MAPS PRESERVING THE LOCAL SPECTRUM OF PRODUCT OF OPERATORS

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Abstract. Let X and Y be infinite-dimensional complex Banach spaces, and $\mathcal{B}(X)$ (resp. $\mathcal{B}(Y)$) be the algebra of all bounded linear operators on X (resp. on Y). For an operator $T \in \mathcal{B}(X)$ and a vector $x \in X$, let $\sigma_T(x)$ denote the local spectrum of T at x. For two nonzero vectors $x_0 \in X$ and $y_0 \in Y$, we show that a map φ from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ satisfies

$$\sigma_{\varphi(T)\varphi(S)}(y_0) = \sigma_{TS}(x_0), \ (T, \ S \in \mathcal{B}(X)),$$

if and only if there exists a bijective bounded linear mapping A from X into Y such that $Ax_0 = y_0$ and either $\varphi(T) = ATA^{-1}$ or $\varphi(T) = -ATA^{-1}$ for all $T \in \mathcal{B}(X)$.

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1. Introduction. Throughout this paper, X and Y denote infinite-dimensional complex Banach spaces, and $\mathcal{B}(X, \ Y)$ denotes the space of all bounded linear maps from X into Y. When $X = Y$, we simply write $\mathcal{B}(X)$ instead of $\mathcal{B}(X, X)$. The local resolvent set, $\rho_T(x)$, of an operator $T \in \mathcal{B}(X)$ at a point $x \in X$ is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\phi : U \to X$ such that $(T - \lambda)\phi(\lambda) = x, \ (\lambda \in U)$. The local spectrum of T at x is defined by $\sigma_T(x) := \mathbb{C} \backslash \rho_T(x)$, and is obviously a closed subset of $\sigma(T)$, the spectrum of T. The local spectral radius of T at x is defined by $r_T(x) := \limsup_{n \to +\infty} \|T^n x\|^{\frac{1}{n}}$, and coincides with maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property (SVEP). Recall that $T \in \mathcal{B}(X)$ is said to have SVEP provided that for every open subset U of $\mathbb{C}$, the equation $(T - \lambda)\phi(\lambda) = 0, \ (\lambda \in U)$ has no nontrivial analytic solution ϕ. Every operator $T \in \mathcal{B}(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property. The remarkable books by P. Aiena [1] and by K.B. Laursen, M.M. Neumann [13] provide an excellent exposition as well as a rich bibliography of the local spectral theory.

Several results on linear or additive preserver problems have been extended to the setting of nonlinear preservers, and, in many cases, their extensions demonstrated to be nontrivial. In particular, the problem of characterizing maps preserving certain

functions, subsets, relations and properties of product of matrices or operators has attracted the attention of several authors; see for instance [**2**, **7**, **10**, **11**, **15**, **14**, **17**] and the references therein. Motivated by problems concerning local automorphisms L. Molnár characterized, in [**15**], maps preserving different spectra of operator or matrix products. In [**11**], Hou and Di described, in particular, maps preserving the numerical range of products of Hilbert space operators. In [**14**], maps preserving the nilpotency of operator products are characterized.

On the other hand, the problem of describing linear or additive maps on $\mathcal{B}(X)$ preserving the local spectra has been initiated by A. Bourhim and T. Ransford in [**5**], and continued by several authors; see for instance [**6**, **8**, **9**] and the references therein. In [**8**], C. Costara described surjective linear maps on $\mathcal{B}(X)$ which preserve operators of local spectral radius zero at points of X. He showed, in particular, that if φ is a surjective linear map on $\mathcal{B}(X)$ such that for every $x \in X$ and $T \in \mathcal{B}(X)$, we have $\mathrm{r}_T(x) = 0$ if and only if $\mathrm{r}_{\varphi(T)}(x) = 0$, then there exists a nonzero scalar c such that $\varphi(T) = c\,T$ for all $T \in \mathcal{B}(X)$. This result has been extended in [**3**] to the nonlinear setting where it was shown that if φ is a surjective (not necessarily linear) map on $\mathcal{B}(X)$ satisfying $\mathrm{r}_{T-S}(x) = 0$ if and only if $\mathrm{r}_{\varphi(T)-\varphi(S)}(x) = 0$, for every $x \in X$ and $S,\ T \in \mathcal{B}(X)$, then there is a nonzero scalar $c \in \mathbb{C}$ and an operator $A \in \mathcal{B}(X)$ such that $\varphi(T) = cT + A$ for all $T \in \mathcal{B}(X)$.

This paper is a continuation of our recent work on nonlinear preserver problems of local spectra of operators [**3**, **4**], and the related works in [**8**]. It can be viewed as a step towards the study of problems on nonlinear preservers of local spectra. Specifically, we are interested in determining the structure of maps preserving the local spectrum at a fixed vector of product of operators. Our main result is the following theorem.

THEOREM 1.1. *Let $x_0 \in X$ and $y_0 \in Y$ be two nonzero vectors. A map φ from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ satisfies*

$$\sigma_{\varphi(T)\varphi(S)}(y_0) \; = \; \sigma_{TS}(x_0), \; (T,\ S \in \mathcal{B}(X)), \tag{1.1}$$

if and only if there exists a bijective bounded linear mapping A from X into Y such that $Ax_0 = y_0$ and either $\varphi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(X)$ or $\varphi(T) = -ATA^{-1}$ for all $T \in \mathcal{B}(X)$.

Our arguments are influenced by ideas from [**14**, **15**] and the approach given therein, but the proof of the main result requires new ingredients such a new spectral characterization of rank one operators in term of the local spectrum. It also relies on a local spectral identity principle that characterizes in term of the local spectrum when two operators are the same. We also would like to point out that if X and Y are isomorphic Banach spaces, then the statements of our main result can be reduced to the case when $X = Y$ and $x_0 = y_0$. But the fact that "X and Y are isomorphic" is a part of the conclusion of this result rather being a part of its hypothesis.

2. Background from local spectral theory and preliminaries. In this section, we fix some notion and gather together some useful lemmas needed for the proof of our main result. The first lemma summarizes some basic properties of the local spectrum which will be used frequently.

LEMMA 2.1. *For an operator $T \in \mathcal{B}(X)$, vectors $x,\ y \in X$ and a nonzero scalar $\alpha \in \mathbb{C}$, the following statements hold.*

(a) *If T has SVEP, then $\sigma_T(x) \neq \emptyset$ provided that $x \neq 0$.*
(b) *$\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha \sigma_T(x)$.*
(c) *$\sigma_T(x + y) \subset \sigma_T(x) \cup \sigma_T(y)$. The equality holds if $\sigma_T(x) \cap \sigma_T(y) = \emptyset$.*
(d) *If T has SVEP, $x \neq 0$ and $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) = \{\lambda\}$.*
(e) *If T has SVEP and $Tx = \alpha y$, then $\sigma_T(y) \subset \sigma_T(x) \subset \sigma_T(y) \cup \{0\}$.*
(f) *If $R \in \mathcal{B}(X)$ commutes with T, then $\sigma_T(Rx) \subset \sigma_T(x)$.*
(g) *$\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ for all $x \in X$ and $n \geq 1$.*

Proof. See for instance [**1**, **13**]. □

For any operator $T \in \mathcal{B}(X)$, let T^* be its adjoint on the dual space X^* of X. For every nonzero $x \in X$ and $f \in X^*$, we sometime write $\langle x, f \rangle$ instead $f(x)$ and let $x \otimes f$ denote the rank one operator defined on X by $(x \otimes f)(y) = f(y)x$, $\sim (y \in X)$. Note that every rank one operator on X can be written in this way and every finite rank operator is a finite sum of rank one operators. Let $\mathcal{F}(X)$ denote the ideal of all finite rank operators on X, and $\mathcal{F}_n(X)$ denote the set of all operators on X of rank of at most n.

For a nonzero vector $x_0 \in X$ and an operator $T \in \mathcal{B}(X)$, we use this notation

$$\sigma_T^*(x_0) := \begin{cases} \{0\} & \text{if } \sigma_T(x_0) = \{0\}, \\ \sigma_T(x_0) \backslash \{0\} & \text{if } \sigma_T(x_0) \neq \{0\}. \end{cases}$$

The second lemma is a useful observation needed to establish the linearity of surjective maps $\varphi : \mathcal{B}(X) \to \mathcal{B}(Y)$ satisfying (1.1).

LEMMA 2.2. *Let x_0 be a nonzero vector in X. For two vectors x and y in X and a linear functional $f \in X^*$, the following statements hold.*

(a)

$$\sigma_{x \otimes f}^*(x_0) = \begin{cases} \{0\} & \text{if } f(x_0) = 0, \\ \{f(x)\} & \text{if } f(x_0) \neq 0. \end{cases}$$

(b) *$\sigma_{(x+y) \otimes f}^*(x_0) = \sigma_{x \otimes f}^*(x_0) + \sigma_{y \otimes f}^*(x_0)$.*
(c) *For all rank one operators $R \in \mathcal{B}(X)$ and all $T, S \in \mathcal{B}(X)$, we have*

$$\sigma_{(T+S)R}^*(x_0) = \sigma_{TR}^*(x_0) + \sigma_{SR}^*(x_0).$$

Proof.

(a) If $f(x_0) = 0$, then $(x \otimes f)x_0 = 0$ and $\sigma_{x \otimes f}(x_0) = \sigma_{x \otimes f}^*(x_0) = \{0\}$. If $f(x_0) \neq 0$, then $(x \otimes f)x_0 = f(x_0)x$ and

$$\{f(x)\} = \sigma_{x \otimes f}(x) \subset \sigma_{x \otimes f}(x_0) \subset \sigma_{x \otimes f}(x) \cup \{0\} = \{0, f(x)\};$$

see Lemma 2.1-(e). It follows that $\sigma_{x \otimes f}^*(x_0) = \{f(x)\}$.
(b) If $f(x_0) = 0$, then $\sigma_{(x+y) \otimes f}^*(x_0) = \sigma_{x \otimes f}^*(x_0) = \sigma_{y \otimes f}^*(x_0) = \{0\}$, and the desired identity holds. If $f(x_0) \neq 0$, then $\sigma_{(x+y) \otimes f}^*(x_0) = \{f(x+y)\} = \{f(x)\} + \{f(y)\} = \sigma_{x \otimes f}^*(x_0) + \sigma_{y \otimes f}^*(x_0)$.

(*c*) Write $R = x \otimes f$. Then, by part (*b*),

$$\sigma^*_{(T+S)R}(x_0) = \sigma^*_{(T+S)(x \otimes f)}(x_0) = \sigma^*_{(T+S)(x) \otimes f}(x_0)$$
$$= \sigma^*_{(Tx+Sx) \otimes f}(x_0) = \sigma^*_{Tx \otimes f}(x_0) + \sigma^*_{Sx \otimes f}(x_0)$$
$$= \sigma^*_{T(x \otimes f)}(x_0) + \sigma^*_{S(x \otimes f)}(x_0) = \sigma^*_{TR}(x_0) + \sigma^*_{SR}(x_0).$$

$\square$

LEMMA 2.3. *Let $x_0 \in X$ and $y_0 \in Y$ be nonzero vectors, and let $A : X \to Y$ and $B : X^* \to Y^*$ be bijective linear transformations. The following statements are equivalent.*

(*a*) *For every $x \in X$ and $f \in X^*$, we have $\sigma_{x \otimes f}(x_0) = \sigma_{Ax \otimes Bf}(y_0)$.*
(*b*) *A is continuous, $B = A^{*-1}$ and $Ax_0 = \alpha y_0$ for some nonzero scalar $\alpha \in \mathbb{C}$.*

Proof. If A is continuous, $B = A^{*-1}$ and $Ax_0 = \alpha y_0$ for some nonzero scalar $\alpha \in \mathbb{C}$, then $\sigma_{Ax \otimes Bf}(y_0) = \sigma_{A(x \otimes f)A^{-1}}(y_0) = \sigma_{x \otimes f}(A^{-1}y_0) = \sigma_{x \otimes f}(\alpha^{-1}x_0) = \sigma_{x \otimes f}(x_0)$ for all $x \in X$ and $f \in X^*$. This establishes the implication (*b*) $\Rightarrow$ (*a*).

Conversely, assume that $\sigma_{x \otimes f}(x_0) = \sigma_{Ax \otimes Bf}(y_0)$ for all $x \in X$ and $f \in X^*$. Let $x \in X$ and $f \in X^*$, and let us show that

$$\langle x, f \rangle = \langle Ax, Bf \rangle. \tag{2.2}$$

Assume first that $f(x_0) \neq 0$, and note that, since $\sigma_{Ax_0 \otimes Bf}(y_0) = \sigma_{x_0 \otimes f}(x_0) = \{f(x_0)\}$, we infer that $Bf(y_0) \neq 0$ and $Ax_0 = \alpha y_0$ for some nonzero $\alpha \in \mathbb{C}$. Since $\sigma_{x \otimes f}(x_0) = \sigma_{Ax \otimes Bf}(y_0)$, we have $\{f(x)\} = \sigma^*_{x \otimes f}(x_0) = \sigma^*_{Ax \otimes Bf}(y_0) = \{Bf(Ax)\}$ and thus (2.2) holds.

Next consider the case when $f(x_0) = 0$ and take a linear functional $g \in X^*$ such that $g(x_0) \neq 0$. It follows from what has been shown previously that $\langle x, g \rangle = \langle Ax, Bg \rangle$, and $\langle x, (f + g) \rangle = \langle Ax, B(f + g) \rangle$. Expanding the last identity, we get $\langle x, f \rangle = \langle Ax, Bf \rangle$ which establishes (2.2) in this case too; as desired.

Now, let us show that A is continuous and $B = A^{*-1}$. Assume that $(x_n)_n \subset X$ is such that $\lim_{n \to \infty} x_n = x \in X$ and $\lim_{n \to \infty} Ax_n = y \in Y$. Then, for every $f \in X^*$, we have $\langle y, Bf \rangle = \lim_{n \to \infty} \langle Ax_n, Bf \rangle = \lim_{n \to \infty} \langle x_n, f \rangle = \langle x, f \rangle = \langle Ax, Bf \rangle$. Since B is bijective and $f \in X^*$ is an arbitrary linear functional, the closed graph theorem shows that A is continuous.

Moreover, if $f \in X^*$ is a fixed linear functional, then for every $x \in X$, we have $\langle x, f \rangle = \langle Ax, Bf \rangle = \langle x, A^*Bf \rangle$, and thus $A^*B = \mathbf{1}_{X^*}$. The proof is therefore complete. $\square$

LEMMA 2.4. *Let $x_0 \in X$ and $y_0 \in Y$ be nonzero vectors, and let $A : X^* \to Y$ and $B : X \to Y^*$ be bijective linear transformations. Then there are $x \in X$ and $f \in X^*$ such that $\sigma_{x \otimes f}(x_0) \neq \sigma_{Af \otimes Bx}(y_0)$.*

Proof. Assume by the way of contradiction that $\sigma_{x \otimes f}(x_0) = \sigma_{Af \otimes Bx}(y_0)$ for all $x \in X$ and $f \in X^*$. Just as in the proof of the previous lemma, one can show that both A and B are continuous and thus both X and Y are reflexive. After identifying X and Y with their second dual spaces X^{**} and Y^{**}, one can also show that $B = A^{*-1}$, and thus $\sigma_R(x_0) = \sigma_{AR^*A^{-1}}(y_0) = \sigma_{R^*}(A^{-1}y_0)$ for all rank one operators $R \in \mathcal{B}(X)$.

Let $x \in X$ be a nonzero element such that $x \notin \mathbb{C}x_0$ and $\langle x, A^{-1}y_0 \rangle = 1$, and let f be a linear functional on X such that $f(x_0) = 0$ and $f(x) = 1$. Set $R := x \otimes f$, and note

that $Rx_0 = 0$ and that $R^* \left(A^{-1} y_0 \right) = f$ and $R^* f = f$. Thus

$$\sigma_R(x_0) = \{0\} \text{ and } \sigma_{R^*} \left(A^{-1} y_0 \right) = \sigma_{R^*} \left(R^* A^{-1} y_0 \right) = \sigma_{R^*} (f) = \{1\}.$$

This contradiction finishes the proof. $\qquad\square$

3. Local spectral identity principle. This section is devoted to some local spectral identity principles, which we believe to be of interest in their own right. The first one characterizes in term of the local spectrum when two operators are linearly dependent.

THEOREM 3.1. *Let $x_0 \in X$ be a nonzero vector, and let $A, B \in \mathcal{B}(X)$. The following statements are equivalent.*

  (a) *$A = \alpha B$ for some nonzero scalar $\alpha \in \mathbb{C}$.*
  (b) *$\sigma_{AT}(x_0) = \{0\} \iff \sigma_{BT}(x_0) = \{0\}$ for all operators $T \in \mathcal{B}(X)$.*
  (c) *$\sigma_{AT}(x_0) = \{0\} \iff \sigma_{BT}(x_0) = \{0\}$ for all rank one operators $T \in \mathcal{B}(X)$.*
  (d) *$\sigma^*_{AT}(x_0) = \{0\} \iff \sigma^*_{BT}(x_0) = \{0\}$ for all rank one operators $T \in \mathcal{B}(X)$.*

Proof. We only need to show that $(d) \Rightarrow (a)$.

Assume that $\sigma^*_{AT}(x_0) = \{0\} \iff \sigma^*_{BT}(x_0) = \{0\}$ for all rank one operators $T \in \mathcal{B}(X)$, and let us first show that Ax_0 and Bx_0 are linearly dependent. Suppose to the contrary that Ax_0 and Bx_0 are linearly independent, and let us show that x_0, Ax_0 and Bx_0 are linearly dependent. If not, let $f_0 \in X^*$ be a linear functional such that $f_0(x_0) = f_0(Ax_0) = 1$ and $f_0(Bx_0) = 0$. For $T_0 := x_0 \otimes f_0$, we have $AT_0 = Ax_0 \otimes f_0$ and $\sigma_{AT_0}(x_0) = \sigma_{AT_0}(x_0 - Ax_0) \cup \sigma_{AT_0}(Ax_0) = \{0, 1\}$. Moreover, $BT_0 = Bx_0 \otimes f_0$ is a nilpotent rank one operator, and $\sigma_{BT_0}(x_0) = \{0\}$. Thus $\sigma^*_{AT_0}(x_0) = \{1\} \neq \{0\} = \sigma^*_{BT_0}(x_0)$, and this contradiction shows that $x_0 = \alpha Ax_0 + \beta Bx_0$ for some scalars $\alpha, \beta \in \mathbb{C}$. Note that either $\alpha \neq 0$ or $\beta \neq 0$ and thus we may and shall assume that $\alpha \neq 0$. Let $f_1 \in X^*$ be a linear functional such that $f_1(Ax_0) = 1$ and $f_1(Bx_0) = 0$. For $T_1 := x_0 \otimes f_1$, we have $AT_1 = Ax_0 \otimes f_1$ and $\{1\} \subset \sigma_{AT_1}(Ax_0) \cup \sigma_{AT_1}(\beta Bx_0) \subset \{0, 1\}$, and thus $\sigma^*_{AT_1}(x_0) = \{1\}$. Note that $BT_1 = Bx_0 \otimes f_1$ is a nilpotent rank one operator and $\sigma_{BT_1}(x_0) = \{0\}$. This shows that $\sigma^*_{AT_1}(x_0) = \{1\} \neq \{0\} = \sigma^*_{BT_1}(x_0)$, and yields to a contradiction. Thus, $Ax_0 = \alpha_{x_0} Bx_0$ for some nonzero scalar α_{x_0}; as desired.

Now, let x be an arbitrary vector in X, and let $S \in \mathcal{B}(X)$ be an operator such that $Sx_0 = x$. Replacing T by ST in the statement (d), we note that $\sigma^*_{AST}(x_0) = \{0\} \iff \sigma^*_{BST}(x_0) = \{0\}$ for all rank one operators $T \in \mathcal{B}(X)$. By what has been shown above, there is α_x that $Ax = ASx_0 = \alpha_x BSx_0 = \alpha_x Bx$ for some nonzero scalar α_x. Thus, there is a nonzero scalar $\alpha \in \mathbb{C}$ such that $A = \alpha B$; as desired. $\qquad\square$

The second result characterizes in term of the local spectrum when two operators are the same.

THEOREM 3.2. *Let $x_0 \in X$ be a nonzero vector, and let $A, B \in \mathcal{B}(X)$. The following statements are equivalent.*

  (a) *$A = B$.*
  (b) *$\sigma_{AT}(x_0) = \sigma_{BT}(x_0)$ for all operators $T \in \mathcal{B}(X)$.*
  (c) *$\sigma_{AT}(x_0) = \sigma_{BT}(x_0)$ for all rank one operators $T \in \mathcal{B}(X)$.*
  (d) *$\sigma^*_{AT}(x_0) = \sigma^*_{BT}(x_0)$ for all rank one operators $T \in \mathcal{B}(X)$.*

Proof. We only need to show that $(d) \Rightarrow (a)$. So, assume that $\sigma^*_{AT}(x_0) = \sigma^*_{BT}(x_0)$ for all rank one operators $T \in \mathcal{B}(X)$, and note that there is a nonzero $\alpha \in \mathbb{C}$ such

that $A = \alpha B$, by Theorem 3.1. To show that such α must be one, assume first that there is $x \in X$ such that Bx and x_0 are linearly independent, and let $f \in X^*$ be a linear functional such that $f(Bx) = f(x_0) = 1$. Set $T := x \otimes f$, and note that $BT = Bx \otimes f$ and thus $\sigma_{BT}(x_0) = \sigma_{BT}(x_0 - Bx) \cup \sigma_{BT}(Bx) = \{0, \ 1\}$. It follows that $\{0, \ 1\} = \sigma_{BT}(x_0) = \sigma_{AT}(x_0) = \sigma_{\alpha BT}(x_0) = \{0, \ \alpha\}$, and $\alpha = 1$.

Now, if Bx and x_0 are linearly dependent for any $x \in X$, then either $B = 0$ and there is nothing to prove, or $B = x_0 \otimes f$ for some $f \in X^*$. If the last case occurs, pick up $x \notin \ker(f)$ such that $f(x) = 1$ and a linear functional $g \in X^*$ such that $g(x_0) = 1$. Let $T := x \otimes g$, and note that $Bx = x_0$ and that $BTx_0 = Bx = x_0$. It follows that $\{1\} = \sigma_{BT}(x_0) = \sigma_{AT}(x_0) = \sigma_{\alpha BT}(x_0) = \{\alpha\}$.   □

4. Local spectral characterization of rank one operators.   The following theorem, which may be of independent interest, gives a spectral characterization of rank one operators in term of local spectrum.

THEOREM 4.1. *For a nonzero vector x_0 of X and a nonzero operator $R \in \mathcal{B}(X)$, the following statements are equivalent.*

  (a) *R has rank one.*
  (b) *$\sigma^*_{RT}(x_0)$ contains at most one element for all $T \in \mathcal{B}(X)$.*
  (c) *$\sigma^*_{RT}(x_0)$ contains at most one element for all $T \in \mathcal{F}_2(X)$.*

Proof. Clearly $(a) \Rightarrow (b) \Rightarrow (c)$, and so we only need to establish the implication $(c) \Rightarrow (a)$. So suppose that $\sigma^*_{RT}(x_0)$ contains at most one element for all $T \in \mathcal{F}_2(X)$, and assume for the sake of contradiction that rank $(R) \geq 2$. Let us first show that x_0 is not in the range of R. Assume to the contrary that x_0 is in the range of R so that there are $x, \ y \in X$ such that $x_0 = Rx$ and $v = Ry$ are linearly independent. Set

$$x_1 = \frac{x - y}{2} \quad \text{and} \quad x_2 = \frac{x + y}{2},$$

and note that $x_0 = Rx_1 + Rx_2$. Now, choose two linear functionals f_1 and f_2 in X^* such that $f_1(x_0) = 1$ and $f_1(v) = -1$, and $f_2(x_0) = f_2(v) = 2$ and thus $f_1(Rx_1) = 1, f_1(Rx_2) = 0$, $f_2(Rx_1) = 0$ and $f_2(Rx_2) = 2$. For $S := x_1 \otimes f_1 + x_2 \otimes f_2$, we have $RS = (Rx_1) \otimes f_1 + (Rx_2) \otimes f_2$, and $RS(Rx_1) = Rx_1$ and $RS(Rx_2) = 2Rx_2$. As $x_0 = Rx_1 + Rx_2$, we have $\sigma_{RS}(x_0) = \sigma_{RS}(Rx_1 + Rx_2) = \sigma_{RS}(Rx_1) \cup \sigma_{RS}(Rx_2) = \{1, \ 2\}$. This contradiction shows x_0 is not in the range of R.

Now, let $u = Rx$ and $v = Ry$ from the range of R so that $x_0, \ u$ and v are linearly independent. Note that, since u and v are linearly independent, we see that either x_0 and x or x_0 and y are linearly independent. Suppose for instance that x_0 and y are linearly independent, and choose two linear functionals f and g in X^* such that $f(x_0) = 1$ and $f(y) = 0$, and $g(x_0) = 0$ and $g(y) = 1$. For $R_1 := x \otimes f + y \otimes g$, we have

$$RR_1 x_0 = Rx = u \quad \text{and} \quad RR_1 y = Ry = v.$$

Set

$$x_3 := \frac{x_0 - y}{2} \quad \text{and} \quad x_4 := \frac{x_0 + y}{2},$$

and note that, since $x_0 = x_3 + x_4$ is not in the range of R, the two sets $\{x_3, \; RR_1x_3\}$ and $\{x_4, \; RR_1x_4\}$ cannot be simultaneously linearly dependent. Now, we shall discuss two cases.

Case 1. One of the two sets is linearly dependent and the other one is linearly independent. Suppose, for instant, that $RR_1x_3 = \alpha x_3$ for some scalar $\alpha \in \mathbb{C}$. On the one hand, we note that, since u and v are linearly independent, such α which must be a nonzero scalar. On the other hand, $\{x_3, \; x_4, \; RR_1x_4\}$ must be linearly independent. If not, since x_3 and x_4 are linearly independent, there scalars $\alpha_1, \; \beta_1 \in \mathbb{C}$ such that $RR_1x_4 = \alpha_1 x_3 + \beta_1 x_4$. Note that, since RR_1x_3 and RR_1x_4 are linearly independent, $\beta_1 \neq 0$ and $R\left(x_4 + (\beta_1 - \alpha_1)\alpha^{-1}x_3\right) = \beta_1(x_3 + x_4) = \beta_1 x_0$. This contradicts the assumption that x_0 is not in the range of R, and shows that x_3, x_4 and RR_1x_4 are linearly independent. Now, choose two linear functionals f_3 and f_4 in X^* such that $f_3(x_3) = 1, f_3(x_4) = f_3(RR_1x_4) = 0, f_4(x_3) = 0$, and $f_4(x_4) = f_4(RR_1x_4) = 2\alpha$. For $T := x_3 \otimes f_3 + x_4 \otimes f_4$, we have $RR_1T = (RR_1x_3) \otimes f_3 + (RR_1x_4) \otimes f_4$ and $RR_1Tx_3 = RR_1x_3 = \alpha x_3$, and

$$RR_1Tx_4 = 2\alpha RR_1x_4, \quad \text{and} \quad RR_1T(RR_1x_4) = 2\alpha RR_1x_4.$$

Thus, $\sigma_{RR_1T}(x_3) = \{\alpha\}$ and $\{2\alpha\} \subset \sigma_{RR_1T}(x_4) \subset \{0, \; 2\alpha\}$. From this, it follows that $\{\alpha, \; 2\alpha\} \subset \sigma_{RR_1T}(x_0) = \sigma_{RR_1T}(x_3) \cup \sigma_{RR_1T}(x_4) \subset \{0, \; \alpha, \; 2\alpha\}$, and $\sigma^*_{RR_1T}(x_0) = \{\alpha, \; 2\alpha\}$. This gives a contradiction and shows that R must have rank one.

Case 2. The two sets $\{x_3, \; RR_1x_3\}$ and $\{x_4, \; RR_1x_4\}$ are linearly independent. If $\{x_3, \; RR_1x_3, \; x_4, \; RR_1x_4\}$ is linearly independent, then choose two linear functionals g_1 and g_2 in X^* such that $g_1(x_3) = g_1(RR_1x_3) = 1, g_1(x_4) = g_1(RR_1x_4) = 0, \; g_2(x_3) = g_2(RR_1x_3) = 0$, and $g_2(x_4) = f_4(RR_1x_4) = 2$. For $T := x_3 \otimes g_1 + x_4 \otimes g_2$, we have $RR_1T = (RR_1x_3) \otimes g_1 + (RR_1x_4) \otimes g_2$. Then $RR_1Tx_3 = RR_1x_3$, and $RR_1T(RR_1x_3) = RR_1x_3$, and thus $\{1\} \subset \sigma_{RR_1T}(x_3) \subset \{0, \; 1\}$. We also have $RR_1Tx_4 = 2RR_1x_4$, and $RR_1T(RR_1x_4) = 2RR_1x_4$, and thus $\{2\} \subset \sigma_{RR_1T}(x_4) \subset \{0, \; 2\}$. From this, it follows that $\{1, \; 2\} \subset \sigma_{RR_1T}(x_0) = \sigma_{RR_1T}(x_3) \cup \sigma_{RR_1T}(x_4) \subset \{0, \; 1, \; 2\}$, and $\sigma^*_{RR_1T}(x_0) = \{1, \; 2\}$. This contradiction shows that $\{x_3, \; RR_1x_3, \; x_4, \; RR_1x_4\}$ is linearly dependent.

Now, suppose that $x_3 = \alpha x_4 + \beta RR_1x_3 + \gamma RR_1x_4$ for some scalars $\alpha, \; \beta, \; \gamma \in \mathbb{C}$, and pick up two linear functionals g_3 and g_4 in X^* such that $g_3(RR_1x_3) = 1, g_3(x_4) = g_3(RR_1x_4) = 0, g_4(RR_1x_3) = 0, g_4(x_4) = -2\gamma/\alpha$, and $g_4(RR_1x_4) = 2$. For $P := x_3 \otimes g_3 + x_4 \otimes g_4$, we have $RR_1P = (RR_1x_3) \otimes g_3 + (RR_1x_4) \otimes g_4$ and $RR_1Px_3 = \beta RR_1x_3$, and $RR_1P(RR_1x_3) = RR_1x_3$. Thus $\{1\} \subset \sigma_{RR_1P}(x_3) \subset \{0, 1\}$. We also have $RR_1Px_4 = -\frac{2\gamma}{\alpha}RR_1x_4$, and $RR_1P(RR_1x_4) = 2RR_1x_4$, and thus $\{2\} \subset \sigma_{RR_1P}(x_4) \subset \{0, 2\}$. It follows that $\{1, \; 2\} \subset \sigma_{RR_1T}(x_0) = \sigma_{RR_1T}(x_3) \cup \sigma_{RR_1T}(x_4) \subset \{0, \; 1, \; 2\}$, and $\sigma^*_{RR_1T}(x_0) = \{1, \; 2\}$. This contradiction shows that R has rank one in this case too. $\square$

5. Proof of theorem 1.1. We only need to establish the 'only if' part whose proof breaks down into several steps. Assume that φ satisfies (1.1), and let us first show that φ is injective and $\varphi(0) = 0$. If $\varphi(A) = \varphi(B)$ for some $A, \; B \in \mathcal{B}(X)$, we get

$$\sigma_{AT}(x_0) = \sigma_{\varphi(A)\varphi(T)}(y_0) = \sigma_{\varphi(B)\varphi(T)}(y_0) = \sigma_{BT}(x_0)$$

for all $T \in \mathcal{B}(X)$. By Theorem 3.2, we see that $A = B$ and φ is injective. But since φ is assumed to be surjective, the map φ is, in fact, bijective. For the second part of the

claim, note that for every $T \in \mathcal{B}(X)$, we have $\{0\} = \sigma_{0 \times T}(x_0) = \sigma_{\varphi(0)\varphi(T)}(y_0)$. Again by Theorem 3.2 and the bijectivity of φ, we see that $\varphi(0) = 0$.

Second let us show that either

$$\sigma_{\varphi(T)}(y_0) = \sigma_T(x_0) \tag{5.3}$$

for all $T \in \mathcal{B}(X)$, or

$$\sigma_{\varphi(T)}(y_0) = -\sigma_T(x_0) \tag{5.4}$$

for all $T \in \mathcal{B}(X)$. To do so, we first prove that either $\varphi(\mathbf{1}) = \mathbf{1}$ or $\varphi(\mathbf{1}) = -\mathbf{1}$. Indeed, for every $T \in \mathcal{B}(X)$, we have $\{0\} = \sigma_T(x_0) = \sigma_{T^2}(x_0) \iff \{0\} = \sigma_{\varphi(T)^2}(y_0) = \sigma_{\varphi(T)}(y_0)$, and thus $\sigma_{\varphi(T)}(y_0) = \{0\} \iff \sigma_T(x_0) = \{0\} = \sigma_{\mathbf{1}T}(x_0) \iff \sigma_{\varphi(\mathbf{1})\varphi(T)}(y_0) = \{0\}$. By the surjectivity of φ and Theorem 3.1, we see that $\varphi(\mathbf{1}) = \alpha\mathbf{1}$ for some nonzero scalar $\alpha \in \mathbb{C}$. Such α must be 1 or -1 since $\{1\} = \sigma_{\mathbf{1}^2}(x_0) = \sigma_{\varphi(\mathbf{1})^2}(y_0) = \sigma_{\alpha^2\mathbf{1}}(y_0) = \{\alpha^2\}$. If $\varphi(\mathbf{1}) = \mathbf{1}$, then for every $T \in \mathcal{B}(X)$, we have

$$\sigma_T(x_0) = \sigma_{\mathbf{1} \times T}(x_0) = \sigma_{\varphi(\mathbf{1})\varphi(T)}(y_0) = \sigma_{\mathbf{1} \times \varphi(T)}(y_0) = \sigma_{\varphi(T)}(y_0),$$

and (5.3) is established. If $\varphi(\mathbf{1}) = -\mathbf{1}$, then

$$\sigma_T(x_0) = \sigma_{\mathbf{1} \times T}(x_0) = \sigma_{\varphi(\mathbf{1})\varphi(T)}(y_0) = \sigma_{-\mathbf{1} \times \varphi(T)}(y_0) = -\sigma_{\varphi(T)}(y_0)$$

for all $T \in \mathcal{B}(X)$. This establish (5.4).

Now, we show that φ is a linear map preserving rank one operators in both directions. Let $R \in \mathcal{B}(X)$ be a rank one operator, and note that, since $\varphi(0) = 0$ and φ is bijective, $\varphi(R) \neq 0$. Moreover, for every operator $S = \varphi(T) \in \mathcal{B}(Y)$, we have $\sigma_{RT}^*(x_0) = \sigma_{\varphi(R)\varphi(T)}^*(y_0) = \sigma_{\varphi(R)S}^*(y_0)$ contains at most one element. By Theorem 4.1, we see that $\varphi(R)$ has rank one. The converse holds in a similar way and thus φ preserves the rank one operators in both directions; as desired.

To establish the linearity of φ, let us first show that φ is homogenous. For every $\lambda \in \mathbb{C}$ and $A, T \in \mathcal{B}(X)$, we have

$$\sigma_{\lambda\varphi(A)\varphi(T)}(y_0) = \lambda\sigma_{\varphi(A)\varphi(T)}(y_0) = \lambda\sigma_{AT}(x_0) = \sigma_{\lambda AT}(x_0) = \sigma_{\varphi(\lambda A)\varphi(T)}(y_0).$$

Since φ is surjective, Theorem 3.2 shows that $\varphi(\lambda A) = \lambda\varphi(A)$; as desired. Now, let us show that φ is additive. Note that, by Theorem 4.1, the map φ preserves rank one operators in both directions. Let $R \in \mathcal{B}(X)$ be a rank one operator and $T, S \in \mathcal{B}(X)$, and note that, by Lemma 2.2, we have

$$\begin{aligned}
\sigma_{\varphi(T+S)\varphi(R)}^*(y_0) &= \sigma_{(T+S)R}^*(x_0) = \sigma_{TR}^*(x_0) + \sigma_{SR}^*(x_0) \\
&= \sigma_{\varphi(T)\varphi(R)}^*(y_0) + \sigma_{\varphi(S)\varphi(R)}^*(y_0) \\
&= \sigma_{(\varphi(T)+\varphi(S))\varphi(R)}^*(y_0).
\end{aligned}$$

By the arbitrariness of the rank one operator R, the bijectivity of φ and Theorem 3.1, we deduce that

$$\varphi(T + S) = \varphi(T) + \varphi(S)$$

for all $T, S \in \mathcal{B}(X)$, and φ is linear.

Finally, let us show that φ takes the desired form. Since φ is a bijective linear map preserving the rank one operators in both directions, either there are bijective linear mappings $A : X \to Y$ and $B : X^* \to Y^*$ such that

$$\varphi(x \otimes f) = Ax \otimes Bf, \ (x \in X, \ f \in X^*), \tag{5.5}$$

or there are bijective linear mappings $A : X^* \to Y$ and $B : X \to Y^*$ such that

$$\varphi(x \otimes f) = Af \otimes Bx, \ (x \in X, \ f \in X^*); \tag{5.6}$$

see for instance [**16**]. By Lemma 2.4, the second form cannot occur, and thus φ takes the form (5.5). Since φ satisfies either (5.3) or (5.4), Lemma 2.3, shows that A is a continuous map for which $Ax_0 = \alpha y_0$ for some nonzero scalar $\alpha \in \mathbb{C}$, and $\varphi(x \otimes f) = \pm A(x \otimes f)A^{-1}$ for all $x \in X$ and $f \in X^*$. Replacing A by $\alpha^{-1}A$, we may and shall assume that $Ax_0 = y_0$. To finishes the proof, note that for every rank one operator $R \in \mathcal{B}(X)$ and every $T \in \mathcal{B}(X)$, we have

$$\sigma_{\pm ATA^{-1}\varphi(R)}(y_0) = \sigma_{ATA^{-1}ARA^{-1}}(y_0) = \sigma_{ATRA^{-1}}(y_0) = \sigma_{TR}(x_0) = \sigma_{\varphi(T)\varphi(R)}(y_0).$$

By Theorem 3.2, we see that $\varphi(T) = \pm ATA^{-1}$ for all $T \in \mathcal{B}(X)$. The proof is now complete.

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