CHARACTERIZATIONS OF QUASI-METRIZABLE BITOPOLITICAL SPACES

T. G. RAGHAVAN and I. L. REILLY

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Abstract

In this paper we prove that a pairwise Hausdorff bitopological space \((X, \mathcal{T}_1, \mathcal{T}_2)\) is quasi-metrizable if and only if for each point \(x \in X\) and for \(i, j = 1, 2, \) \(i \neq j\), one can assign \(\mathcal{T}_i\) nbd bases \(\{S(n, i; x) | n = 1, 2, \ldots\}\) such that (i) \(y \notin S(n - 1, i; x)\) implies \(S(n, i; x) \cap S(n, j; y) = \emptyset\), (ii) \(y \in S(n, i; x)\) implies \(S(n, i; y) \subset S(n - 1, i; x)\). We derive two further results from this.


Keywords and phrases: quasi-metric, quasi-uniformity, bitopological space.

The concept of quasi-metric spaces was first introduced by Wilson [11]. The fact that a quasi-metric gives rise to a conjugate quasi-metric was noticed by Kelly [1], thus leading to the study of bitopological spaces. Since then one of the main problems in this area has been to find necessary and sufficient conditions for quasi-metrization. This problem was considered by Kelly [1] Patty [5], Lane [2], Reilly [6], Salbany [9] and later by Pareek [4] and Romaguera [7, 8].

The related notion of quasi-uniform spaces and their properties have been discussed in great detail in Murdheswar and Naimpally [3] and Stoltenberg [10]. In the proof of Theorem 1 we make use of the quasi-uniform analogue of the metrization theorem of Alexandroff and Urysohn, namely, a pairwise Hausdorff quasi-uniform space \((X, \mathcal{V}_1, \mathcal{V}_2)\) is quasi-metrizable if and only if \(\mathcal{V}_1\) has a countable base. From Theorem 1 we derive Theorems 2 and 3 as corollaries. It must be noted that Theorem 2 has been proved by Pareek [4].

We write nbd for neighbourhood. If \(A\) is a subset of \(X\) and \(\mathcal{T}_i\) is a topology on \(X\), then \(\mathcal{T}_i \text{ cl } A(\mathcal{T}_i \text{ int } A)\) is the closure (interior) of \(A\) in the space \((X, \mathcal{T}_i)\).
The letters \( m, n, n_i, m_j, n_j \) represent positive integers. The letters \( i, j \) always take the values \( i, j = 1, 2; \ i \neq j \). \( S(n, i; x) \) represents a \( \mathcal{T}_i \) nbd of \( x \) where \( n \) is a positive integer.

1. **Theorem.** A pairwise Hausdorff bitopological space \((X, \mathcal{T}_1, \mathcal{T}_2)\) is quasi-
   metrizable if and only if for each point \( x \in X \) one can assign \( \mathcal{T}_i \) neighbourhood bases \( \{S(n, i; x) \mid n = 1, 2, \ldots \} \) such that
   
   (i) \( y \notin S(n - 1, i; x) \) implies \( S(n, i; x) \cap S(n, j; y) = \emptyset \),
   (ii) \( y \in S(n, i; x) \) implies \( S(n, i; y) \subset S(n - 1, i; x) \) (\( i, j = 1, 2; \ i \neq j \)).

**Proof.** To prove that the conditions are sufficient, we show first that
\((X, \mathcal{T}_1, \mathcal{T}_2)\) is pairwise regular. If \( S(n, i; x) \ni y \) \( \neq \emptyset \), then \( \mathcal{T}_i \) cl\( S(n, i; x) \) \( \subset X - \mathcal{T}_j \) int\( S(n, j; y) \). Thus if \( y \notin S(n - 1, i; x) \), then \( y \notin \mathcal{T}_j \) cl\( S(n, i; x) \) so that
\[
x \in S(n, i; x) \subset \mathcal{T}_j \text{ cl} S(n, i; x) \subset S(n - 1, i; x).
\]

Furthermore the space is pairwise normal. Indeed, if \( A \) and \( B \) are \( \mathcal{T}_1 \) closed and \( \mathcal{T}_2 \) closed subsets (of \( X \)) respectively such that \( A \cap B = \emptyset \) and \( y \in B \), then there exists a positive integer \( n(y) \) such that \( A \cap \mathcal{T}_2 \text{ cl} S(n(y), 1; y) = \emptyset \). Since \( x \notin S(n(y), 1; y) \) for each \( x \in A \), \( S(n(y) + 1, 1; y) \cap S(n(y) + 1, 2; x) = \emptyset \) for all \( x \in A \). If \( Q_n(y) = \{S(n(y) + 1, 2; x) \mid x \in A \} \), then \( Q_n(y) \supseteq A \) and \( Q_n(y) \cap \mathcal{T}_2 \text{ cl} S(n(y) + 1, 1; y) = \emptyset \). If we write \( \bigcup \mathcal{T}_2 \text{ int} S(n(y) + 1, 1; y) \cap n(y) = k \} = W(k, 1) \), then \( \mathcal{T}_2 \text{ cl} W(k, 1) = \emptyset \) so that we get a \( \mathcal{T}_1 \) open covering \( \{W(k, 1) \mid k = 1, 2, \ldots \} \) of \( A \) such that \( A \cap \mathcal{T}_2 \text{ cl} W(k, 2) = \emptyset \) for each \( k \). Then a standard argument produces disjoint sets \( W_1 \subset \mathcal{T}_1 \) and \( W_2 \subset \mathcal{T}_2 \) such that \( W_1 \supset B \) and \( W_2 \supset A \).

Let \( \mathcal{K}(m, i) = \{S(m, i; x) \mid y \in X \} \). Let \( \mathcal{J}(x, \mathcal{K}(m, i)) = U\{\mathcal{T}_i \text{ int} S(m, i; y) \mid y \in X \} \). Let \( \mathcal{B}(i; x) = \{S(x, \mathcal{K}(m, i)) \mid m = 1, 2, \ldots \} \). We claim \( \mathcal{B}(i; x) \) is a \( \mathcal{T}_i \) local base at \( x \). If \( x \) is fixed initially and \( U(i; x) \) are arbitrary \( \mathcal{T}_i \) nbds of \( x \) then there exists \( n_i \) such that \( x \in S(n_i - 1, i; x) \subset U(i; x) \). Consider \( m = \max(n_1 + 1, n_2 + 1) \). Then clearly \( S(m, i; x) \subset S(n_i, i; x) \). In order to avoid confusion, let us now prove specifically \( \mathcal{B}(2; x) \) is a \( \mathcal{T}_2 \) local base at \( x \). Let \( y \) be such that \( x \in \mathcal{T}_1 \text{ int} S(m, 1; y) \). Then \( S(m, 1; y) \cap S(m, 2; x) \neq \emptyset \) so that \( y \in S(m - 2, 2; x) \subset S(n_2, 2; x) \). Hence \( S(n_2, 2; y) \subset S(n_2 - 1, 2; x) \). Since \( m = \max(n_1 + 1, n_2 + 1) \), \( \mathcal{T}_2 \text{ int} S(m, 2; y) \subset S(n_2, 2; y) \subset S(n_2 - 1, 2; x) \subset U(2; x) \). Thus \( \mathcal{B}(2; x) \) is a \( \mathcal{T}_2 \) local base at \( x \).

If \( x \in \mathcal{T}_1 \text{ int} S(n + 2, 1; y) \), then \( S(n + 2, 1; y) \cap S(n + 2, 2; x) \neq \emptyset \) so that by (i) \( y \in S(n + 1, 2; x) \). Hence \( S(n + 2, 2; y) \subset S(n, 2; x) \) so that \( \mathcal{T}_2 \text{ int} S(n + 2, 2; y) \subset \mathcal{T}_1 \text{ int} S(n + 1, 1; y) \subset \mathcal{T}_2 \text{ int} S(n, 2; x) \). If we define \( \mathcal{L}(m, i) = \{S(x, \mathcal{K}(m, i)) \mid x \in X \} \), then \( \mathcal{L}(n + 2, i) \subset \mathcal{L}(n, i) \) for all
n = 1, 2, 3 \ldots \text{ If we write } V(m, i) = \bigcup \{ \mathcal{S}_j \text{ int } S(m, j; y) \times \mathcal{S}_i \text{ int } S(m, i; y) \mid y \in X \}, \text{ then } (x, y) \in V(m + 2, i) \ast V(m + 2, i) \text{ implies, for some } z \in X \text{ that } (x, z) \in V(m + 2, i) \text{ and } (z, y) \in V(m + 2, i).

Indeed \( x \in V(m + 2, j)[z] \subset \mathcal{S}_j \text{ int } S(m, j; z) \) and \( y \in V(m + 2, i)[z] \subset \mathcal{S}_i \text{ int } S(m, i; z) \) so that \((x, y) \in V(m, i)\). Also notice that \((V(m, i))^{-1} = V(m, j)\).

Thus the conditions are sufficient.

The necessity is proved as follows. Let \( p_1 \) be the quasi-metric that induces \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) be induced by its conjugate \( p_2 \). Let us write \( S(n, i; x) = \{ y \mid p_i(x, y) < (\frac{1}{2})^n \} \). If \( x \not\in S(n - 1, i; x) \) and \( S(n, i; x) \cap S(n, j; y) \neq \emptyset \), then there exists \( y \in X \) such that \( p_i(x, y) < (\frac{1}{2})^n \) and \( p_j(z, y) < (\frac{1}{2})^n \). Hence \( p_i(x, z) \leq p_i(y, z) < (\frac{1}{2})^{n-1} \), a contradiction. Also, if \( y \in S(n, i; x) \) and \( z \in S(n, i; y) \), then \( p_i(x, y) < (\frac{1}{2})^n \) and \( p_j(y, z) < (\frac{1}{2})^n \) so that \( p_i(x, z) < (\frac{1}{2})^{n-1} \) and hence \( z \in S(n - 1, i; x) \).

2. \textbf{Theorem.} A pairwise Hausdorff space \((X, \mathcal{S}_1, \mathcal{S}_2)\) is quasi-metrizable if and only if for each \( x \in X \) one can assign \( \mathcal{S}_i \) nbd bases \( \{ S(n, i; x) \mid n = 1, 2, \ldots \} \) such that

(i) \( y \not\in S(n - 1, i; x) \) implies \( S(n, i; x) \cap S(n, j; y) = \emptyset \),

(ii) \( y \in S(n, i; x) \) implies \( x \in S(n, j; y) \) (\( i, j = 1, 2; i \neq j \)).

\textbf{Proof.} We have to verify only condition (ii) of Theorem 1. Now

\( y \not\in S(n - 1, i; x) \)

implies \( S(n, i; x) \cap S(n, j; y) = \emptyset \) so that if \( z \in S(n, i; x) \), then \( z \not\in S(n, j; y) \).

Thus \( y \not\in S(n, i; z) \). The necessity is obvious.

3. \textbf{Theorem.} A pairwise Hausdorff space \((X, \mathcal{S}_1, \mathcal{S}_2)\) is quasi-metrizable if and only if for each \( x \in X \) one can assign \( \mathcal{S}_i \) nbd bases \( \{ S(n, i; x) \mid n = 1, 2, \ldots \} \) such that

(i) \( y \in S(n, i; x) \) implies \( S(n, i; y) \subset S(n - 1, i; x) \),

(ii) \( y \in S(n, i; x) \) implies \( x \in S(n, j; y) \) (\( i, j = 1, 2; i \neq j \)).

\textbf{Proof.} We only have to verify condition (i) of Theorem 1. If

\( S(n, i; x) \cap S(n, j; y) \neq \emptyset \),

then there is a point \( z \in S(n, i; x) \) and \( z \in S(n, j; y) \) so that \( S(n, i; z) \subset S(n - 1, i; x) \) and \( y \in S(n, i; z) \). Thus \( y \in S(n - 1, i; x) \).

The necessity is obvious.

\textbf{References}


Department of Mathematics
University of Auckland
Private Bag, Auckland
New Zealand