}}(E \otimes_R R_\mathfrak{p}) = 0$ because $depth(R_X \otimes_R R_\mathfrak{p}) = 2$.

Therefore, $E \otimes_R R_{\mathfrak{p}}$ is a free module and the exact sequence (7) splits. But this contradicts the fact that a column of d_2 has all its entries in the ideal $\mathfrak{p}$.

Case 2: height $(\varphi_1, \varphi_2, \varphi_3) = 2$.

Let us assume that $(\varphi_1, \varphi_2, \varphi_3) = (\varphi_1, \varphi_2)$ (the other cases are similar). Then (φ_1, φ_2) is the ideal of a line in X.

From (3) we see that in $R/(\varphi_1, \varphi_2)$, $(\overline{l_1}, \overline{l_2}, \ldots, \overline{l_{3a-3}})\overline{D^t} = 0$. The ideal $(\overline{l_1}, \overline{l_2}, \ldots, \overline{l_{3a-3}})$ has height at most 2 in $R/(\varphi_1, \varphi_2)$. Performing a base change if necessary we can assume that $(\overline{l_1}, \overline{l_2}, \ldots, \overline{l_{3a-3}}) = (\overline{l_1}, \overline{l_2})$. The case where $\overline{l_1} \in (\overline{l_2})$ or vice versa is easier and left to the reader.

The columns of $\overline{D^t}$ are syzygies of $(\overline{l_1}, \overline{l_2}, \ldots, \overline{l_{3a-3}})$. This means that $\overline{D^t} = Z.T$ where Z is the following $(3a - 3) \times (3a - 4)$ matrix

$$
\begin{pmatrix}
\overline{l_2} & z_1^2 & z_1^3 & \cdots & z_1^{3a-4} \\
-\overline{l_1} & z_2^2 & z_2^3 & \cdots & z_2^{3a-4} \\
0 & z_3^2 & 0 & \cdots & 0 \\
0 & 0 & z_4^3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & z_{3a-3}^{3a-4}
\end{pmatrix},
$$

$z_j^i \in k$, and T is a $(3a - 4) \times (2a)$ matrix of polynomials in $R/(\varphi_1, \varphi_2)$. Considering the following constant vector

$$
\mu = \left(1, 0, -\frac{z_1^2}{z_3^2}, -\frac{z_1^3}{z_4^3}, \ldots, -\frac{z_1^{3a-4}}{z_{3a-3}^{3a-4}} \right),
$$

we have that $\overline{D}\mu^t = T^t Z^t \mu^t$ is equal to

$$
T^t \begin{pmatrix} -\overline{l_2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
$$

Therefore $D\mu^t$ has each entry in the ideal $(\varphi_1, \varphi_2, l_2)$. After doing a base change we can assume that the entries of one of the columns in D all lie in the ideal $(\varphi_1, \varphi_2, l_2)$ (or any other linear form instead of l_2 if the reader bothers about too many base changes). We consider the point p defined by this ideal which is a point in the line given by φ_1, φ_2. It cannot be a point of $Z_{o(a)}$ because the points of $Z_{o(a)}$ were general points in X, so none of them can lie in one of the 27 lines of X. The rest of the proof in this case follows by localizing at p, exactly as we did in case 1.

Therefore, in both cases we obtain that the rank of σ must be 3, which in turn implies that the term $R(-a-2)^3$ in the resolution of $H_*^0(\mathcal{J}_{Z_{o(a)},C})$ is redundant. ∎

Now we are ready to prove Proposition 3.7.

Proof of Proposition 3.7 $n = o(a)$ and $n' = p(a)$ satisfy the hypotheses of Lemma 3.3 with $d = 3a$ and $g = 1/2(3a^2 - 3a)$ because $n + n' = \deg((2a - 1)H_C - K_C)$ for a curve $C \sim \Gamma + (a - 1)H$. Therefore, there exists a smooth curve $C \sim \Gamma + (a - 1)H$ and an arithmetically Gorenstein scheme $G \sim (2a - 1)H_C - K_C$ so that $Z_{o(a)}$ and $Z_{p(a)}$ are linked by G on C. Then on the curve C, $Z_{o(a)} + Z_{p(a)}$ is linearly equivalent to $(2a - 1)H_C - K_C$. From Lemma 3.10 we have a minimal free resolution of the R-module $M = H^0_*(\mathcal{O}_C(-Z_{o(a)}))$:

$$0 \to R(-a - 3)^a \to R(-a - 1)^{3a} \to R(-a)^{2a} \to H^0_*(\mathcal{O}_C(-Z_{o(a)})) \to 0.$$

As M is a Cohen–Macaulay module we can dualize applying $\mathrm{Hom}(\,\cdot\,, R(-4))$ and we obtain a minimal free resolution of $\mathrm{Ext}^2(M, R(-4))$:

(8) $$0 \to R(a - 4)^{2a} \to R(a - 3)^{3a} \to R(a - 1)^a \to \mathrm{Ext}^2(M, R(-4)) \to 0.$$

By local duality $\mathrm{Ext}^2(M, R(-4))$ is isomorphic to $H^2_{\mathfrak{m}}(M)^{\vee}$, which at the same time is isomorphic to $H^1_*(\mathcal{O}_C(-Z_{o(a)}))^{\vee}$. By Serre duality we have $H^1_*(\mathcal{O}_C(-Z_{o(a)}))^{\vee} \cong H^0_*(\mathcal{O}_C(Z_{o(a)} + K_C))$. But as $Z_{o(a)} + K_C \sim -Z_{p(a)} + (2a - 1)H_C$, twisting (8) by $R(-2a + 1)$ we obtain a minimal free resolution of $H^0_*(\mathcal{O}_C(-Z_{p(a)}))$. Now we apply the horseshoe lemma to the following diagram:

$$
\begin{array}{ccccccccc}
0 & \to & H^0_*(\mathcal{J}_{C,X}) & \longrightarrow & H^0_*(\mathcal{J}_{Z_{p(a)},X}) & \longrightarrow & H^0_*(\mathcal{J}_{Z_{p(a)},C}) & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & R(-a-1)^3 & & & & R(-a)^a & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & R(-a-2)^3 & & & & R(-a-2)^{3a} & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & R(-a-3)^{2a} & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$

and we obtain the desired minimal free resolution (note that it is already minimal because no terms can be split off.) ∎

Before proving Proposition 3.8 we need the following result.

Lemma 3.11 *Let $C \subset X$ be a smooth curve linearly equivalent to $C_0 + aH$ containing $Z_{p(a)}$. If the minimal free resolution of $I_{Z_{p(a)},X}$ is*

$$0 \to R(-a - 3)^{2a} \to R(-a - 2)^{3a+3} \to R(-a)^a \oplus R(a - 1)^3 \to I_{Z_{p(a)},X} \to 0,$$

then the minimal free resolution of $I_{Z_{p(a)},C}$ is:

$$0 \to R(-a-3)^{2a+2} \to R(-a-2)^{3a+4} \to R(-a)^a \oplus R(-a-1)^2 \to I_{Z_{p(a)},C} \to 0.$$

Proof We know both the minimal free resolution of $I_{C,X}$ and $I_{Z_{p(a)},X}$, so we apply the mapping cone procedure to the following commutative diagram:

$$
\begin{array}{ccccccc}
0 \to & I_{C,X} & \xrightarrow{\ i\ } & I_{Z_{p(a)},X} & \to & I_{Z_{p(a)},C} & \to 0 \\
& e_1 \uparrow & & d_1 \uparrow & & & \\
& R(-a-1) \oplus R(-a-2) & \xrightarrow{\xi_1} & R(-a)^a \oplus R(-a-1)^3 & & & \\
& e_2 \uparrow & & d_2 \uparrow & & & \\
& R(-a-3)^2 & \xrightarrow{\xi_2} & R(-a-2)^{3a+3} & & & \\
& \uparrow & & d_3 \uparrow & & & \\
& 0 & & R(-a-3)^a & & & \\
& & & \uparrow & & & \\
& & & 0 & & &
\end{array}
$$

Note that the top row is exact because $H^1_*(\mathcal{J}_{C,X}) = 0$. We need to prove that there exists a non-zero scalar entry in the first column of ξ_1. Let us assume that the scalar entries in ξ_1 are zero. Then if $I_{Z_{p(a)},X} = (\overline{f_1}, \ldots, \overline{f_a}, \overline{f_{a+1}}, \overline{f_{a+2}}, \overline{f_{a+3}}) \subset R_X$, where $f_{a+1}, f_{a+2}, f_{a+3}$ are forms of degree $a+1$, we have that $I_{C,X}$ is generated by two forms $\overline{x}, \overline{y}$ whith $\overline{x} \in J := (\overline{f_1}, \ldots, \overline{f_a})$. Now if $\left(\begin{smallmatrix} Q_1 & Q_2 \\ L_1 & L_2 \end{smallmatrix} \right)$ is the matrix corresponding to e_2 in certain basis, we have that $\overline{x}Q_1 + \overline{y}L_1$ is 0 in $R/(f)$. Hence, as $\overline{x}$ is in J we obtain that $\overline{y}L_1$ is in J too and thus $(\overline{x}, \overline{y}L_1) \subset J$. As C is an irreducible curve, $\overline{y}$ and $\overline{L_1}$ are coprimes in $R_X = R/(f)$ and thus $(\overline{x}, \overline{y}L_1) = (\overline{x}, \overline{y}) \cap (\overline{x}, \overline{L_1}) \subset J$. But J is formed by the generators of degree a of the ideal of a *general* set of points in C, so we can deduce that $(\overline{x}, \overline{y}) \subset J$. If we denote by Z the zero set of J, this means that Z is inside C. But this contradicts the fact that J is formed by removing three generators of the ideal of a general set of points in C, and thus Z cannot lie inside C. Therefore there is a non-zero scalar entry in ξ_1, which implies that a summand $R(-a-1)$ is redundant in the resolution of $I_{Z_{p(a)},C}$ obtained by the mapping cone. ■

Proof of Proposition 3.8 $n = p(a)$ and $n' = m(a+1)$ satisfy the hypotheses of Lemma 3.3 with $d = 3a+2$, $g = \frac{1}{2}(3a^2 + a - 2)$ (which are the degree and genus of $C \sim C_0 + aH$) because $n + n' = \deg(2aH_C - K_C)$. Therefore we perform a link on a smooth curve $C \sim C_0 + aH$ with an arithmetically Gorenstein scheme $G \sim 2aH_C - K_C$. By Lemma 3.9, the minimal free resolution of $I_{G,C}$ is

$$0 \to R(-2a-4) \to R(-a-2) \oplus R(-a-3) \oplus R(-2a-1) \to R(-a-1)^2 \to I_{G,C} \to 0.$$

On the other hand, by Lemma 3.11 we know the minimal free resolution of $I_{Z_{p(a)},C}$, so

we can apply the mapping cone procedure to the following commutative diagram:

$$
\begin{array}{ccccc}
0 \to & I_{G,C} & \xrightarrow{\ i\ } & I_{Z_{p(a)},C} & \to I_{Z_{p(a)},C}/I_{G,C} \to 0 \\
& e_1 \uparrow & & d_1 \uparrow & \\
& R(-a-1)^2 & \xrightarrow{\ \xi_1\ } & R(-a)^a \oplus R(-a-1)^2 & \\
& e_2 \uparrow & & d_2 \uparrow & \\
& \begin{array}{c} R(-a-2) \oplus R(-a-3) \\ \oplus R(-2a-1) \end{array} & \xrightarrow{\ \xi_2\ } & R(-a-2)^{3a+4} & \\
& e_3 \uparrow & & d_3 \uparrow & \\
& R(-2a-4) & \xrightarrow{\ \xi_3\ } & R(-a-3)^{2a-2} & \\
& \uparrow & & \uparrow & \\
& 0 & & 0 &
\end{array}
$$

We shall prove that the first column of ξ_2 has a non-zero scalar entry and that, if we write $\xi_1 = \left(\begin{smallmatrix} M \\ \sigma \end{smallmatrix} \right)$ where σ is a 2×2 matrix with scalar entries, then σ has rank 2.

We start by proving that the first column of ξ_2 has a non-zero scalar entry.

We write $e_2 = \left(\begin{smallmatrix} L_1 & Q_1 & H_1 \\ L_2 & Q_2 & H_2 \end{smallmatrix} \right)$. The first two columns form the transpose of the resolution of $I_{C,X}$ over R (see the proof of Lemma 3.9), so L_1 and L_2 are two linear forms defining a line in X. If the first column of ξ_2 was 0, then $\xi_1 \cdot \left(\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix} \right) = 0$ and in particular $\sigma \cdot \left(\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix} \right) = 0$. But L_1 and L_2 are linearly independent over k, so this implies that σ is the zero matrix. If we denote by $(\overline{f_1}, \ldots, \overline{f_a}, \overline{f_{a+1}}, \overline{f_{a+2}})$ the generators of $I_{Z_{p(a)},C}$ in R/I_C, where $deg\,\overline{f_i} = a$ if $i \le a$ and $deg\,\overline{f_{a+1}} = deg\,\overline{f_{a+2}} = a+1$, then $\sigma = 0$ implies that $I_{G,C} \subseteq (\overline{f_1}, \ldots, \overline{f_a}) \subset I_{Z_{p(a)},C}$. But this is impossible because $Z_{p(a)}$ is a set of general points in C, and so it cannot be contained in the set of zeroes $Z(\overline{f_1}, \ldots, \overline{f_a}) \subset G \subset C$. Therefore the first column of ξ_2 must have a non-zero scalar entry.

Now we prove that σ has rank 2. Indeed, if $\sigma = 0$, we have just seen how to get to a contradiction. If the rank of σ was 1, then we can assume that the second row of σ is 0 (changing basis if necessary). But this implies $I_{G,C} \subseteq (\overline{f_1}, \ldots, \overline{f_a}, \overline{f_{a+1}}) \subset I_{Z_{p(a)},C}$ which leads to a contradiction as above.

Therefore, the mapping cone construction tells us that the minimal free resolution of $I_{Z_{p(a)},C}/I_{G,C} \cong I_{Z_{p(a)}}/I_G$ is

$$
0 \to R(-2a-4) \to R(-a-3)^{2a+3} \oplus R(-2a-1) \to R(-a-2)^{3a+3}
$$
$$
\to R(-a)^a \to I_{Z_{p(a)}}/I_G \to 0.
$$

By Lemma 2.7, the resolution of the residual $I_{Z_{m(a+1)}}$ is obtained by dualizing this resolution and twisting by $R(-2a-4)$. So this gives the desired free resolution, which is minimal because no terms can be split off. ∎

References

[1]   E. Ballico and A. V. Geramita, *The minimal free resolution of the ideal of s general points in $\mathbb{P}^3$*. Proceedings of the 1984 Vancuver conference on algebraic geometry, CMS Conf. Proc. 6, American Mathematical Society, Providence, RI, 1986, pp. 1–10.

[2]   M. P. Broadmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications.* Cambridge studies in advanced mathematics 60, Cambridge University Press, 1998.

[3]   M. Casanellas, E. Drozd, and R. Hartshorne, *Gorenstein liaison and ACM sheaves.* J. Reine Angew. Math. **584**(2005), 149–171.

[4]   D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations.* Trans. Amer. Math. Soc. **260**(1980), no. 1, 35–64.

[5]   ———, *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics 150, Springer-Verlag, New-York, 1995.

[6]   D. Eisenbud, S. Popescu, F.-O. Schreyer, and C. Walter, *Exterior algebra methods for the minimal resolution conjecture.* Duke Math. J. **112**(2002), no. 2, 379–395.

[7]   G. Farkas, M. Mustaţǎ, and M. Popa, *Divisors on $\mathcal{M}_{g,g+1}$ and the minimal resolution conjecture for points on canonical curves.* Ann. Sci. École Norm. Sup. **36**(2003), no. 4, 553–581.

[8]   A. V. Geramita, T. Harima, J. C. Migliore, and Y. S. Shin, *The Hilbert function of a level algebra.* Mem. Amer. Math. Soc. **186**(2007), no. 872.

[9]   S. Giuffrida, R. Maggioni, and A. Ragusa, *Resolutions of generic points lying on a smooth quadric.* Manuscripta Math. **91**(1996), no. 4, 421–444.

[10]   R. Hartshorne, *Some examples of Gorenstein liaison in codimension three.* Collect. Math. **53**(2002), no. 1, 21–48.

[11]   A. Hirschowitz and C. Simpson, *La résolution minimale de l'idéal d'un arrangement général d'un grand nombre de points dans $\mathbf{P}^n$.* Invent. Math. **126**(1996), no. 3, 467–503.

[12]   A. Lorenzini, *The minimal resolution conjecture.* J. Algebra **156**(1993), no. 1, 5–35.

[13]   S. MacLane, *Homology.* Die Grundlehren der mathematischen Wissenschaften 114, Springer-Verlag, Berlin, 1963.

[14]   J. C. Migliore, *Introduction to liaison theory and deficiency modules.* Progress in Mathematics 165, Birkhäuser, Boston, MA, 1998.

[15]   M. Mustaţǎ, *Graded Betti numbers of general finite subsets of points on projective varieties.* Matematiche **53** (1998), suppl., 53–81.

[16]   C. Walter, *The minimal free resolution of the homogeneous ideal of s general points in $\mathbb{P}^4$,* Math. Z. **219**(1995), no. 2, 231–234.

[17]   C. A. Weibel, *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, Cambridge, 1994.

Departament Matemàtica Aplicada I, ETSEIB UPC., Av. Diagonal 647, 08028-Barcelona. Spain
e-mail: marta.casanellas@upc.edu