Proceedings of the Edinburgh Mathematical Society (1978), 21, 167–173 ©

THE RADON-NIKODYM THEOREM FOR MULTIMEASURES

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(Received 14th March 1977)

1. Introduction

Let $(S, \mathcal{M})$ be a *measurable space* (that is, a set S in which is defined a σ-algebra $\mathcal{M}$ of subsets) and X a locally convex space. A map M from $\mathcal{M}$ to the family of all non-empty subsets of X is called a *multimeasure* iff for every sequence of disjoint sets $A_n \in \mathcal{M}$ $(n = 1, 2, \ldots)$ with $\bigcup_{n=1}^{\infty} A_n = A$, the series $\Sigma_{n=1}^{\infty} M(A_n)$ converges (in the sense of (**6**), p. 3) to $M(A)$.

The concept of multimeasure with values in $\mathbb{R}^n$ was first introduced by Vind (**15**, p. 174) in order to solve some problems in economics. In (**14**, Théorème 23, p. 292), Valadier has proved the Radon-Nikodym theorem for multimeasures taking values in $\mathbb{R}^n$ (or $\mathbb{R}^\infty$) using the notion of scalar integrability of set-valued functions. (For further results in this aspect, see (**3**) and (**11**).)

In Section 2 of this paper, we shall define integrability for a special class of set-valued functions, which we shall call *perfectly measurable multifunctions*. Then we prove a theorem (Theorem 1) that serves as an example of a multimeasure. In Section 3 we prove the main result, the Radon-Nikodym theorem for multimeasures taking values in a locally convex space; this, however, is not a generalisation of Théorème 23 of (**14**), nor a consequence of that.

I should like to express my gratitude to Professor A. P. Robertson for suggesting this problem and for many helpful discussions.

2. Integrable multifunctions

Henceforth, $(S, \mathcal{M})$ is a measurable space, μ is a finite positive measure on $\mathcal{M}$ and X is a Hausdorff locally convex space with (topological) dual X', except where otherwise specified.

Let F be a map that assigns to each $t \in S$, a non-empty set $F(t) \subseteq X$. Then F is called a *multifunction* (or a set-valued function) from S to X. A point-valued function f from S to X is called a *selector* for F iff $f(t) \in F(t)$ for every $t \in S$. For any subset B of X, we put

$$F^{-1}(B) = \{t \in S : F(t) \cap B \neq \phi\}.$$

The multifunction F is called *measurable* iff $F^{-1}(B) \in \mathcal{M}$ for each closed subset B of X. We say that F is *perfectly measurable* iff it is measurable and, for every closed subset B of X, the multifunction F_B (called the *refinement* of F by B), defined on $F^{-1}(B)$ by $F_B(t) = F(t) \cap B$, has a measurable selector.

The following Lemma 1 that assures a measurable selector for F is due to Leese

(7) (for other results on the existence of measurable selectors for a multifunction, we refer to (1), (4), (10) and (12)). Because this work does not yet seem to have been published, a brief proof is included.

Lemma 1. *Suppose that X' contains a sequence (x'_n), $n = 1, 2, \ldots$, which separates the points of X. Then every compact-valued measurable multifunction F from S to X has a measurable selector.*

Proof. For each $t \in S$, let $F_0(t) = F(t)$ and define $F_n(t)$ $(n = 1, 2, \ldots)$ inductively as follows

$$F_n(t) = \{x \in F_{n-1}(t) : \langle x, x'_n \rangle \text{ maximal}\}.$$

Then it can be shown that each F_n is a compact-valued measurable multifunction from S to X. Moreover, it is clear that $\bigcap_{n=1}^{\infty} F_n(t)$ consists of a single point, $f(t)$ say, and that for every closed set B in X,

$$f^{-1}(B) = \bigcap_{n=1}^{\infty} F_n^{-1}(B).$$

Therefore f is a measurable selector for F.

Lemma 2. *Suppose that X' contains a sequence which separates the points of X. Then every compact-valued measurable multifunction F from S to X is perfectly measurable.*

Proof. Let B be a closed subset of X. Let F_B be the refinement of F by B, which is defined on $F^{-1}(B)$ by $F_B(t) = F(t) \cap B$. Then for every closed set C in X,

$$F_B^{-1}(C) = F^{-1}(B \cap C),$$

which is measurable. Thus F_B is a compact-valued measurable multifunction from $F^{-1}(B)$ to X. Hence, by Lemma 1, F_B has a measurable selector. Therefore F is perfectly measurable.

Before going on, let us recall that a (point-valued) function f from $(S, \mathcal{M}, \mu)$ to X is called *scalarly integrable* iff for every $x' \in X'$, the function $\langle x', f \rangle = x' \circ f$ is integrable. Then, for any $A \in \mathcal{M}$, we denote by $\int_A f \, d\mu$ the linear form on X' defined by

$$\left\langle x', \int_A f \, d\mu \right\rangle = \int_A \langle x', f \rangle \, d\mu.$$

A measurable function f from $(S, \mathcal{M}, \mu)$ to X is said to be *integrable* iff f is scalarly integrable and $\int_A f \, d\mu \in X$ for every $A \in \mathcal{M}$ (see for example (9)).

Now let F be a multifunction from $(S, \mathcal{M}, \mu)$ to X and let $\mathcal{S}(F)$ denote the set of all measurable selectors of F. The multifunction F is called *integrable* iff F is perfectly measurable and every $f \in \mathcal{S}(F)$ is integrable. We denote, for any $A \in \mathcal{M}$,

$$\int_A F \, d\mu = \left\{ \int_A f \, d\mu : f \in \mathcal{S}(F) \right\},$$

which is a subset of X. Note that we require F to be perfectly measurable so that every refinement of F (by any closed subset B of X) contributes to the integral (of course provided that $A \cap F^{-1}(B)$ has a non-zero measure). Otherwise, it may happen that $F(t) = G(t) \cup \{x\}$ for each $t \in S$, where $G(t)$ is contained in a fixed closed subset B of X, G has no measurable selector, and $x \in X \backslash B$. In such a case, the integral of F does not reflect the full range of values taken by F at all (For the basic properties of integrable multifunctions, see (**13**)).

Theorem 1. *Let F be an integrable multifunction from S to X. Then the set-valued map M from $\mathcal{M}$ to X, defined by*

$$M(A) = \int_A F\,d\mu \quad (A \in \mathcal{M}),$$

is a multimeasure.

Proof. Let (A_n), $n = 1, 2, \ldots$, be a sequence of disjoint sets in $\mathcal{M}$ and let $A = \bigcup_{n=1}^{\infty} A_n$. We prove that

$$M(A) = \sum_{n=1}^{\infty} M(A_n).$$

For each n, let $x_n \in M(A_n)$. Then there exist $f_n \in \mathcal{S}(F)$ such that $x_n = \int_{A_n} f_n\,d\mu$, $n = 1, 2, \ldots$. Let us define a function f by

$$f = \begin{cases} f_n \text{ on } A_n, & n = 1, 2, \ldots \\ f_1 \text{ on } S \backslash A. \end{cases}$$

Certainly $f \in \mathcal{S}(F)$, hence f is integrable and $x_n = \int_{A_n} f\,d\mu$. Now, for every $x' \in X'$ and every positive integer N,

$$\left\langle x', \sum_{n=1}^{N} x_n \right\rangle = \sum_{n=1}^{N} \int_{A_n} \langle x', f \rangle\,d\mu,$$

which converges, as $N \to \infty$, to

$$\int_A \langle x', f \rangle\,d\mu = \left\langle x', \int_A f\,d\mu \right\rangle.$$

This means that the series $\Sigma_{n=1}^{\infty} x_n$ converges weakly to $x = \int_A f\,d\mu$, and a similar property holds for every subseries of $\Sigma_{n=1}^{\infty} x_n$. Hence, by the Orlicz-Pettis Theorem (see for example (**6**), p. 4), the series $\Sigma_{n=1}^{\infty} x_n$ converges (unconditionally) to x, which belongs to $M(A)$. Thus we have proved that the series $\Sigma_{n=1}^{\infty} M(A_n)$ is (unconditionally) convergent and is contained in $M(A)$.

To prove the reverse inclusion let $x \in M(A)$; then $x = \int_A f\,d\mu$ for some $f \in \mathcal{S}(F)$. Then, as before, the series $\Sigma_{n=1}^{\infty} \int_{A_n} f\,d\mu$ converges to x. This shows that $x \in \Sigma_{n=1}^{\infty} M(A_n)$, and completes the proof.

3. The Radon-Nikodym theorem

Let K be a convex closed subset of X and let $x' \in X'$. We write $\varphi(x', K) = \sup\{\langle x', x \rangle : x \in K\}$. Following Meyer (**8**, p. 32), we denote by $\mathcal{L}^{\infty}(S, \mathcal{M})$ (resp. $\mathcal{L}^{1}(S, \mathcal{M}, \mu)$) the vector space of all measurable bounded (resp. integrable) real-valued

https://doi.org/10.1017/S0013091500016138 Published online by Cambridge University Press

functions on S and by $L^\infty(S, \mathcal{M}, \mu)$ (resp. $L^1(S, \mathcal{M}, \mu)$) the associated quotient space under the relation of equality μ-almost everywhere.

We first prove the following lemma.

Lemma 3. *Suppose that X is semireflexive. Let ρ be a real-valued function, defined on X', satisfying*:

(i) $\rho(x'+y') \leqslant \rho(x')+\rho(y')$ *and* $\rho(\lambda x') = \lambda\rho(x')$ *for* $\lambda \geqslant 0$,

(ii) *for every* $\epsilon > 0$, $\rho^{-1}((-\infty, \epsilon))$ *is a neighbourhood of* 0 *in* X'.

Then ρ is $\sigma(X', X)$-lower semicontinuous.

Proof. Let $\alpha \in \mathbf{R}$; we prove that the set

$$A = \{x' \in X' : \rho(x') \leqslant \alpha\}$$

is $\sigma(X', X)$-closed. Since X is semireflexive and A is convex, it is sufficient to prove that A is strongly closed. Let $y' \in \bar{A}$ and let $\epsilon > 0$. By (ii), there exists a balanced neighbourhood U of 0 in X' such that $z' \in U$ implies $\rho(z') < \epsilon$. Then there is $x' \in y' + U$ such that $\rho(x') \leqslant \alpha$. It follows that

$$\rho(y') \leqslant \rho(y'-x') + \rho(x') < \epsilon + \alpha.$$

Therefore $y' \in A$, which completes the proof.

Theorem 2. *Let $(S, \mathcal{M}, \mu)$ be a probability space (i.e. $\mu(S) = 1$) and X a locally convex space that is semireflexive. Assume that X' contains a sequence which separates the points of X. Also let M be a convex compact-valued multimeasure from $\mathcal{M}$ to X. Suppose that there exist a convex compact metrizable subset K of X and a positive measure $\nu \ll \mu$ such that for every $A \in \mathcal{M}$,*

$$M(A) \subseteq \nu(A)K.$$

Then there is a convex compact-valued integrable multifunction F from S to X such that

$$M(A) = \int_A F\, d\mu,$$

for every $A \in \mathcal{M}$.

Proof. We may suppose that K is balanced without loss of generality. For every $x' \in X'$, we define for each $A \in \mathcal{M}$,

$$\mu_{x'}(A) = \varphi(x', M(A)).$$

Then each $\mu_{x'}$ is a real-valued bounded measure and these measures satisfy the following properties:

(i) $\mu_{x'+y'} \leqslant \mu_{x'} + \mu_{y'}$,

(ii) $\mu_{\lambda x'} = \lambda \mu_{x'}$ for $\lambda \geqslant 0$.

Moreover, for each $x' \in X'$, we have $\mu_{x'} \ll \mu$; hence there is $\psi_{x'} \in L^1(S, \mathcal{M}, \mu)$ such that for every $A \in \mathcal{M}$,

$$\mu_{x'}(A) = \int_A \psi_{x'}\, d\mu.$$

Certainly the functions $\psi_{x'}$ satisfy the conditions similar to (i) and (ii).

Now we want to find, for each $x' \in X'$, a function $\Psi_{x'}$ in the class $\psi_{x'}$ such that for every $t \in S$, the map $x' \to \Psi_{x'}(t)$ satisfies the conditions of Lemma 3. Let θ be the density function of ν with respect to μ. For every $x' \in X'$ and every $A \in \mathcal{M}$, since $M(A) \subseteq \nu(A)K$, we have

$$\varphi(x', M(A)) \leq \varphi(x', \nu(A)K) = \nu(A)\varphi(x', K).$$

Hence, putting $k_{x'} = |\varphi(x', K)|$, we have $|\mu_{x'}|(A) \leq \nu(A)k_{x'}$, for every $A \in \mathcal{M}$. Therefore, for every $x' \in X'$,

$$|\psi_{x'}| \leq k_{x'}\theta.$$

Let us choose a non-negative member Θ in the class θ and put, for $n = 0, 1, 2, \ldots$,

$$S_n = \{t \in S : n \leq \Theta(t) < n+1\}.$$

Thus each $S_n \in \mathcal{M}$ and the S_n form a partition for S. For each $n = 0, 1, 2, \ldots$, let $\psi_{x',n}$ be the restriction of $\psi_{x'}$ on S_n and define $\mathcal{M}_n$, μ_n analogously. Then $\psi_{x',n} \in L^\infty(S_n, \mathcal{M}_n, \mu_n)$. Therefore, by the Lifting Theorem (**8**, Théorème 12, p. 195), each $\psi_{x',n}$ can be lifted to a function $\Psi_{x',n} \in \mathcal{L}^\infty(S_n, \mathcal{M}_n)$ (note that the lifting map is linear, positive and isometric). We obtain the function $\Psi_{x'}$ by gluing the functions $\Psi_{x',n}$ together. It is clear that $\Psi_{x'} \in \mathcal{L}^1(S, \mathcal{M}, \mu)$ and

(iii) $\Psi_{x'+y'} \leq \Psi_{x'} + \Psi_{y'}$,

(iv) $\Psi_{\lambda x'} = \lambda \Psi_{x'}$ for $\lambda \geq 0$.

Now, let t be chosen and fixed in S; then $t \in S_n$ for some $n = 0, 1, 2, \ldots$. For every $x' \in X'$, since $\|\psi_{x',n}\|_\infty \leq k_{x'}(n+1)$, we have $\|\Psi_{x',n}\| \leq k_{x'}(n+1)$ and hence

$$|\Psi_{x'}(t)| \leq k_{x'}(n+1).$$

According to Hörmander (**5**, Théorème 7), the function $x' \to k_{x'}$ is (strongly) continuous. Hence the function $x' \to \Psi_{x'}(t)$ is continuous at 0. This fact, combined with (iii) and (iv), implies that the function $x' \to \Psi_{x'}(t)$ is $\sigma(X', X)$-lower semicontinuous (Lemma 3). Therefore (by Théorème 5 of (**5**)), there is a convex closed subset $F(t)$ of X such that

$$\Psi_{x'}(t) = \varphi(x', F(t)),$$

for every $x' \in X'$. Moreover if $x' \in K^\circ$, the polar set of K, then $k_{x'} \leq 1$. It follows that

$$F(t) \subseteq (n+1)K^{\infty} = (n+1)K.$$

Hence $F(t)$ is compact for each $t \in S$.

Next, we prove that the multifunction F is integrable. Note first that for every $x' \in X'$, the function $t \to \varphi(x', F(t))$ is measurable and that $F(t)$ is contained in the convex compact metrizable set $(n+1)K$ whenever $t \in S_n$. Thus (by Proposition 8 of (**14**)), the restriction of F on each S_n $(n = 0, 1, 2, \ldots)$ is measurable. Therefore F is measurable. Then, by Lemma 2, F is perfectly measurable. Now let $f \in \mathcal{S}(F)$; then for every $x' \in X'$,

$$-\Psi_{-x'} \leq \langle x', f \rangle \leq \Psi_{x'}.$$

This shows that f is scalarly integrable. Furthermore, for each $n = 0, 1, 2, \ldots$, $f(S_n) \subseteq$

$(n+1)K$ which is convex compact and balanced. Therefore (by Théorème 1 of (2)), f is integrable (that is, $\int_A f\,d\mu \in X$ for every $A \in \mathcal{M}$). This means that F is integrable.

Finally, since X is semireflexive and because every scalarly measurable selector of F is measurable, we obtain from (2, Théorème 2)

$$\varphi\left(x', \int_A F\,d\mu\right) = \int_A \varphi(x', F(\cdot))\,d\mu,$$

for every $A \in \mathcal{M}$ and every $x' \in X'$. Yet the right-hand side is the same as

$$\int_A \Psi_{x'}\,d\mu = \int_A \psi_{x'}\,d\mu = \varphi(x', M(A)).$$

Therefore, by Théorème 1 of (5),

$$M(A) = \int_A F\,d\mu,$$

for every $A \in \mathcal{M}$. This completes the proof.

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https://doi.org/10.1017/S0013091500016138 Published online by Cambridge University Press