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CHAOTICITY FOR MULTICLASS SYSTEMS AND EXCHANGEABILITY WITHIN CLASSES

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Abstract

Classical results for exchangeable systems of random variables are extended to multiclass systems satisfying a natural partial exchangeability assumption. It is proved that the conditional law of a finite multiclass system, given the value of the vector of the empirical measures of its classes, corresponds to *independent* uniform orderings of the samples within *each* class, and that a family of such systems converges in law *if and only if* the corresponding empirical measure vectors converge in law. As a corollary, convergence within *each* class to an infinite independent and identically distributed system implies asymptotic independence between *different* classes. A result implying the Hewitt–Savage 0–1 law is also extended.

Keywords: Interacting particle system; multiclass; multitype; multispecies; mixtures; partial exchangeability; chaoticity; convergence of empirical measures; de Finetti's theorem; directing measure; Hewitt–Savage 0–1 law

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1. Introduction

Among many others, Kallenberg [9], Kingman [10], Diaconis and Freedman [4], and Aldous [1] studied exchangeable random variables (RVs) with Polish state spaces. The related notion of chaoticity (convergence in law to independent and identically distributed (i.i.d.) RVs) appears in many contexts, such as statistical estimation, or the asymptotic study of interacting particle systems or communication networks. It is behind many fruitful heuristics, such as the 'molecular chaos assumption' (*Stosszahlansatz*) used by Ludwig Boltzmann to derive the Boltzmann equation; see [3, Sections 2 and 4].

A sequence of finite exchangeable systems converges in law to an infinite system if and only if the corresponding sequence of empirical measures converges to the directing measure of the limit infinite system, given by the de Finetti theorem. Hence, chaoticity is equivalent to the fact that the empirical measures satisfy a weak law of large numbers, for which Sznitman [13] developed a compactness-uniqueness method of proof, yielding propagation of chaos results for varied models of interest. Sznitman also devised a coupling method for proving chaoticity directly. See [13] for a survey, and [5], [6], and [11] for some developments.

The above notions pertain to the study of *similar* random objects, but many systems in stratified sampling, statistical mechanics, chemistry, communication networks, biology, etc. involve *varied* classes of similar objects (which we call 'particles'). See, for instance, [3, p. 454] and the review papers [2], [6], [8], and [12].

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In this paper we consider natural notions of multi-exchangeability and chaoticity for such multiclass systems, and extend the above results. These notions are explicit in [5, pp. 78, 81], and implicit in [2], [3], [8], and [12], where the corresponding limit equations are directly considered. Graham and Robert [7] extended Sznitman's coupling method in this context. For infinite classes, Aldous calls multi-exchangeability 'internal exchangeability' just before [1, Corollary 3.9].

We prove that the conditional law of a finite multiclass system, given the value of the vector of the empirical measures of its classes, corresponds to choosing *independent* uniform orderings of the samples within *each* class, and that a family of such systems converges in law *if and only if* the corresponding empirical measure vectors converge in law. We conclude by extending a result which implies the Hewitt–Savage 0–1 law.

As a corollary, for a multi-exchangeable system, chaoticity *within* classes implies asymptotic independence *between* classes; see Theorem 3, below. This striking result allows rigorous derivation of limit macroscopic models from microscopic dynamics using Sznitman's compactness-uniqueness methods, and was a major goal of this paper.

We state as a 'proposition' any known result and as a 'theorem' any result we believe to be new. All state spaces $\mathcal{S}$ are Polish, and the weak topology is used for the space of probability measures $\mathcal{P}(\mathcal{S})$, which is also Polish, as are the products of Polish spaces. For $k \geq 1$, we denote by $\Sigma(k)$ the set of permutations of $\{1, \dots, k\}$.

2. Some classical results

2.1. Finite and infinite exchangeable systems

For $N \geq 1$, a finite system $(X_n^N)_{1 \leq n \leq N}$ of RVs with state space $\mathcal{S}$ is *exchangeable* if

$$\mathcal{L}(X_{\sigma(1)}^N, \dots, X_{\sigma(N)}^N) = \mathcal{L}(X_1^N, \dots, X_N^N) \quad \text{for all } \sigma \in \Sigma(N).$$

Then, the conditional law of such a system, given the value of its empirical measure defined by

$$\Lambda^N = \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n^N}, \tag{1}$$

corresponds to a uniform ordering of the N (possibly repeated) values occurring in Λ^N (its atoms, counted according to their multiplicity); see [1, Lemma 5.4, p. 38].

An *infinite* system $(X_n)_{n \geq 1}$ is *exchangeable* if every finite subsystem $(X_n)_{1 \leq n \leq N}$ is exchangeable. De Finetti's theorem (see, e.g. [1], [4], [9], and [10]) states that such a system is a mixture of i.i.d. sequences: its law is of the form

$$\int \mathrm{P}^{\otimes\infty}\, \mathcal{L}_\Lambda(\mathrm{dP}),$$

where $\mathcal{L}_\Lambda$ is the law of the (random) *directing measure* Λ, which can be obtained as

$$\Lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n} \quad \text{almost surely (a.s.).} \tag{2}$$

Thus, laws of infinite exchangeable systems with state space $\mathcal{S}$ and laws of random measures with state space $\mathcal{P}(\mathcal{S})$ are in one-to-one correspondence.

All this leads to the following fact; see [1, Proposition 7.20(b)] and [9, Theorem 1.2].

Proposition 1. *Let* $(X_n^N)_{1\le n\le N}$ *for* $N \ge 1$ *be finite exchangeable systems, and let* Λ^N *be their empirical measures (1). Then, we have*

$$\lim_{N\to\infty} (X_n^N)_{1\le n\le N} = (X_n)_{n\ge 1} \quad \textit{in law},$$

where the limit is necessarily infinite exchangeable and its directing measure is denoted by Λ*, if and only if we have*

$$\lim_{N\to\infty} \Lambda^N = \Lambda \quad \textit{in law}.$$

A sequence $(X_n^N)_{1\le n\le N}$ for $N \ge 1$ is P-*chaotic*, where $\mathrm{P} \in \mathcal{P}(\mathcal{S})$, if

$$\lim_{N\to\infty} \mathcal{L}(X_1^N, \dots, X_k^N) = \mathrm{P}^{\otimes k} \quad \text{for all } k \ge 1,$$

i.e. if it converges in law to an i.i.d. system of RVs of law P. The following corollary of Proposition 1 is proved directly in [11, Proposition 4.2] and [13, Proposition 2.2].

Proposition 2. *Let* $(X_n^N)_{1\le n\le N}$ *for* $N \ge 1$ *be finite exchangeable systems, let* Λ^N *be their empirical measures (1), and let* $\mathrm{P} \in \mathcal{P}(\mathcal{S})$*. Then, the sequence is* P*-chaotic if and only if*

$$\lim_{N\to\infty} \Lambda^N = \mathrm{P} \quad \textit{in law}$$

and, hence, in probability, since the limit is deterministic.

2.2. Multi-exchangeable systems

We assume that $C \ge 1$ and that the state spaces $\mathcal{S}_i$ for $1 \le i \le C$ are fixed. For a multi-index $\boldsymbol{N} = (N_i)_{1\le i\le C} \in \mathbb{N}^C$, we consider a multiclass system

$$(X_{n,i}^{\boldsymbol{N}})_{1\le n\le N_i,\,1\le i\le C}, \qquad X_{n,i}^{\boldsymbol{N}} \text{ with state space } \mathcal{S}_i, \tag{3}$$

where $X_{n,i}^{\boldsymbol{N}}$ is the nth particle, or object, of class i, and we say that it is *multi-exchangeable* if its law is invariant under permutation of the particles *within* classes:

$$\mathcal{L}((X_{\sigma_i(n),i}^{\boldsymbol{N}})_{1\le n\le N_i,\,1\le i\le C}) = \mathcal{L}((X_{n,i}^{\boldsymbol{N}})_{1\le n\le N_i,\,1\le i\le C}) \quad \text{for all } \sigma_i \in \Sigma(N_i).$$

This natural assumption means that particles of a class are statistically indistinguishable, and obviously implies that $(X_{n,i}^{\boldsymbol{N}})_{1\le n\le N_i}$ is exchangeable for $1 \le i \le C$. It is sufficient to check that it is true when all σ_i but one are the identity. The *empirical measure vector*, $(\Lambda_i^{\boldsymbol{N}})_{1\le i\le C}$, with samples in $\mathcal{P}(\mathcal{S}_1) \times \cdots \times \mathcal{P}(\mathcal{S}_C)$, is given by

$$\Lambda_i^{\boldsymbol{N}} = \frac{1}{N_i} \sum_{n=1}^{N_i} \delta_{X_{n,i}^{\boldsymbol{N}}}. \tag{4}$$

We say that the multiclass system $(X_{n,i})_{n\ge 1,\,1\le i\le C}$ with *infinite* classes is *multi-exchangeable* if every finite subsystem $(X_{n,i})_{1\le n\le N_i,\,1\le i\le C}$ is multi-exchangeable. Particles of class i form an exchangeable system, which has a directing measure Λ_i, and we call $(\Lambda_i)_{1\le i\le C}$ the *directing measure vector*.

The following result is given in [1] and is attributed to de Finetti. A remarkable fact is conditional independence between *different* classes.

Proposition 3. ([1, Corollary 3.9].) *Let* $(X_{n,i})_{n\ge 1,\,1\le i\le C}$ *be an infinite multi-exchangeable system, and let* Λ_i *be the directing measure of* $(X_{n,i})_{n\ge 1}$*. Given the directing measure vector* $(\Lambda_i)_{1\le i\le C}$*, the* $X_{n,i}$ *for* $n \ge 1$ *and* $1 \le i \le C$ *are conditionally independent, and* $X_{n,i}$ *has conditional law* Λ_i*.*

3. The extended results

We will extend the main results for exchangeable systems to multi-exchangeable systems, which hold even though the symmetry assumption and resulting structure is much weaker. Indeed, the symmetry order of the multi-exchangeable system, (3), is $N_1! \dots N_C!$, whereas the symmetry order of an exchangeable system of the same size is the much larger $(N_1 + \cdots + N_C)!$.

The following extension of [1, Lemma 5.4] (stated in words at the beginning of Section 2) shows that, for a finite multi-exchangeable system, the classes are *conditionally independent* given the vector of the empirical measures within each class. Hence, *no further information* can be attained on its law by cleverly trying to involve what happens for different classes.

A statistical interpretation of this remarkable fact is that the empirical measure vector is a *sufficient statistic* for the law of the system, the family of all such laws being trivially parameterized by the laws themselves.

Theorem 1. *Let $(X^{N}_{n,i})_{1\le n\le N_i,\,1\le i\le C}$ be a finite multi-exchangeable system, as in (3). Then its conditional law, given the value of the empirical measure vector $(\Lambda^{N}_i)_{1\le i\le C}$ defined in (4), corresponds to* independent *uniform orderings for $1 \le i \le C$ of the N_i values of the particles of class i (possibly repeated), which are the atoms of the value of Λ^{N}_i (counted with their multiplicities).*

Proof. Multi-exchangeability and the obvious fact that

$$\Lambda^{N}_j = \frac{1}{N_j}\sum_{n=1}^{N_j}\delta_{X^{N}_{n,j}} = \frac{1}{N_j}\sum_{n=1}^{N_j}\delta_{X^{N}_{\sigma(n),j}} \quad \text{for all } \sigma \in \Sigma(N_j)$$

imply that, for all $g\colon \mathcal{P}(\mathcal{S}_1)\times\cdots\times\mathcal{P}(\mathcal{S}_C)\to\mathbb{R}_+$ and $f_i\colon \mathcal{S}_i^{N_i}\to\mathbb{R}_+$, we have

$$\begin{aligned}
&\mathrm{E}\bigg[g((\Lambda^{N}_j)_{1\le j\le C})\prod_{i=1}^{C} f_i(X^{N}_{1,i},\dots,X^{N}_{N_i,i})\bigg]\\
&\quad = \frac{1}{N_1!}\sum_{\sigma_1\in\Sigma(N_1)}\cdots\frac{1}{N_C!}\sum_{\sigma_C\in\Sigma(N_C)} \mathrm{E}\bigg[g((\Lambda^{N}_j)_{1\le j\le C})\prod_{i=1}^{C} f_i(X^{N}_{\sigma_i(1),i},\dots,X^{N}_{\sigma_i(N_i),i})\bigg]\\
&\quad = \mathrm{E}\bigg[g((\Lambda^{N}_j)_{1\le j\le C})\prod_{i=1}^{C}\frac{1}{N_i!}\sum_{\sigma\in\Sigma(N_i)} f_i(X^{N}_{\sigma(1),i},\dots,X^{N}_{\sigma(N_i),i})\bigg]\\
&\quad = \mathrm{E}\bigg[g((\Lambda^{N}_j)_{1\le j\le C})\prod_{i=1}^{C}\bigg\langle f_i, \frac{1}{N_i!}\sum_{\sigma\in\Sigma(N_i)}\delta_{(X^{N}_{\sigma(1),i},\dots,X^{N}_{\sigma(N_i),i})}\bigg\rangle\bigg],
\end{aligned} \tag{5}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket between functions and measures, and the empirical measure

$$\frac{1}{N_i!}\sum_{\sigma\in\Sigma(N_i)}\delta_{(X^{N}_{\sigma(1),i},\dots,X^{N}_{\sigma(N_i),i})}$$

corresponds to exhaustive uniform draws without replacement among the atoms $X^{N}_{1,i},\dots,X^{N}_{N_i,i}$ of Λ^{N}_i counted according to multiplicity, and, hence, is a function of Λ^{N}_i. Since g is arbitrary, the characteristic property of conditional expectation yields

$$\mathrm{E}\bigg[\prod_{i=1}^{C} f_i(X^{N}_{1,i},\dots,X^{N}_{N_i,i}) \,\bigg|\, (\Lambda^{N}_i)_{1\le i\le C}\bigg] = \prod_{i=1}^{C}\bigg\langle f_i, \frac{1}{N_i!}\sum_{\sigma\in\Sigma(N_i)}\delta_{(X^{N}_{\sigma(1),i},\dots,X^{N}_{\sigma(N_i),i})}\bigg\rangle,$$

which completes the proof, since the f_i are arbitrary and the spaces Polish.

This result and Proposition 3 lead to the following extension of Proposition 1. We denote by $\lim_{\boldsymbol{N}\to\infty}$ the limit along a fixed arbitrary subsequence of $\boldsymbol{N} \in \mathbb{N}^C$ such that $\min_{1\le i\le C} N_i$ goes to ∞.

Theorem 2. *We consider a family of finite multi-exchangeable multiclass systems*

$$(X_{n,i}^{\boldsymbol{N}})_{1\le n\le N_i,\,1\le i\le C}, \qquad \boldsymbol{N} \in \mathbb{N}^C,$$

all of the form (3) with the same $C \ge 1$ and state spaces $\mathcal{S}_i$, and the corresponding empirical measure vectors $(\Lambda_i^{\boldsymbol{N}})_{1\le i\le C}$ given in (4). Then we have

$$\lim_{\boldsymbol{N}\to\infty} (X_{n,i}^{\boldsymbol{N}})_{1\le n\le N_i,\,1\le i\le C} = (X_{n,i})_{n\ge 1,\,1\le i\le C} \quad \text{in law},$$

where the limit is necessarily infinite multi-exchangeable and its directing measure vector is denoted by $(\Lambda_i)_{1\le i\le C}$, if and only if we have

$$\lim_{\boldsymbol{N}\to\infty} (\Lambda_i^{\boldsymbol{N}})_{1\le i\le C} = (\Lambda_i)_{1\le i\le C} \quad \text{in law.}$$

Proof. Since the state spaces are Polish, it is enough to prove that, for arbitrary $k \ge 1$ and bounded continuous $f_i : \mathcal{S}_i^k \to \mathbb{R}$ for $1 \le i \le C$, we have

$$\lim_{\boldsymbol{N}\to\infty} \mathrm{E}\Bigg[\prod_{i=1}^{C} f_i(X_{1,i}^{\boldsymbol{N}}, \dots, X_{k,i}^{\boldsymbol{N}})\Bigg] = \mathrm{E}\Bigg[\prod_{i=1}^{C} f_i(X_{1,i}, \dots, X_{k,i})\Bigg] \tag{6}$$

if and only if

$$\lim_{\boldsymbol{N}\to\infty} \mathrm{E}\Bigg[\prod_{i=1}^{C} \langle f_i, (\Lambda_i^{\boldsymbol{N}})^{\otimes k}\rangle\Bigg] = \mathrm{E}\Bigg[\prod_{i=1}^{C} \langle f_i, \Lambda_i^{\otimes k}\rangle\Bigg]. \tag{7}$$

Let $(m)_k = m!/(m-k)! = m(m-1)\cdots(m-k+1)$ for $m \ge 1$ and, for $N_i \ge k$, let

$$\Lambda_i^{\boldsymbol{N},k} = \frac{1}{(N_i)_k} \sum_{\substack{1\le n_1,\dots,n_k\le N_i \\ \text{distinct}}} \delta_{(X_{n_1,i}^{\boldsymbol{N}},\dots,X_{n_k,i}^{\boldsymbol{N}})}$$

denote the empirical measure for distinct k-tuples in class i, corresponding to sampling k times *without* replacement among $X_{1,i}^{\boldsymbol{N}}, \dots, X_{N_i,i}^{\boldsymbol{N}}$. Theorem 1 implies that

$$\begin{aligned}\mathrm{E}\Bigg[\prod_{i=1}^{C} f_i(X_{1,i}^{\boldsymbol{N}}, \dots, X_{k,i}^{\boldsymbol{N}})\Bigg] &= \mathrm{E}\Bigg[\mathrm{E}\Bigg[\prod_{i=1}^{C} f_i(X_{1,i}^{\boldsymbol{N}}, \dots, X_{k,i}^{\boldsymbol{N}}) \,\Bigg|\, (\Lambda_i^{\boldsymbol{N}})_{1\le i\le C}\Bigg]\Bigg] \\ &= \mathrm{E}\Bigg[\prod_{i=1}^{C} \langle f_i, \Lambda_i^{\boldsymbol{N},k}\rangle\Bigg] \end{aligned}\tag{8}$$

(which follows directly from (5) with $g = 1$ and the extensions of f_i on $\mathcal{S}_i^{N_i}$), and Proposition 3 similarly implies that

$$\mathrm{E}\Bigg[\prod_{i=1}^{C} f_i(X_{1,i}, \dots, X_{k,i})\Bigg] = \mathrm{E}\Bigg[\prod_{i=1}^{C} \langle f_i, \Lambda_i^{\otimes k}\rangle\Bigg]. \tag{9}$$

The corresponding empirical measure for sampling *with* replacement is given by

$$(\Lambda_i^N)^{\otimes k} = \frac{1}{N_i^k} \sum_{1\le n_1,\dots,n_k\le N_i} \delta_{(X^N_{n_1,i},\dots,X^N_{n_k,i})}$$
$$= \frac{(N_i)_k}{N_i^k}\Lambda_i^{N,k} + \frac{1}{N_i^k} \sum_{\substack{1\le n_1,\dots,n_k\le N_i \\ \text{not distinct}}} \delta_{(X^N_{n_1,i},\dots,X^N_{n_k,i})},$$

and in total variation norm $\|\mu\| = \sup\{\langle \phi, \mu\rangle : \|\phi\|_\infty \le 1\}$ we have

$$\|(\Lambda_i^N)^{\otimes k} - \Lambda_i^{N,k}\| \le 2\frac{N_i^k - (N_i)_k}{N_i^k} \le \frac{k(k-1)}{N_i}, \tag{10}$$

where we bound $N_i^k - (N_i)_k$ by counting $k(k-1)/2$ possible positions for two identical indices with N_i choices and N_i^{k-2} choices for the other $k-2$ positions. Hence, if (6) holds then (8), (9), and (10) imply (7), and conversely, if (7) holds then (8), (9), and (10) imply (6). This completes the proof.

Let $\mathrm{P}_i \in \mathcal{P}(\mathcal{S}_i)$ for $1 \le i \le C$. We say that the family of finite multiclass systems such as in Theorem 2 is $(\mathrm{P}_1, \dots, \mathrm{P}_C)$-*chaotic* if

$$\lim_{N\to\infty} \mathcal{L}((X^N_{n,i})_{1\le n\le k,\, 1\le i\le C}) = \mathrm{P}_1^{\otimes k} \otimes \cdots \otimes \mathrm{P}_C^{\otimes k} \quad \text{for all } k \ge 1.$$

This means that the multiclass systems converge to an *independent* system, in which particles of class i have law P_i. We state a striking corollary of Theorem 2.

Theorem 3. *We consider a family of finite multi-exchangeable multiclass systems such as in Theorem 2, and let* $\mathrm{P}_i \in \mathcal{P}(\mathcal{S}_i)$ *for* $1 \le i \le C$. *Then the family is* $(\mathrm{P}_1, \dots, \mathrm{P}_C)$*-chaotic if and only if the* $(X^N_{n,i})_{1\le n\le N_i}$ *are* P_i*-chaotic for* $1 \le i \le C$.

Proof. Since $(\mathrm{P}_i)_{1\le i\le C}$ is deterministic, $\lim_{N\to\infty}(\Lambda_i^N)_{1\le i\le C} = (\mathrm{P}_i)_{1\le i\le C}$ in law if and only if $\lim_{N\to\infty} \Lambda_i^N = \mathrm{P}_i$ in law for $1 \le i \le C$. We conclude using Theorem 2.

We finish with the following extension of [1, Corollary 3.10] and of the Hewitt–Savage 0–1 law. For $k \ge 1$, we say that a set

$$B \subset \mathcal{S}_1^\infty \times \cdots \times \mathcal{S}_C^\infty$$

is k-multi-exchangeable if, for all permutations σ_i of $\{1, 2, \dots\}$ leaving $\{k+1, k+2, \dots\}$ invariant, $1 \le i \le C$, we have

$$(x_{n,i})_{n\ge 1,\, 1\le i\le C} \in B \iff (x_{\sigma_i(n),i})_{n\ge 1,\, 1\le i\le C} \in B.$$

We define the multi-exchangeable σ-algebra

$$\mathcal{E} = \bigcap_{k\ge 1} \mathcal{E}_k, \qquad \mathcal{E}_k = \{\{(X_{n,i})_{n\ge 1,\, 1\le i\le C} \in B\} : B \text{ is } k\text{-multi-exchangeable}\},$$

and the multitail σ-algebra

$$\mathcal{T} = \bigcap_{k\ge 1} \mathcal{T}_k, \qquad \mathcal{T}_k = \sigma((X_{n,i})_{n\ge k,\, 1\le i\le C}).$$

Clearly, $\mathcal{T}_{k+1} \subset \mathcal{E}_k$ and, hence, $\mathcal{T} \subset \mathcal{E}$.

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Theorem 4. *Let $(X_{n,i})_{n\geq 1,\,1\leq i\leq C}$ be an infinite multi-exchangeable system with directing measure vector $(\Lambda_i)_{1\leq i\leq C}$. Then*

$$\sigma((\Lambda_i)_{1\leq i\leq C}) = \mathcal{T} = \mathcal{E} \quad a.s.$$

If, moreover, the $X_{n,i}$ are independent then $\mathrm{P}(A) \in \{0,1\}$ for all $A \in \mathcal{E}$.

Proof. Consideration of (2) yields $\sigma((\Lambda_i)_{1\leq i\leq C}) \subset \mathcal{T}$ a.s., and we have seen that $\mathcal{T} \subset \mathcal{E}$; hence, the first statement is true if $\mathcal{E} \subset \sigma((\Lambda_i)_{1\leq i\leq C})$ a.s. Now, let $A \in \mathcal{E}$. For every $k \geq 1$, since $A \in \mathcal{E}_k$, there is some k-multi-exchangeable set B_k such that

$$A = \{(X_{n,j})_{n\geq 1,\,1\leq j\leq C} \in B_k\},$$

and, hence, for all permutations σ_i of $\{1, 2, \dots\}$ leaving $\{k+1, k+2, \dots\}$ invariant,

$$\begin{aligned}(\mathbf{1}_A, X_{\sigma_i(n),i})_{n\geq 1,\,1\leq i\leq C} &= (\mathbf{1}_{B_k}((X_{n,j})_{n\geq 1,\,1\leq j\leq C}), X_{\sigma_i(n),i})_{n\geq 1,\,1\leq i\leq C} \\ &= (\mathbf{1}_{B_k}((X_{\sigma_j(n),j})_{n\geq 1,\,1\leq j\leq C}), X_{\sigma_i(n),i})_{n\geq 1,\,1\leq i\leq C}\end{aligned}$$

and the multi-exchangeability of $(X_{n,i})_{n\geq 1,\,1\leq i\leq C}$ implies that

$$\begin{aligned}\mathcal{L}((\mathbf{1}_A, X_{\sigma_i(n),i})_{n\geq 1,\,1\leq i\leq C}) &= \mathcal{L}((\mathbf{1}_{B_k}((X_{\sigma_j(n),j})_{n\geq 1,\,1\leq j\leq C}), X_{\sigma_i(n),i})_{n\geq 1,\,1\leq i\leq C}) \\ &= \mathcal{L}((\mathbf{1}_{B_k}((X_{n,j})_{n\geq 1,\,1\leq j\leq C}), X_{n,i})_{n\geq 1,\,1\leq i\leq C}) \\ &= \mathcal{L}((\mathbf{1}_A, X_{n,i})_{n\geq 1,\,1\leq i\leq C}).\end{aligned}$$

Thus, $(\mathbf{1}_A, X_{n,i})_{n\geq 1,\,1\leq i\leq C}$ is infinite multi-exchangeable, and Proposition 3 implies that the $(\mathbf{1}_A, X_{n,i})$ are conditionally independent given $(\hat{\Lambda}_i)_{1\leq i\leq C}$ and have conditional laws $\hat{\Lambda}_i$, where considering (2) we have

$$\hat{\Lambda}_i = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \delta_{(\mathbf{1}_A, X_{n,i})} = \delta_{\mathbf{1}_A} \otimes \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\delta_{X_{n,i}} = \delta_{\mathbf{1}_A}\otimes\Lambda_i \quad \text{a.s.}$$

Hence, for arbitrary $k \geq 1$ and Borel sets $B_{n,i} \subset \mathcal{S}_i$ for $1 \leq n \leq k$ and $1 \leq i \leq C$,

$$\begin{aligned}&\mathrm{P}(X_{n,i} \in B_{n,i} : 1 \leq n \leq k,\ 1 \leq i \leq C \mid A, (\Lambda_i)_{1\leq i\leq C}) \\ &\quad = \mathrm{P}(X_{n,i} \in B_{n,i} : 1 \leq n \leq k,\ 1 \leq i \leq C \mid (\hat{\Lambda}_i)_{1\leq i\leq C}) \\ &\quad = \prod_{1\leq n\leq k,\,1\leq i\leq C} \Lambda_i(B_{n,i})\end{aligned}$$

is a function of $(\Lambda_i)_{1\leq i\leq C}$, conditionally to which A and $(X_{n,i})_{n\geq 1,\,1\leq i\leq C}$ are thus independent. Since $A \in \mathcal{E}$ is arbitrary, we deduce that $\mathcal{E} \subset \sigma((X_{n,i})_{n\geq 1,\,1\leq i\leq C})$ and $(X_{n,i})_{n\geq 1,\,1\leq i\leq C}$ are conditionally independent given $(\Lambda_i)_{1\leq i\leq C}$, which implies that $\mathcal{E} \subset \sigma((\Lambda_i)_{1\leq i\leq C})$ a.s. This proves the first statement, from which the second follows since $\mathcal{T}$ is a.s. trivial if the $X_{n,i}$ are independent; see the Kolmogorov 0–1 law.

4. Concluding remarks

The important bound (10) is a combinatorial estimate of the difference between sampling with and without replacement; see [1, Proposition 5.6] and [4, Theorem 13] for related results. It is used in [4] to prove de Finetti's theorem.

The result stated in Theorem 3 can be used for proving $(\mathrm{P}_1, \ldots, \mathrm{P}_C)$-chaoticity results, in conjunction with Proposition 2 and Sznitman's compactness-uniqueness methods for proof that the empirical measures Λ_i^N converge in law to P_i for $1 \le i \le C$. This was the main motivation for this paper, as can be seen by its title. In the reviewing process, the referee's suggestions led to a much improved and fuller study of multi-exchangeable systems.

The techniques developed in this paper could also extend convergence results, such as [1, Proposition 7.20(a)] and [9, Theorem 1.3], suited for a family of multi-exchangeable systems of fixed possibly infinite class sizes depending on a parameter. We refrain from doing so for the sake of coherence.

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