ON REAL ALMOST HERMITIAN STRUCTURES SUBORDINATE TO ALMOST TANGENT STRUCTURES

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1. Introduction. Some of the most important G-structures of the first kind (1) are those defined by linear operators satisfying algebraic relations. Let J be a linear operator acting on the complexified space of a differentiable manifold V, and satisfying a relation of the form

$$J^2 = \lambda^2 I$$

where λ is a complex constant and I is the identity operator. In the case $\lambda \neq 0$ the manifold has an almost product structure (2) which in the case $\lambda = i$ reduces to an almost complex structure (3). In the remaining case, $\lambda = 0$, the manifold has an almost tangent structure (4).

It has been shown that the fibre bundle of the tangent vectors to a differentiable manifold is endowed in a natural manner with an almost tangent structure under "admissible" coordinate changes (5). The natural occurrence of such structures recommends them for further investigation.

Almost hermitian structures subordinate to an almost complex structure have been studied in (3) and this notion

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has been generalized to the almost product structures in (2). In the present paper the author extends this treatment to the almost tangent structures.

Certain degenerate riemannian structures subordinate to the almost tangent structures have been studied in (6) under the name of almost euclidean structures. We will see that the "subordinate almost hermitian structures" are non-degenerate riemannian structures.

2. Almost tangent structures. We consider a differentiable manifold V_{2n} of class C^{∞}. Let T_x be the tangent space at any point $x \in V_{2n}$. We shall assume that a field of class C^{∞} of linear operators J_x is given on V_{2n} such that at each point $x \in V_{2n}$, J_x maps T_x into itself; also we suppose that J_x is of rank n everywhere in V_{2n} and satisfies the relation

$$J_x^2 = 0$$

for any $x \in V_{2n}$, where 0 is the null operator. We then say that V_{2n} is endowed with an almost tangent structure (4).

Let $J(T_x)$ be the image of T_x under J_x. It is a vector space of dimension n and it coincides with $\operatorname{Ker} J_x$, the kernel of the linear operator J_x. If V_x is a supplementary space of $\operatorname{Ker} J_x$ with respect to T_x, we have

$$T_x = V_x \oplus \operatorname{Ker} J_x$$

and as a result J_x induces an isomorphism of V_x with $\operatorname{Ker} J_x$. In the sequel we shall write simply J, T, and V for J_x, T_x and V_x respectively.

Let (e_i), $i = 1, 2, \ldots, 2n$ be a basis of T where (e_α), $\alpha = 1, 2, \ldots, n$ is a basis of $\operatorname{Ker} J$ and (e_{α^*}),

$\alpha^* = \alpha + n$ a basis of V. In general, any Greek index will take the values 1, 2, ..., n and any latin index the values 1, 2, ..., 2n. If (e_i) is a basis such that $Je_{\alpha^*} = e_\alpha$ we call (e_i) a basis adapted to the almost tangent structure or briefly an AT-adapted basis. If $(e_{j'})$ is another AT-adapted basis we have

$$e_{\alpha'} = A^{\beta}_{\alpha'} e_{\beta}$$

$$e_{\alpha'^*} = B^{\beta}_{\alpha'^*} e_{\beta} + A^{\beta^*}_{\alpha'^*} e_{\beta^*}$$

with $A^{\beta^*}_{\alpha'^*} = A^{\beta}_{\alpha'}$. The transformation matrices for the AT-adapted bases are of the form:

$$(2.1) \qquad (A^{i}_{j'}) = \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$$

where $A \in GL(n, R)$, B is an (n, n) matrix, and 0 is the (n, n) null matrix.

With respect to the AT-adapted bases the tensor F^i_j, associated to J, has components given by the matrix

$$(F^i_j) = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}$$

where I_n is the (n, n) identity matrix. In other words

$$(2.2) \qquad F^i_j = \delta^{i+n}_j \quad .$$

The set of all matrices of the form (2.1) is an algebraic subgroup of $GL(2n, R)$ and thus is a Lie group. It will be denoted by G_T. G_T can be intrinsically characterised as the set of matrices of $GL(2n, R)$ which commutes with J.

The set $E_T(V_{2n})$ of all the AT-adapted bases at the different points of V_{2n} has a natural structure as a principal fibre bundle with base space V_{2n}, structural group G_T, and projection mapping $p: E_T(V_{2n}) \to V_{2n}$ which assigns to an AT-adapted basis at x the point x itself.

3. Real almost Hermitian structures. Let us suppose we have defined on V_{2n} a riemannian metric of class C^∞, that is, a real symmetric tensor $G = (g_{ij})$ for which the components in a system of local coordinates (x^i) are functions of class C^∞ of the (x^i) and for which the determinant is everywhere different from 0.

DEFINITION 3.1. We will say that G is hermitian with respect to J if

(3.1) $$G\,{}^tJ + JG = 0$$

where ${}^tJ$ is the transpose of the matrix J.

In the above case we shall say that V_{2n} is endowed with an almost hermitian structure subordinate to the almost tangent structure.

The hermitian condition (3.1) can be written in the form

(3.2) $$g_{ij}F^j_k + g_{jk}F^j_i = 0 .$$

THEOREM 3.1. G is hermitian with respect to J if and only if for any $u, v \in T$ we have

(3.3) $$(u, Jv) + (Ju, v) = 0$$

where $(\,,\,)$ is the inner product defined by G. Condition (3.3) says that J is skew-symmetric with respect to $(\,,\,)$.

Proof. Let $u, v \in T$; then

$$(u, Jv) + (Ju, v) = g_{ij}u^i(Jv)^j + g_{ij}(Ju)^i v^j$$

$$= g_{ij}u^i(F^j_k v^k) + g_{ij}(F^i_k u^k)v^j$$

$$= (g_{ij}F^j_k + g_{jk}F^j_i)\, u^i v^k$$

$$= 0$$

for all u, v if and only if (3.2) is satisfied.

If we multiply the condition (3.1) by J we obtain

$$JG^tJ = 0 \quad \text{since} \quad J^2 = 0.$$

In other words

$$g_{kl}F^k_i F^l_j = 0.$$

It then follows that for any $u, v \in T$ we get

$$(Ju, Jv) = g_{kl}(Ju)^k(Jv)^l$$

$$= g_{kl}(F^k_i u^i)(F^l_j v^j)$$

$$= (g_{kl}F^k_i F^l_j)u^i v^j$$

$$= 0.$$

We may then state:

COROLLARY 3.1. If G is hermitian with respect to J then the image space $J(T) = \text{Ker } J$ is completely isotropic, that is, for any $u, v \in T$ we have

(3.4) $$(Ju, Jv) = 0.$$

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According to (3.3) we have for any $u \in T$ that $(u, Ju) + (Ju, u) = 0$ and thus $(u, Ju) = 0$. This gives us

COROLLARY 3.2. If G is hermitian with respect to J then any vector is orthogonal to its transform by J, that is, for any $u \in T$ we have $(u, Ju) = 0$.

In the remainder of the paper we shall assume that G is hermitian with respect to J and shall investigate the resulting subordinate almost hermitian structure.

To obtain an expression for G relative to an AT-adapted basis we note that from (3.4) we get

$$g_{\alpha\beta} = (e_\alpha, e_\beta) = (Je_{\alpha*}, Je_{\beta*}) = 0$$

and from (3.3) we get

$$g_{\alpha*\beta} = (e_{\alpha*}, e_\beta) = (e_{\alpha*}, Je_{\beta*}) = -(Je_{\alpha*}, e_{\beta*}) = -g_{\alpha\beta*}.$$

These conditions are equivalent to the condition that

$$(3.5) \qquad G = \begin{pmatrix} 0 & G_1 \\ -G_1 & G_2 \end{pmatrix}$$

where $G_1 = (g_{\alpha\beta*})$ and $G_2 = (g_{\alpha*\beta*})$. We see from (3.5) that $\det G = (\det G_1)^2 \neq 0$ and thus G_1 is regular and so is of rank n. Further, since G is symmetric we have

$$G = {}^tG = \begin{pmatrix} 0 & -{}^tG_1 \\ {}^tG_1 & {}^tG_2 \end{pmatrix} ,$$

that is, $G_1 = -{}^tG_1$ and $G_2 = {}^tG_2$. Hence G_1 is skew-symmetric and G_2 is symmetric.

COROLLARY 3.3. In order that an almost tangent structure admit a subordinate almost hermitian structure it is necessary that V_{2n} have dimension $4m$ for some integer m, that is, for some m we have $n = 2m$.

Proof. This follows immediately since G_1 is an (n, n) skew-symmetric matrix over a field of characteristic different than 2 and is of rank n.

THEOREM 3.2. *If ϕ is symmetric and non-singular on Ker J, then we can always define ψ such that ψ is skew-symmetric and non-singular on T; indeed, it suffices to define $\psi(u, v) = \phi(Ju, v) - \phi(u, Jv)$.*

Proof. The symmetry of ϕ implies the skew-symmetry of ψ.

If $\psi(u, v) = 0$ for every $v \in T$, then $\phi(Ju, v) - \phi(u, Jv) = 0$ for every $v \in T$. In particular $\phi(Ju, v) = 0$ for every $v \in \text{Ker } J$. Since ϕ is non-singular on Ker J, $Ju = 0$, that is $u \in \text{Ker } J$. But then $\phi(u, Jv) = 0$ for every $v \in T$. Again, since ϕ is non-singular on Ker J, $u = 0$. Thus ψ is non-singular on T.

THEOREM 3.3. *If ϕ is skew-symmetric and non-singular on Ker J, then we can always define a riemannian structure $(,)$ on T which is hermitian with respect to J; indeed, it suffices to define $(u, v) = \phi(u, Jv) + \phi(v, Ju)$.*

Proof. Clearly $(,)$ is symmetric. Also, by an argument similar to the above we can show that $(,)$ is non-singular. Finally we note that for any $u, v \in T$ we have

$$(u, Jv) + (Ju, v) = \phi(u, J^2 v) + \phi(Jv, Ju) + \phi(Ju, Jv) + \phi(v, J^2 u) = 0 ;$$

thus $(,)$ is hermitian with respect to J.

From (3.2) we may write

$$g_{ij} F^j_k F^k_m + g_{jk} F^j_i F^k_m = 0$$

or

$$g_{jk} F^j_i F^k_m = 0 . \qquad (3.6)$$

DEFINITION 3.2. Let $F_{ik} = g_{jk} F^j_i$; then from (3.2) we have $F_{ji} + F_{ij} = 0$ and the (F_{ij}) are components of a real exterior 2-form F which we will call the fundamental form of the almost hermitian structure. We note that this definition means $F(u, v) = (Ju, v)$.

Let the matrix (g^{ij}) be the inverse of (g_{ij}). Then from (3.6) we obtain $F_{ik}F^{k}_{m} = 0$ or

(3.7) $$F_{ik}F_{mj}g^{jk} = 0$$

THEOREM 3.4. Given a real exterior 2-form (F_{ij}) of rank n and a riemannian metric (g_{ij}) both of class C^{∞} and defined on V_{2n} such that (3.7) is satisfied, one can always define an almost tangent structure and a subordinate almost hermitian structure with hermitian metric (g_{ij}) and fundamental form (F_{ij}).

Proof. Let us define a linear operator J on T by the tensor

$$F^{j}_{i} = F_{ik}g^{kj} .$$

Then $F^{h}_{i}F^{j}_{h} = F_{ik}g^{kh}F_{hl}g^{lj} = (F_{ik}F_{hl}g^{kh})g^{lj} = 0$; thus $J^{2} = 0$. Also rank J = rank (F^{j}_{i}) = rank $(FG^{-1}) = n$ since (g^{kj}) is regular and (F_{ik}) is of rank n. Hence, J defines an almost tangent structure on V_{2n}. Moreover, we have

$$F^{k}_{j}g_{ik} + F^{k}_{i}g_{kj} = F_{jl}g^{lk}g_{ki} + F_{il}g^{lk}g_{kj} = F_{jl}\delta^{l}_{i} + F_{il}\delta^{l}_{j} = F_{ji} + F_{ij} = 0 .$$

Hence (3.2) is satisfied and (g_{ij}) is hermitian with respect to J. The resulting almost hermitian structure has fundamental form $F^{k}_{i}g_{kj} = F_{ij}$ as asserted.

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4. H-adapted bases. Let us consider the matrix H defined by

(4.1)
$$H = (h_{ij}) = \begin{pmatrix} 0 & 0 & 0 & I_m \\ 0 & 0 & -I_m & 0 \\ 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{pmatrix} \; ; \; n = 2m$$

where I_m is the (m, m) identity matrix. We will examine the question of whether it is possible to transform a riemannian metric (g_{ij}), hermitian with respect to J, into a matrix of the form (4.1) by a transformation of AT-adapted bases. If (e_i) is an AT-adapted basis let us consider the vectors $v_i \in T$ such that $v_\alpha = x^\beta_\alpha e_\beta$ and $v_{\alpha*} = x^\beta_\alpha e_{\beta*} + x^\beta_{\alpha*} e_\beta$. Clearly, the (v_i) form an AT-adapted basis. Our question may be restated in the following manner: do scalars (x^j_i), where $x^{\beta*}_{\alpha*} = x^\beta_\alpha$, $x^{\beta*}_\alpha = 0$, exist such that the equations

(4.2)
$$(v_i, v_j) = g_{kl} x^k_i x^l_j = h_{ij}$$

will be satisfied.

The system (4.2) is equivalent to the systems:

(4.3)
$$g_{\alpha\beta*} x^\alpha_\lambda x^\beta_\mu = h_{\lambda\mu*}$$

(4.4)
$$g_{\alpha\beta*} x^\alpha_{\lambda*} x^\beta_\mu + g_{\alpha*\beta} x^\alpha_\lambda x^\beta_{\mu*} + g_{\alpha*\beta*} x^\alpha_\lambda x^\beta_\mu = h_{\lambda*\mu*} .$$

Let $K = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, $G_1 = (g_{\lambda\mu*})$, $G_2 = (g_{\lambda*\mu*})$, $X = (x^\alpha_\lambda)$, and $Y = (x^\alpha_{\lambda*})$. Then we may write (4.3) and

(4.4) in matrix form as follows:

(4.5) $$X G_1 {}^tX = K$$

(4.6) $$Y G_1 {}^tX - X G_1 {}^tY + X G_2 {}^tX = 0$$

Our question now is whether there exists solutions X and Y satisfying the systems (4.5) and (4.6).

In (4.5) we may regard K as the normal form of the congruence class of skew-symmetric matrices of rank $2m$. It then follows that one can always find a regular matrix X satisfying this system (7).

Given a regular matrix X satisfying (4.5) we may write (4.6) in the form

(4.7) $$B {}^tY + Y {}^tB = C$$

where $B = XG_1$ and $C = XG_2 {}^tX$. Let $Z = B {}^tY$; then (4.7) becomes

(4.8) $$Z + {}^tZ = C$$

which will have infinitely many solutions if C is symmetric. Since G_2 is symmetric then $XG_2 {}^tX = C$ is symmetric and (4.8) always admits solutions. Moreover both X and G_1 are regular so that B is regular and $X = {}^t(B^{-1}Z)$ will be a solution of (4.7). Hence we can always find scalars (x_i^j) such that (4.2) is satisfied. We thus have proven

THEOREM 4.1. <u>If</u> (g_{ij}) <u>is a riemannian metric hermitian with respect to</u> J <u>there always exists an AT-adapted basis such that</u> (g_{ij}) <u>will have the form</u> (4.1).

DEFINITION 4.1. We shall say that any AT-adapted basis satisfying the conditions (4.2) is adapted to the almost hermitian structure, or briefly is H-adapted.

Let (e_i) be an H-adapted basis and let (θ^i) be the corresponding dual basis. We will denote by ds^2 the quadratic form defined by G. One then has

$$ds^2 = g_{ij}\theta^i\theta^j = g_{\alpha\beta*}(\theta^\alpha\theta^{\beta*} - \theta^{\alpha*}\theta^\beta)$$

$$= \delta_{\alpha(2)\beta(2)}(\theta^{\alpha(1)}\theta^{\beta(2)*} - \theta^{\alpha(1)*}\theta^{\beta(2)})$$

$$- \delta_{\alpha(1)\beta(1)}(\theta^{\alpha(2)}\theta^{\beta(1)*} - \theta^{\alpha(2)*}\theta^{\beta(1)})$$

where $\alpha(1) = 1, 2, \ldots, m$, $\alpha(2) = \alpha(1)+m$, $\alpha(1)* = \alpha(1)+n$, and $\alpha(2)* = \alpha(2)+n = \alpha(1)+m+n$. We then can write

$$ds^2 = 2\sum_{\alpha(1)=1}^{m}(\theta^{\alpha(1)}\theta^{\alpha(2)*} - \theta^{\alpha(1)*}\theta^{\alpha(2)}) .$$

Also the fundamental form can be written

$$F = F_{ij}\theta^i \wedge \theta^j (i<j) = g_{kj}F^k_i\theta^i \wedge \theta^j = g_{\alpha j}\theta^{\alpha*} \wedge \theta^j (\alpha*<j)$$

$$= g_{\alpha(1)\beta(2)*}\theta^{\alpha(1)*} \wedge \theta^{\beta(2)*} = \sum_{\alpha(1)=1}^{m}\theta^{\alpha(1)*} \wedge \theta^{\alpha(2)*}$$

Suppose now that (e_i) and $(e_{j'})$ are both H-adapted bases; then

(4.9) $$g_{k'l'} = A^i_{k'}A^j_{l'}g_{ij}$$

where $A_o = (A^i_{k'}) = \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$, $A \in GL(2m, R)$, and $G = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}$.

We may then write (4.9) in the form

$$G = A_o G {}^tA_o$$

$$\text{or} \quad \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & A \end{pmatrix} \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \begin{pmatrix} {}^tA & {}^tB \\ 0 & {}^tA \end{pmatrix} = \begin{pmatrix} 0 & AK\,{}^tA \\ -AK\,{}^tA & BK\,{}^tA - AK\,{}^tB \end{pmatrix}$$

Hence,

$$(4.10) \qquad AK\,{}^tA = K$$

$$(4.11) \qquad AK\,{}^tB - BK\,{}^tA = 0 \, .$$

We thus see that a transformation matrix between two H-adapted bases is of the form $\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$ where A, B satisfy the conditions (4.10), (4.11) respectively.

Let G_H be the set of matrices of the form $\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$ where A, B satisfy (4.10), (4.11) respectively. The set G_H is the subset of G_T consisting of matrices orthogonal with respect to G, that is matrices M such that $MG\,{}^tM = G$.

THEOREM 4.2. G_H is a subgroup of $GL(4m, R)$.

<u>Proof</u>. It suffices to show that if $M, M' \in G_H$ then $MM' \in G_H$ and $M^{-1} \in G_H$. Let $M, M' \in G_H$; then

$$(MM')G\,{}^t(MM') = MM'G\,{}^tM'\,{}^tM = MG\,{}^tM = G \text{ and}$$

$$M^{-1}G\,{}^t(M^{-1}) = M^{-1}(MG\,{}^tM)({}^tM)^{-1} = G.$$

It follows that $MM' \in G_H$ and $M^{-1} \in G_H$.

Since the conditions (4.10), (4.11) define G_H as an algebraic subgroup of the Lie group G_T, then necessarily G_H is itself a Lie group.

Let $E_H(V_{2n})$ be the set of the H-adapted bases at the different points of V_{2n} and let $p: E_H(V_{2n}) \to V_{2n}$ be the mapping which to an H-adapted basis at a point $x \in V_{2n}$ makes correspond x itself. $E_H(V_{2n})$ then has, with respect to p, a natural structure of a principal fibre bundle with base space V_{2n} and structural group G_H.

5. H-connections. We will call H-connection any infinitesimal connection (3) defined on the fibre bundle $E_H(V_{2n})$. Given a covering of V_{2n} by neighbourhoods endowed with local cross sections of $E_H(V_{2n})$ an H-connection may be defined in each neighbourhood U by a form w_U with values in the Lie algebra $L(G_H)$ of the group G_H. Such a form may be represented at $x \in V_{2n}$ by means of a matrix of order 2n whose elements are real valued linear forms at x; it will be denoted locally by $w_U = (w^i_j)$ where $(w^i_j) \in L(G_H)$.

To determine the form of the elements of $L(G_H)$ we recall that G_H consists of matrices of $GL(4m, R)$ which commute with J and are orthogonal with respect to G. The Lie algebra of G_H consists of tangent vectors at the identity to curves in G_H. Thus, by differentiation of the relations defining elements of G_H, we find that $L(G_H)$ consists of (4m, 4m) matrices which commute with J and are skew-symmetric with respect to G. Hence $L(G_H)$ consists of matrices of the form

$$(5.1) \qquad M = \begin{pmatrix} A & 0 \\ B & A \end{pmatrix} \quad \text{where } {}^tGM + MG = 0.$$

With respect to an H-adapted basis the conditions (5.1) can be written

$$\begin{pmatrix}0&K\\-K&0\end{pmatrix}\begin{pmatrix}{}^tA&{}^tB\\0&{}^tA\end{pmatrix}+\begin{pmatrix}A&0\\B&A\end{pmatrix}\begin{pmatrix}0&K\\-K&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

or

$$K\,{}^tA + AK = 0 \tag{5.2}$$

and

$$-K\,{}^tB + BK = 0. \tag{5.3}$$

Now (5.2) can be written

$$\begin{pmatrix}0&I_m\\-I_m&0\end{pmatrix}\begin{pmatrix}{}^tA_1&{}^tA_3\\{}^tA_2&{}^tA_4\end{pmatrix}+\begin{pmatrix}A_1&A_2\\A_3&A_4\end{pmatrix}\begin{pmatrix}0&I_m\\-I_m&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\begin{aligned}&{}^tA_1 + A_4 = 0 & & A^{\alpha(1)}_{\beta(1)} + A^{\beta(2)}_{\alpha(2)} = 0\\ \text{or}\quad &{}^tA_2 - A_2 = 0 &\quad\text{or}\quad & A^{\alpha(1)}_{\beta(2)} - A^{\beta(2)}_{\alpha(1)} = 0\ .\\ &{}^tA_3 - A_3 = 0 & & A^{\alpha(2)}_{\beta(1)} - A^{\beta(1)}_{\alpha(2)} = 0\end{aligned}$$

Also, (5.3) can be written

$$-\begin{pmatrix}0&I_m\\-I_m&0\end{pmatrix}\begin{pmatrix}{}^tB_1&{}^tB_3\\{}^tB_2&{}^tB_4\end{pmatrix}+\begin{pmatrix}B_1&B_2\\B_3&B_4\end{pmatrix}\begin{pmatrix}0&I_m\\-I_m&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\begin{aligned}&{}^tB_1 - B_4 = 0 & & A^{\alpha(1)}_{\beta(1)*} - A^{\beta(2)}_{\alpha(2)*} = 0\\ \text{or}\quad &{}^tB_2 + B_2 = 0 \quad \text{where } B = (A^{\beta}_{\alpha*}), &\quad\text{or}\quad & A^{\alpha(2)}_{\beta(1)*} + A^{\beta(2)}_{\alpha(1)*} = 0\\ &{}^tB_3 + B_3 = 0 & & A^{\alpha(1)}_{\beta(2)*} + A^{\beta(1)}_{\alpha(2)*} = 0\end{aligned}$$

Clearly, $E_H(V_{2n})$ may be considered as a sub-bundle of the fibre bundle $E(V_{2n})$ of all bases. Thus any H-connection

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defines canonically a linear connection with which it may be identified.

Conversely, let us be given a linear connection and a covering of V_{2n} by open sets, each furnished with a local section of $E_H(V_{2n})$. This connection may be defined on each neighbourhood by a local form, with values in gl (4m, R), represented by a matrix (w^i_j) whose elements are real valued local Pfaffian forms. In order that the given connection be able to be identified with an H-connection it is necessary and sufficient that (w^i_j) belongs to $L(G_H)$; that is

$$(5.4) \qquad w^{\beta*}_{\alpha} = 0, \quad w^{\beta*}_{\alpha*} = w^{\beta}_{\alpha}$$

(5.5)

$$w^{\beta(1)}_{\alpha(1)} + w^{\alpha(2)}_{\beta(2)} = 0, \quad w^{\beta(2)}_{\alpha(1)} - w^{\alpha(2)}_{\beta(1)} = 0, \quad w^{\beta(1)}_{\alpha(2)} - w^{\alpha(1)}_{\beta(2)} = 0$$

(5.6)

$$w^{\beta(1)}_{\alpha(1)*} - w^{\alpha(2)}_{\beta(2)*} = 0, \quad w^{\beta(2)}_{\alpha(1)*} + w^{\alpha(2)}_{\beta(1)*} = 0, \quad w^{\beta(1)}_{\alpha(2)*} + w^{\alpha(1)}_{\beta(2)*} = 0$$

As shown by H. Eliopoulos (4) the conditions (5.4) express that the tensor $J = (F^i_j)$ has absolute differential zero (which is a necessary and sufficient condition that the given connection be an AT-connection).

To interpret the conditions (5.5) and (5.6) let us introduce the absolute differential of the metric tensor, assuming the conditions (5.4). We have

$$\nabla g_{ij} = - w^k_i g_{kj} - w^k_j g_{ik}$$

which leads to

$$\nabla g_{\alpha\beta} = 0$$

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$$\nabla g_{\alpha\beta*} = - w^{\lambda}_{\alpha} g_{\lambda\beta*} - w^{\lambda}_{\beta} g_{\alpha\lambda*}$$ from which we obtain:

$$\nabla g_{\alpha(1)\beta(1)*} = w^{\beta(2)}_{\alpha(1)} - w^{\alpha(2)}_{\beta(1)}$$

$$\nabla g_{\alpha(2)\beta(1)*} = w^{\beta(2)}_{\alpha(2)} + w^{\alpha(1)}_{\beta(1)}$$

$$\nabla g_{\alpha(1)\beta(2)*} = -w^{\beta(1)}_{\alpha(1)} - w^{\alpha(2)}_{\beta(2)}$$

$$\nabla g_{\alpha(2)\beta(2)*} = -w^{\beta(1)}_{\alpha(2)} + w^{\alpha(1)}_{\beta(2)}$$

$$\nabla g_{\alpha*\beta} = \overline{\nabla g_{\alpha\beta*}}$$

$$\nabla g_{\alpha*\beta*} = -w^{\lambda}_{\alpha*} g_{\lambda\beta*} - w^{\lambda}_{\beta*} g_{\alpha*\lambda}$$ from which we obtain:

$$\nabla g_{\alpha(1)*\beta(1)*} = w^{\beta(2)}_{\alpha(1)*} + w^{\alpha(2)}_{\beta(1)*}$$

$$\nabla g_{\alpha(1)*\beta(2)*} = -w^{\beta(1)}_{\alpha(1)*} + w^{\alpha(2)}_{\beta(2)*}$$

$$\nabla g_{\alpha(2)*\beta(1)*} = w^{\beta(2)}_{\alpha(2)*} - w^{\alpha(1)}_{\beta(1)*}$$

$$\nabla g_{\alpha(2)*\beta(2)*} = -w^{\beta(1)}_{\alpha(2)*} - w^{\alpha(1)}_{\beta(2)*} .$$

It can be seen from the above equations that the condition $\nabla g_{\alpha\beta*} = 0$ is equivalent to the conditions (5.5) and the condition $\nabla g_{\alpha*\beta*} = 0$ is equivalent to the conditions (5.6). Hence the conditions (5.5) and (5.6) are equivalent to the vanishing of the absolute differential of (g_{ij}) provided we assume (5.4). We may then state

THEOREM 5.1. <u>A necessary and sufficient condition that a given linear connection on V_{2n} be an H-connection is that the tensors (F^i_j) and (g_{ij}) have absolute differential zero.</u>

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We say that a linear connection defined on a differentiable manifold furnished with a metric (g_{ij}) is euclidean if $\nabla g_{ij} = 0$. The theorem 5.1 then says that we may identify the H-connections with the euclidean AT-connections.

6. Holonomy groups of the H-connections. Let us consider an H-connection; any horizontal path constructed on $E(V_{2n})$ relative to the linear connection identified with the given H-connection and beginning at an H-adapted basis z ends at an H-adapted basis z'. One deduces from this that the holonomy group (3) at z of this connection is a subgroup of G_H.

Conversely, let V_{2n} be a differentiable manifold furnished with a linear connection and let us suppose that at a point x of V_{2n} there is a basis z such that the holonomy group ψ_z of the connection at z is a subgroup of G_H. Let us consider at the point x the tensors (g_{ij}) and (F^i_j) for which the components with respect to the basis z are defined by:

$$(g_{ij}) = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}, \quad (F^i_j) = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}.$$

These tensors are invariant under ψ_z (since ψ_z is a subgroup of G_H). By parallel transport on V_{2n} we obtain the tensors (g_{ij}) and (F^i_j) defined on the whole manifold. Now at the point x we have $F^i_h F^h_j = 0$ and $F^k_j g_{ki} + F^k_i g_{kj} = 0$ and these relations remain true at any point of V_{2n}. Thus V_{2n} may be endowed with an almost hermitian structure subordinate to an almost tangent structure. Since the tensors (g_{ij}) and (F^i_j) are invariant under ψ_z they have absolute differential zero (3); thus the given connection may be identified with an H-connection. We may then state

THEOREM 6.1. A necessary and sufficient condition that a linear connection in V_{2n} be an H-connection of an almost hermitian structure subordinate to an almost tangent structure is that the holonomy group of the connection be a subgroup of G_H.

In general, the holonomy group is independent of position only if V_{2n} is connected.

Suppose now that V_{2n} is a differentiable manifold furnished with a metric (g_{ij}). We will say that a basis z defined at a point of V_{2n} is adapted to the metric if the components of the metric tensor with respect to z are

$$(g_{ij}) = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} .$$

There will be such an adapted basis only if (g_{ij}) has the proper signature. Let us be given on V_{2n} a euclidean connection and let us suppose that there exists at the point x of V_{2n} a basis z, adapted to the metric, such that the holonomy group ψ_z in this connection is a subgroup of G_T. By assumption $\nabla g_{ij} = 0$; the metric tensor is thus invariant under ψ_z. It follows that ψ_z is a subgroup of G_H. Then, as in theorem 6.1, we may endow V_{2n} with an almost hermitian structure subordinate to an almost tangent structure for which the metric coincides with the initial metric. The given connection can then be identified with an H-connection. We have thus proven:

THEOREM 6.2. A necessary and sufficient condition that a euclidean connection in V_{2n} be an H-connection of an almost hermitian structure subordinate to an almost tangent structure is that the holonomy group of the connection be a subgroup of G_T.

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