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PULL-BACKS IN HOMOTOPY THEORY

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Introduction. The (based) homotopy category consists of (based) topological spaces and (based) homotopy classes of maps. In these categories, pull-backs and push-outs do not generally exist. For example, no essential map between Eilenberg-MacLane spaces of different dimensions has a kernel. In this paper we define homotopy pull-backs and push-outs, which do exist and which behave like pull-backs and push-outs, and we give some of their properties. Applications may be found in [**3**; **5**; **6** and **14**].

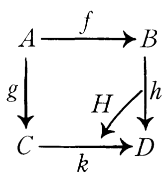
I would like to thank Peter Fantham and Marshall Walker for their help with this paper. They have worked with these techniques [**3**; **14**] and helped me organise my ideas.

We work throughout in the topological category Top or the based topological category Top*. In fact our descriptions will generally be given in Top*. To change to Top simply omit references to the base point. We will denote these categories ambiguously by T.

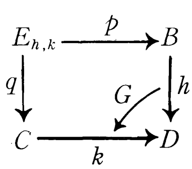
1. Homotopy pull-backs and push-outs.

Let

(1)



denote a square with a homotopy H from $h \circ f$ to $k \circ g$. There is another square



where $E_{h,k} = \{(b, \theta, c) \in B \times D^I \times C; h(b) = \theta(0), k(c) = \theta(1)\}$, D^I is taken in the unbased sense, and with the compact-open topology, $E_{h,k}$ is topologised as a subset of $B \times D^I \times C$, p and q are the restrictions of the projections, and

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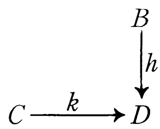
$G((b, \theta, c), t) = \theta(t)$. There is also a map, which we call a *whisker map*, $w : A \rightarrow E_{h,k}$ given by $w(a) = (f(a), H|a \times I, g(a))$. This satisfies:

(i) $p \circ w = f$

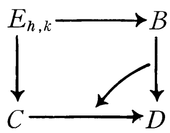
(ii) $q \circ w = g$

(iii) $G \circ w = H$.

(We prefer writing w rather than $w \times 1$ in (iii), where a map is composed with a homotopy.) We call the square (1) a *homotopy pull-back* if w is a homotopy equivalence. We sometimes call the space A, rather than the whole square, a homotopy pull-back for



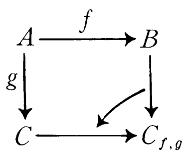
The square



is itself called the *standard* homotopy pull-back.

It is obvious that homotopy pull-backs exist for all such pairs of maps (h, k), and that the space A is unique up to homotopy equivalence. Later we will discuss the pull-back property of this construction, and demonstrate a more precise form of uniqueness. Note that the concept of homotopy pull-back is symmetric in B and C.

The Hilton-Eckmann dual of this definition is just as good. In this case we let $C_{f,g} = B \cup A \times I \cup C/a \times 0 \sim f(a), a \times 1 \sim g(a), * \times I$, and get a square



and a dual *whisker map* $w' : C_{f,g} \rightarrow D$. Then (1) is called a *homotopy push-out* if w' is a homotopy equivalence. (This square is called the *standard* homotopy push-out.)

Examples. (1) If h is a fibration, then the topological pull-back with the static homotopy is a homotopy pull-back. Indeed, in this case the topological pull-back is a strong deformation retract of $E_{h,k}$.

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(2) In particular

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\big\downarrow & & \big\downarrow \\ Y & \longrightarrow & * \end{array}$$

is a homotopy pull-back. (We use $*$ to denote a one-point space.)

(3)

$$\begin{array}{ccc} \Omega B & \longrightarrow & * \\ \big\downarrow & {\scriptstyle G}\swarrow & \big\downarrow \\ * & \longrightarrow & B, \end{array}$$

where $G(\omega, t) = \omega(t)$, is a homotopy pull-back.

(4) If f is a cofibration, then the topological push-out is a homotopy push-out.

(5) In particular, in Top*,

$$\begin{array}{ccc} * & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \vee Y \end{array}$$

is a homotopy push-out. (Note, however, that in the unbased theory, the inclusions $* \subset X$ and $* \subset Y$ may fail to be cofibrations, in which case $X \vee Y$ must be replaced by the long wedge, namely

$$X \cup I \cup Y/*_X \sim 0,\ *_Y \sim 1.$$

Of course, in Top*, $* \subset X$ and $* \subset Y$ are always cofibrations, by definition of a cofibration.)

(6)

$$\begin{array}{ccc} B & \longrightarrow & * \\ \big\downarrow & {\scriptstyle G}\swarrow & \big\downarrow \\ * & \longrightarrow & \Sigma B, \end{array}$$

where $G(b, t) = [(b, t)]$, is a homotopy push-out.

(7) The standard homotopy push-out of

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow & & \\ * & & \end{array}$$

is, of course, the mapping cone of f.

(8) If $* \subset X$ and $* \subset Y$ are closed unbased cofibrations then $X \vee Y \subset$

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$X \times Y$ is a cofibration. (See, for example, Spanier [**10**, Ex 1.E7, p. 58]). In this case

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \wedge Y \end{array}$$

is a homotopy push-out. Otherwise it would seem appropriate to define $X \wedge Y$ to be the standard homotopy push-out, i.e., the mapping cone of $X \vee Y \rightarrow X \times Y$.

(9) $X * Y$ is the standard homotopy push-out of

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \\ Y & & \end{array}$$

2. Homotopy commutative diagrams. In order to be able to discuss the properties of homotopy pull-backs and homotopy push-outs, we need to define what constitutes a homotopy commutative diagram.

Let $f, g : X \rightarrow Y$ be maps and let $F, G : X \times I \rightarrow Y$ be homotopies from f to g. Then F and G are called *equivalent* if there is a map $H : X \times I \times I \rightarrow Y$ such that

(i) $H(x, 0, t) = f(x)$
(ii) $H(x, 1, t) = g(x)$
(iii) $H(x, s, 0) = F(x, s)$
(iv) $H(x, s, 1) = G(x, s)$ for all $(x, s, t) \in X \times I \times I$.

If this happens we write $F \sim G$. This is clearly an equivalence relation.

Remark. It is our general philosophy to study equivalence classes of homotopies, and to ignore differences between higher homotopies. For most purposes this is the appropriate consideration. In an earlier version of this paper I used higher coherent homotopies. This approach has now been discussed more thoroughly by Vogt [**13**]. (It should also be noted that Vogt allows, for example, a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in which f and g are not homotopic. We do not do this.)

Remark. Let $+$ denote the usual track addition of homotopies (given by

$$\begin{aligned} (G + H)(x, t) &= G(x, 2t) \quad \text{if } t \leqq \tfrac{1}{2} \\ &= H(x, 2t - 1) \quad \text{if } t \geqq \tfrac{1}{2}) \end{aligned}$$

and let $-$ denote the reverse, given by

$$(-G)(x, t) = G(x, 1 - t).$$

These operations induce the structure of a groupoid on sets of equivalence classes of homotopies (in the sense of a category in which every morphism is invertible, not in the sense of a set with a binary operation).

A *homotopy commutative diagram* is defined to consist of

*HCD*1. A set of objects of T and morphisms between them, together with the compositions of these morphisms. (This set of objects may include more than one "copy" of an object Y of T. In this case each morphism to or from Y must specify which copy is meant, and compositions may not confuse two different copies. For example, the diagram $Y_1 \xrightarrow{f} Y_2$, where Y_1 and Y_2 are different copies of Y, does not include $f \circ f$.)

*HCD*2. For each pair $\beta, \gamma : B \to C$ in the diagram, a homotopy $H_{\beta,\gamma}$ from β to γ such that

*HCD*3. $H_{\beta,\beta}$ is equivalent to the static homotopy.

*HCD*4. If $\beta, \gamma, \delta : B \to C$ then $H_{\beta,\gamma} + H_{\gamma,\delta} \sim H_{\beta,\delta}$, and

*HCD*5. If $\alpha : A \to B, \beta, \gamma : B \to C$ and $\epsilon : C \to D$ then $H_{\epsilon\circ\beta\circ\alpha,\, \epsilon\circ\gamma\circ\alpha} \sim \epsilon \circ H_{\beta,\gamma} \circ \alpha$.

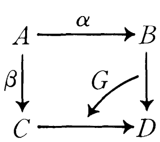
A commutative diagram becomes a homotopy commutative diagram when provided with the appropriate static homotopies. Such a diagram is called *flat*.

We will specify a homotopy commutative diagram by giving the set of objects and maps together with enough homotopies so that the others, at least up to equivalence, can be deduced from *HCD*4 & 5. (In most cases there is, in fact an obvious choice for the missing homotopies. Also, we may omit mention of some or all homotopies if there is no need to be specific about them.)

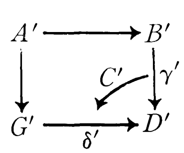
Examples. (1) The squares we have already been dealing with, which have a homotopy across them, are homotopy commutative diagrams.

(2) The cube mentioned in the following definition.

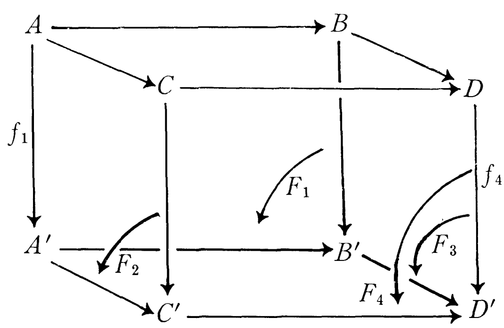
We say that a homotopy commutative square



is *equivalent* to



if there is a homotopy commutative cube



with the given squares as upper and lower faces and with all the vertical maps homotopy equivalences. (The condition on the homotopies for homotopy commutativity is

$$f_4 \circ G + F_4 \circ \beta + \delta' \circ F_2 \sim F_3 \circ \alpha + \gamma' \circ F_1 + G' \circ f_1.)$$

We define *equivalence* of other diagrams similarly.

We will need the following eleven results, which we prove in Appendix 1.

LEMMA 1. *On squares, this is indeed an equivalence relation.*

LEMMA 2. *Let $f : A \to B$ be a homotopy equivalence, and suppose we are given two maps $g, h : X \to A$ and a homotopy H from $f \circ g$ to $f \circ h$. Then there is a homotopy G from g to h such that $f \circ G \sim H$.*

COROLLARY 3. *If $f : A \to B$ is a homotopy equivalence, $g, h : X \to A$ are maps and G, H are homotopies from g to h such that $f \circ G \sim f \circ H$, then $G \sim H$.*

The next two results are the duals of the last two.

LEMMA 4. *Let $f : A \to B$ be a homotopy equivalence, and suppose we are given two maps $g, h : B \to Y$ and a homotopy H from $g \circ f$ to $h \circ f$. Then there is a homotopy G from g to h such that $G \circ f \sim H$.*

COROLLARY 5. *If $f : A \to B$ is a homotopy equivalence, $g, h : B \to Y$ are maps and G, H are homotopies from g to h such that $G \circ f \sim H \circ f$, then $G \sim H$.*

LEMMA 6. *If a square is equivalent to a homotopy pull-back then it is a homotopy pull-back.*

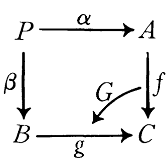
COROLLARY 7. *If, in the homotopy commutative cube above, the upper and lower faces are homotopy pull-backs and the last three vertical maps are homotopy equivalences, then so is the first.*

The next two results are the duals of the last two.

LEMMA 8. *If a square is equivalent to a homotopy push-out, then it is a homotopy push-out.*

COROLLARY 9. *If, in the homotopy commutative cube above, the upper and lower faces are homotopy push-outs and the first three vertical maps are homotopy equivalences then so is the last.*

We say that a homotopy commutative square



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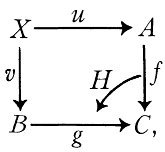
has the *pull-back property* if, given another square

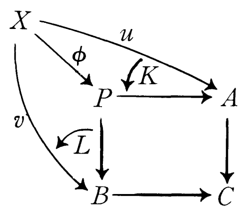


then

*PB*1. There is a map $\phi : X \to P$ (also called a *whisker map*) and

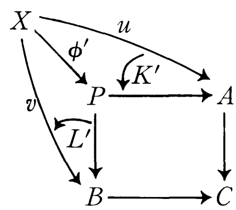
*PB*2. There are the necessary extra homotopies so that, with G and H,

*PB*3. The diagram



is homotopy commutative and, further,

*PB*4. If



is another such homotopy commutative diagram, then there is a homotopy M from ϕ to ϕ' such that $K + \alpha \circ M \sim K'$ and $\beta \circ M + L' \sim L$.

We refer to *PB*4 by saying that the diagram of *PB*3 is *essentially unique*.

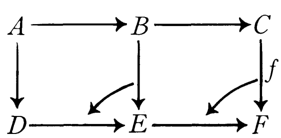
The *push-out property* is defined dually.

THEOREM 10. *A square has the pull-back property if and only if it is a homotopy pull-back.*

THEOREM 11. *A square has the push-out property if and only if it is a homotopy push-out.*

3. Elementary properties of homotopy pull-backs and push-outs.

LEMMA 12. *Let*

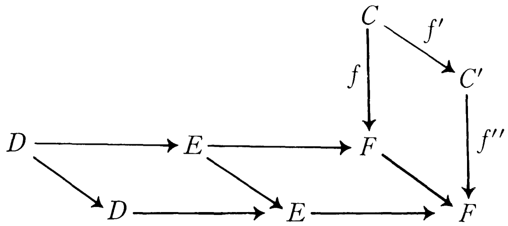


be a homotopy commutative diagram. If the left and right squares are homotopy pull-backs then so is the large square.

Proof. It is well known (Spanier [**10**, p. 99], for example) that $f : C \to F$ may be factored as

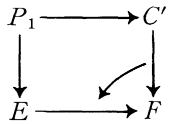
$$C \xrightarrow{f'} C' \xrightarrow{f''} F$$

in such a way that f' is a homotopy equivalence and f'' is a fibration. Then there is a commutative diagram

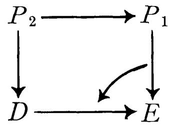


in which the diagonal maps are the identity except for $f' : C \to C'$.

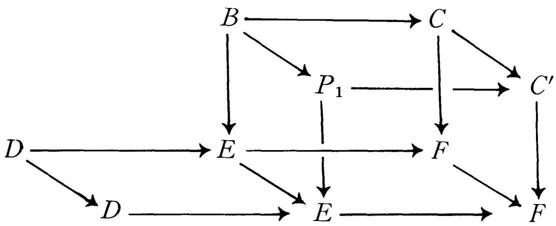
Let



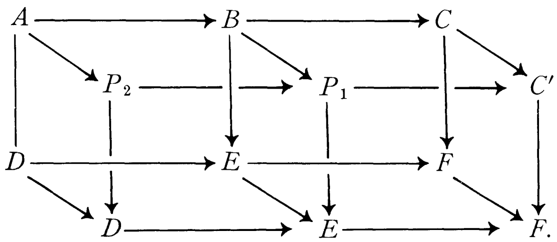
and



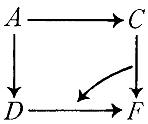
be the fibred pull-backs (i.e., $P_1 \to E$ and $P_2 \to D$ are the induced fibrations). These are flat homotopy pull-backs. Hence, by Theorem 10, there is a homotopy commutative diagram



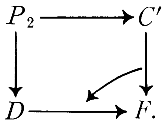
and, for the same reason, this extends to a homotopy commutative diagram



By Corollary 7, $B \to P_1$ is a homotopy equivalence and, similarly, so is $A \to P_2$. Thus



is equivalent to



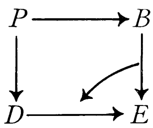
But the latter is a fibred pull-back and hence a homotopy pull-back. Thus, by Lemma 6, the large square of the original diagram is a homotopy pull-back.

LEMMA 13. *If, in the same diagram, the left and right squares are homotopy push-outs then so is the large square.*

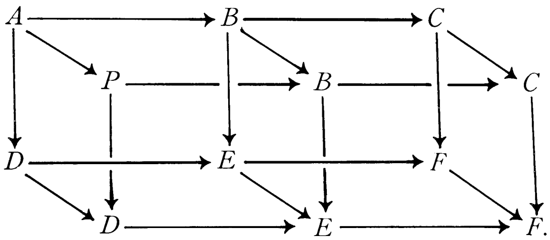
Proof. This is the dual of the last result, and has the dual proof.

LEMMA 14. *If, in the same diagram, the right and large squares are homotopy pull-backs, then so is the left square.*

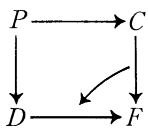
Proof. Let



be a homotopy pull-back. Then, by Theorem 10, we have a homotopy commutative diagram



Applying Lemma 12,



is a homotopy pull-back and hence, applying Corollary 7 to the outside squares, $A \to P$ is a homotopy equivalence. We now apply Lemma 6 to the left hand half of the diagram to get the required result.

Note: Reference in [**5**; **6**] to Theorem 14 of this paper should be to Theorem 47.

Lemma 15. *If, in the same diagram, the left and large squares are homotopy push-outs then so is the right square.*

Proof. This is the dual of the last result, and has the dual proof.

Lemma 16. *If, in the same diagram, the left and large squares are homotopy pull-backs, it does not follow that the right square is a homotopy pull-back, even if E is path-connected.* (It does so follow if we also assume that all the spaces are CW-complexes. See Lemma 37 below.)

Proof. If we are in Top*, we obtain an example as follows.

Let $S = \{(0,0), (1,0), (\frac{1}{2}, 0), (\frac{1}{3}, 0), \ldots\}$ in R^2, and let X be the join in R^2 of $(0, 1)$ with S. We use $(0, 0)$ as base point.

Now ΩX is contractible, since the homotopy type of ΩX does not depend on the choice of base point in X and X is itself contractible if $(0, 1)$ is the base point. On the other hand, X with $(0, 0)$ as base point is not contractible, as follows. Let $H : X \times I \rightarrow X$ be a contraction. Then $H^{-1}(0, 1)$ contains points arbitrarily close to $(0, 0) \times I$ but no point of $(0, 0) \times I$ and so is not closed.

Thus the following diagram, with trivial homotopies and identity maps, is an example:

$$\begin{array}{ccccc} * & \longrightarrow & * & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X & \longrightarrow & X. \end{array}$$

In Top we replace X by $X \vee X$. We leave the details to the reader.

Lemma 17. *If, in the same diagram, the right and large squares are homotopy push-outs, it does not follow that the left square is a homotopy push-out, even if A, B and D are connected and all the spaces are CW-complexes.* (It does so follow if we also assume that B and D are simply connected. See Lemma 41.)

Proof. Let L be Epstein's space [**2**] .This is too complicated to describe here, but has the following properties:

(i) L is a connected CW-complex;

(ii) L is not contractible;

(iii) the suspension ΣL of L is contractible.

Thus

$$\begin{array}{ccccc} L & \longrightarrow & L & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & * & \longrightarrow & * \end{array}$$

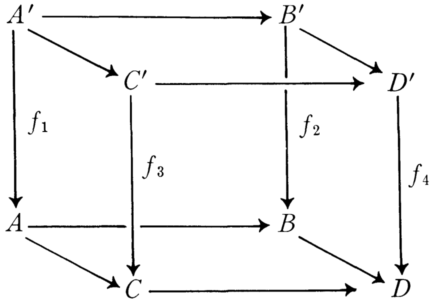
(with trivial homotopies and identity maps) provides an example.

Note: Reference in [**6**] to Theorem 17 of this paper should be to Theorem 50.

4. The first cube theorem. The purpose of this section is to state and prove the following theorem. We draw attention to the fact that we place no restriction on the spaces involved. We work throughout this section in Top.

THEOREM 18. *Suppose that we have a homotopy commutative diagram*

(2)



in which

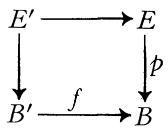
(i) *the left and rear faces are homotopy pull-backs, and*
(ii) *the top and bottom faces are homotopy push-outs.*

Then the front and right faces are homotopy pull-backs.

We will need several lemmas to prove this result. The main technique is to use the weak covering homotopy property (*WCHP*). (See Dold [**1**].) We first remind the reader of the definition of this property.

A map $p : E \to B$ is said to have the *WCHP* if, given a map $f : X \to E$ and a homotopy $\bar{H} : X \times I \to B$ such that $\bar{H}(x, t) = pf(x)$ whenever $0 \leqq t \leqq \frac{1}{2}$, there is a homotopy $H : X \times I \to E$ with $p \circ H = \bar{H}$ and $f = H|X \times 0$.

LEMMA 19. *Let*



be a topological pull-back and let p have the WCHP. Then this square, with the static homotopy, is a homotopy pull-back.

Proof. Let $\bar{E}_{p,f} \subset E_{p,f}$ be the subset given by

$$\bar{E}_{p,f} = \{(e, \theta, b'); p(e) = \theta(t) \text{ all } t \in [0, \tfrac{1}{2}]\}.$$

Then $\bar{E}_{p,f}$ is a weak deformation retract of $E_{p,f}$ and E' is a weak deformation retract of $\bar{E}_{p,f}$. Hence the result.

Let $p : E' \to B$ be a map and $E \subset E'$. Then a *weak deformation retraction of E' to E over B* is defined to be a homotopy $H : E' \times I \to E'$ such that

(i) $1_{E'} = H|E' \times 0$
(ii) $H(E \times I) \subset E$
(iii) $H(E' \times 1) \subset E$
(iv) $p \circ H$ is static.

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LEMMA 20. *Let $p : E' \to B$ be a map, $E \subset E'$, and suppose that there is a weak deformation retraction of E' to E over B. If $p|E : E \to B$ has the CHP then p has the WCHP.*

Proof. This is obvious.

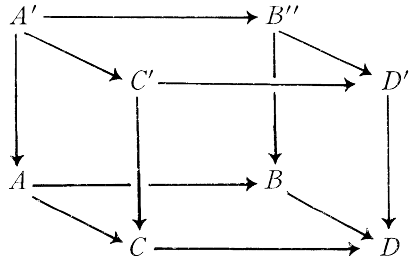
LEMMA 21. *In order to prove Theorem* 18, *it is sufficient to prove the theorem in the case where f_2 and f_3 are fibrations, where the rear face is a topological pull-back* (*with the static homotopy*), *and where the left face is flat.*

Proof. We first show that f_2 may be assumed to be a fibration. Again, by Spanier [**10**, p. 99], f_2 may be factored as

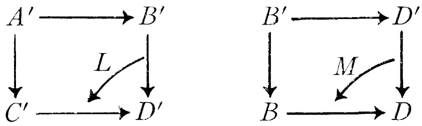
$$B' \xrightarrow{f_2'} B'' \xrightarrow{f_2''} B,$$

in which f_2' is a homotopy equivalence and f_2'' is a fibration. Form a cube

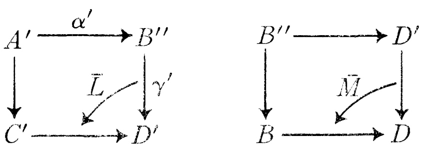
(3)



as follows. Let $f_2'^{-1}$ be a homotopy inverse for f_2', and let K be a homotopy from $f_2'^{-1} \circ f_2'$ to $1_{B'}$. Then the maps in (3) are the same as in (2) except those involving B''. $A' \to B''$ is the composition of $A' \to B' \to B''$, $B'' \to B$ is f_2'', and $B'' \to D'$ is the composition of $B'' \xrightarrow{f_2'^{-1}} B' \to D'$. The homotopies across the front, rear, left, and bottom squares are the same as in (2), and if



are the other two faces in (2), then the corresponding faces in (3) are



where $\bar{L} = \gamma' \circ K \circ \alpha'$ and $\bar{M}$ is given by Lemma 4.

Now (2) is clearly equivalent to (3). Hence we may assume that f_2 is a fibration.

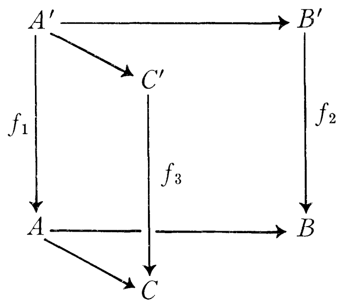
Symmetrically, we may assume that f_3 is a fibration.

It follows easily from Lemma 4 and Corollary 5 that we may assume that the rear face is a topological pull-back (with the static homotopy).

By altering the map $A' \to C'$ by a homotopy, we may also assume that the left face is flat. This proves the lemma.

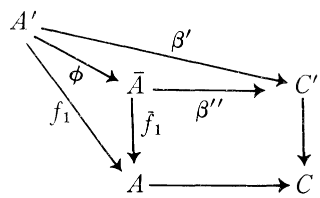
We work from now on in the case described by this lemma.

Suppose we are given



in which f_2 and f_3 are fibrations, the rear face is a topological pull-back, and the left face is flat. We make the following construction.

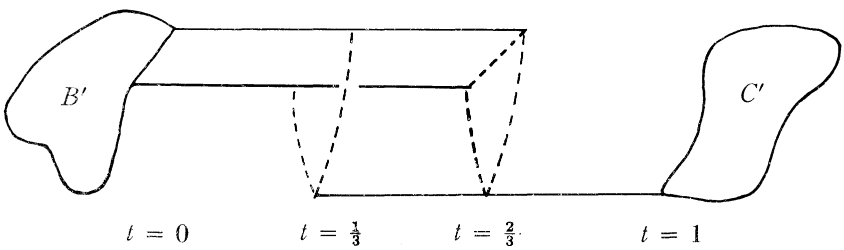
Let $\bar{A}$ be the topological pull-back in the left face, so that we get a commutative diagram



in which, of course, ϕ is a homotopy equivalence by Theorem 10. Let M be the mapping cylinder of ϕ (with $\bar{A}$ at the "one" end). Let $A'' \subset M \times I$ be given by

$$A'' = \{(m, t); m \in A' \times 0 \text{ if } t \leqq \tfrac{1}{3}, m \in \bar{A} \text{ if } t \geqq \tfrac{2}{3}\}.$$

Now let $D_1' = B' \cup A'' \cup C'/(a', 0, 0) \sim \alpha'(a'), (\bar{a}, 1) \sim \beta''(\bar{a})$.



Picture of D_1'

Define a homotopy $K : A' \times I \to D_1'$ by

$$\begin{aligned} K(a', t) &= (a', 0, 3t/2) && \text{for } t \leqq \tfrac{1}{3} \\ &= (a', 3t/2 - 1, \tfrac{1}{2}) && \text{for } \tfrac{1}{3} \leqq t \leqq \tfrac{2}{3} \\ &= (\phi(\mathrm{a}'), 1, 3t/2 - \tfrac{1}{2}) && \text{for } \tfrac{2}{3} \leqq t. \end{aligned}$$

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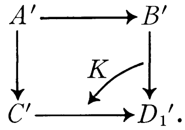
Then we have a homotopy commutative square



LEMMA 22. *This is a homotopy push-out.*

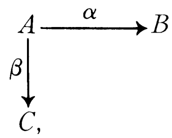
Proof. There is a weak deformation retraction of A'' to the subset

$$A_1 = \{(m, t); m \in A' \times 0 \text{ if } t < \tfrac{1}{2}, m \in \bar{A} \text{ if } t > \tfrac{1}{2}\}$$

keeping ends fixed. Hence there is a weak deformation retraction of D_1' to $B' \cup A_1 \cup C'$, keeping B' and C' fixed.

Now, similarly to the proof that the mapping cones of homotopic maps are homotopy equivalent, $B' \cup A_1 \cup C'$ is homotopy equivalent to the standard homotopy push-out. Hence the result.

Let D_1 be the standard homotopy push-out of

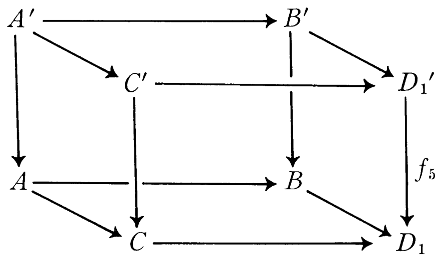


that is to say, $D_1 = B \cup A \times I \cup C/(a, 0) \sim \alpha(a), \quad (a, 1) \sim \beta(a)$. Let $f_5 : D_1' \to D_1$ be f_2 on B', f_3 on C' and, on A'', let

$$f_5(a', s, t) = (f_1(a'), t)$$
$$f_5(\bar{a}, t) = (\bar{f}_1(\bar{a}), t).$$

Then we have a homotopy commutative diagram

(4)



in which the vertical faces are flat.

LEMMA 23. *f_5 has the WCHP.*

Proof. Let $E_1, E_2 \subset D_1$ be the open sets given by

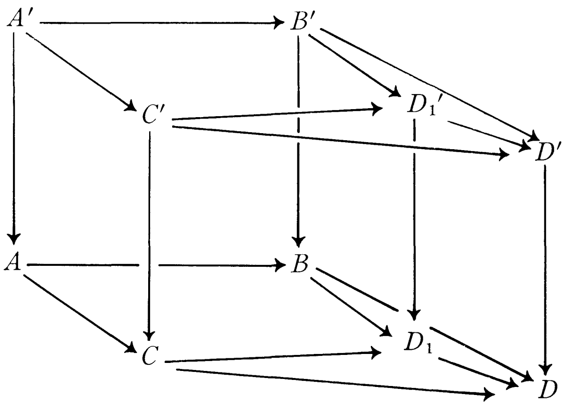
$$E_1 = B \cup A \times [0, \tfrac{2}{3})$$
$$E_2 = A \times (\tfrac{1}{3}, 1] \cup C.$$

By Lemma 20, f_5 has the *WCHP* over each of E_1 and E_2. $\{E_1, E_2\}$ is obviously a numerable covering of D_1. Hence, by Dold [**1**, Theorem 5.12 (a)], f_5 has the *WCHP*.

COROLLARY 24. *The front face of* (4) *is a homotopy pull-back.*

Proof. This follows from Lemma 23 and Lemma 19.

Proof of Theorem 18. Under the circumstances given by Lemma 21, we clearly have a homotopy commutative diagram



in which the maps $D_1' \to D'$ and $D_1 \to D$ are homotopy equivalences. Therefore

$$\begin{array}{ccc} D_1' & \longrightarrow & D' \\ \downarrow & & \downarrow \\ D_1 & \longrightarrow & D \end{array}$$

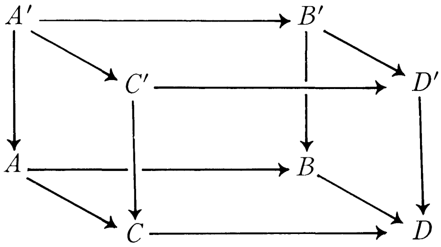
is a homotopy pull-back. Hence so is

$$\begin{array}{ccc} C' & \longrightarrow & D' \\ \downarrow & & \downarrow \\ C & \longrightarrow & D. \end{array}$$

The right hand square follows similarly. Thus we have proved the theorem.

5. The second cube theorem. The purpose of this section is to state and prove the following theorem. Again we remark that there are no restrictions on the spaces involved. Note also that, in view of Lemma 37 below, if we consider only the case where all the spaces are CW-complexes, Theorem 25 is an easy corollary of Theorem 18.

THEOREM 25. *Suppose we have a homotopy commutative diagram*



in which

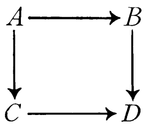
(i) *all the vertical faces are homotopy pull-backs, and*

(ii) *the lower face is a homotopy push-out.*

Then the upper face is a homotopy push-out.

We break the proof into several lemmas.

We say that a homotopy push-out



is in *standard form* if there are subsets $\bar{B} \subset B$ and $\bar{C} \subset C$ such that

(i) $B \cong \bar{B} \cup A \times [0, \frac{1}{2}]/a \times 0 \sim \alpha(a), * \times [0, \frac{1}{2}]$ where α is some map from A to $\bar{B}$;

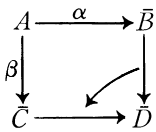
(ii) $C \cong A \times [\frac{1}{2}, 1] \cup \bar{C}/a \times 1 \sim \beta(a), * \times [\frac{1}{2}, 1]$ where β is some map from A to $\bar{C}$;

(iii) $D \cong \bar{B} \cup A \times [0, 1] \cup \bar{C}/a \times 0 \sim \alpha(a), a \times 1 \sim \beta(a), * \times [0, 1]$;

(iv) the maps are the obvious inclusions (where A is identified with $A \times \frac{1}{2}$) and the homotopy is static.

LEMMA 26. *A homotopy push-out is equivalent to one in standard form.*

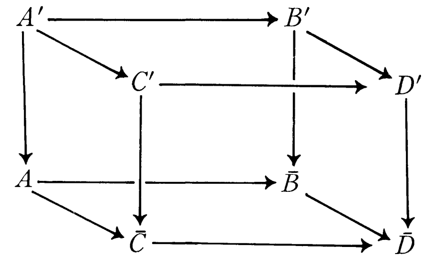
Proof. Given a homotopy push-out



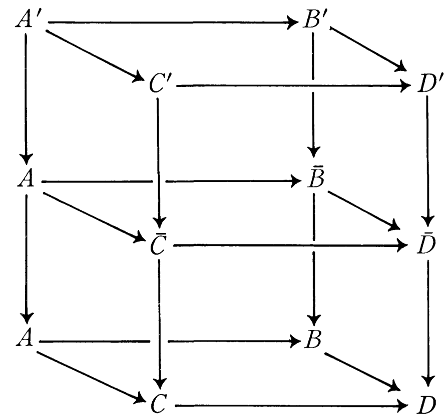
the definition constructs a homotopy push-out in standard form and the two are obviously equivalent.

LEMMA 27. *It suffices to prove the theorem in the case where the lower face is in standard form.*

Proof. Given a homotopy commutative cube



satisfying the conditions of the theorem we obtain, by the previous lemma, a homotopy commutative diagram



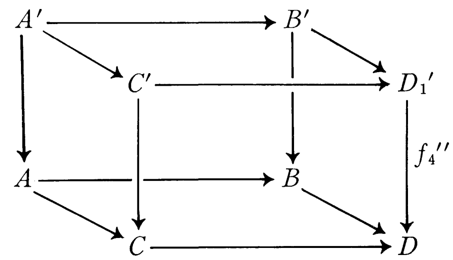
in which the lowest face is in standard form, and the lower vertical maps are homotopy equivalences. It follows readily from Lemma 6 that the large vertical squares are homotopy pull-backs. Hence, by considering the large cube, it suffices to prove the theorem when the lower face is in standard form.

LEMMA 28. *It suffices to prove the theorem when we also suppose that the map $D' \to D$ is a fibration.*

Proof. Let $D' \to D$ be factored into a homotopy equivalence and a fibration

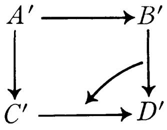
$$D' \xrightarrow{f_4'} D_1' \xrightarrow{f_4''} D,$$

again as in Spanier. Then we obtain a homotopy commutative cube

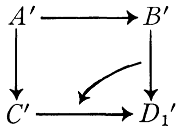


in which the maps and the homotopy into D' have been composed with f_4' and the other maps (except f_4'') and homotopies are unchanged.

Clearly



is equivalent to

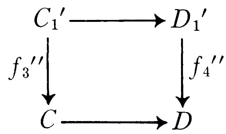


so it suffices to prove that the latter is a homotopy push-out.

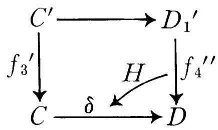
LEMMA 29. *It suffices to prove the theorem when we also assume that the vertical faces are topological pull-backs with static homotopies.*

Note, however, that although the top face is therefore commutative, we put no restriction at this stage on the homotopy across that face.

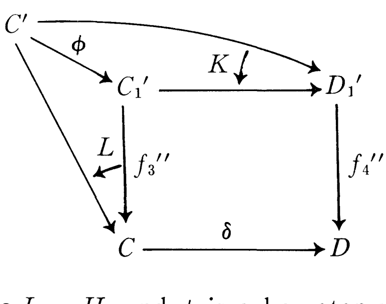
Proof. Let



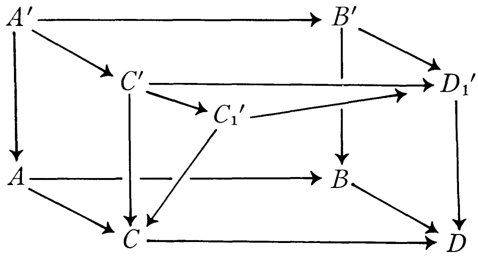
be the topological pull-back. Since f_4'' is a fibration this is a homotopy pull-back (with the static homotopy) and hence has the pull-back property. Thus, if



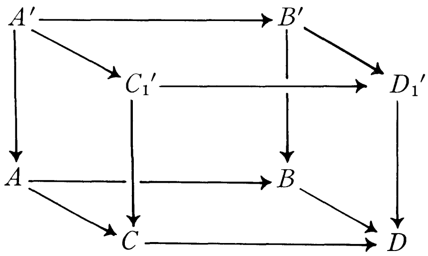
is the front face of the cube, there is a homotopy commutative diagram



in which $f_4'' \circ K + \delta \circ L \sim H$, and ϕ is a homotopy equivalence. Then we obtain a homotopy commutative diagram

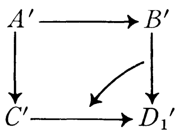


from which we can extract the homotopy commutative cube

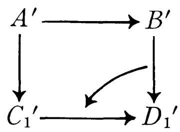


and the front face has the required form.

Clearly



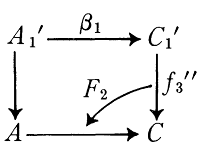
is equivalent to



so we may assume that the front face has the required form.

Similarly we may assume that the right and rear faces have the required form.

Now the left face is, say,



in which f_3'' is a fibration, the square is a topological pull-back, and there is some homotopy across the square.

Since f_3'' is a fibration there is a homotopy $H : A_1' \times I \to C_1'$ such that
(i) $f_3'' \circ H = F_2$
(ii) $H|A_1' \times 0 = \beta_1$.

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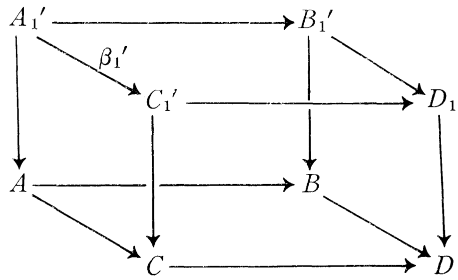
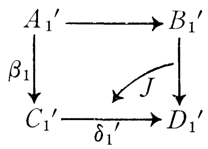
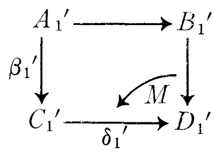
Let $H|A_1' \times 1$ be β_1'. Then we get a homotopy commutative cube



in which the left face is flat, and if the top face was



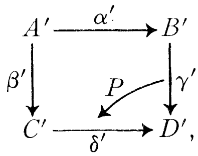
it is now



where $M = J - \delta_1' H$. This completes the lemma.

LEMMA 30. *It suffices to prove the theorem when we also assume that the top face is flat.*

Proof. Let the top face be



where we note that $\gamma' \circ \alpha' = \delta' \circ \beta'$. Thus $P|A' \times 0 = P|A' \times 1$.

We claim that there is a homotopy $P' : A' \times I \to C'$ from β' to β' such that $\delta' \circ P' \sim P$, as follows. Since the cube is homotopy commutative, $f_4 \circ P$ is equivalent to the static homotopy. Hence, since f_4 is a fibration, P is equivalent to a homotopy P'' with the property that $f_4 \circ P''$ is static. Then the image of P'' lies in the image of δ', so that P'' factors through C' as, say,

$$A' \times I \xrightarrow{P'} C' \xrightarrow{\delta'} D'$$

as required.

Now, by performing the homotopy $-P'$ on β', we see that the cube is equivalent to the same cube but with the static homotopy across the top and $f_3 \circ P'$ across the left face. But $\delta \circ f_3 \circ P'$ is static, and hence $f_3 \circ P'$ is static. Thus the whole cube is equivalent to the flat cube. This completes the lemma.

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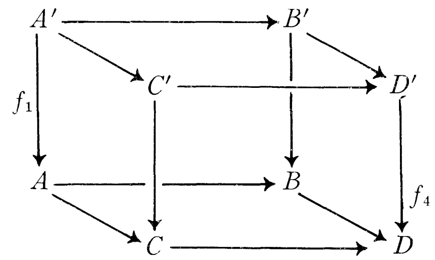
Completion of Theorem 25. We may assume that we have a flat cube



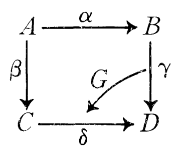
in which the lower face is in standard form, $D' \to D$ is a fibration, and the vertical faces are topological pull-backs. Further, by the particular way that f_4 was constructed, over $A \times [\frac{1}{4}, \frac{3}{4}]$, f_4 is topologically equivalent to the map $f_1 \times 1 : A' \times [\frac{1}{4}, \frac{3}{4}] \to A \times [\frac{1}{4}, \frac{3}{4}]$. Thus

$$D' = f_4^{-1}(\bar{B} \cup A \times [0, \tfrac{1}{3}]) \cup (A' \times [\tfrac{1}{3}, \tfrac{2}{3}]) \cup f_4^{-1}(A \times [\tfrac{2}{3}, 1] \cup \bar{C}).$$

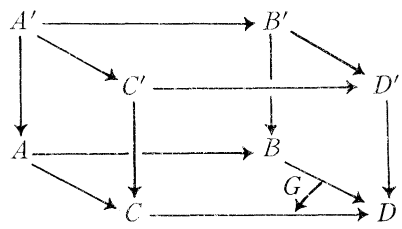
But $f_4^{-1}(\bar{B} \cup A \times [0, \frac{1}{3}]) \simeq B'$ and $f_4^{-1}(A \times [\frac{2}{3}, 1] \cup \bar{C}) \simeq C'$.
Hence the top face is a homotopy push-out, as required.

To complete this section we give a result which helps to make this theorem useful.

LEMMA 31. *Let*



be homotopy commutative, and let $f_4 : D' \to D$ *be a map. Then there is a homotopy commutative cube*



in which the vertical faces are homotopy pull-backs.

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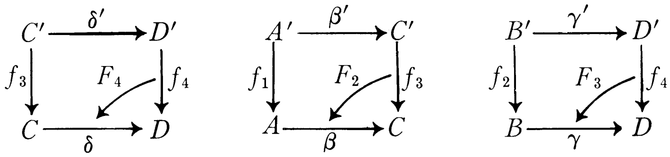
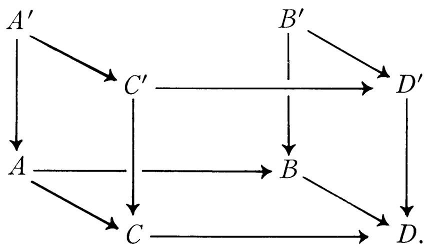
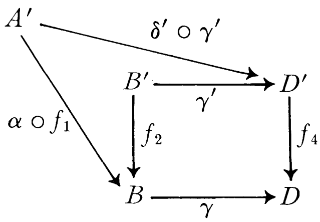
Proof. Take three homotopy pull-backs



thereby defining f_1, f_2, f_3, β', γ', δ', F_2, F_3 and F_4. Form the diagram



Then we have a diagram



with homotopies F_3 from $f_4 \circ \gamma'$ to $\gamma \circ f_2$ and $F_4 \circ \beta + \delta \circ F_2 + G \circ f_1$ from $f_4 \circ \delta' \circ \gamma'$ to $\gamma \circ \alpha \circ f_1$. Hence, by the pull-back property, there is:

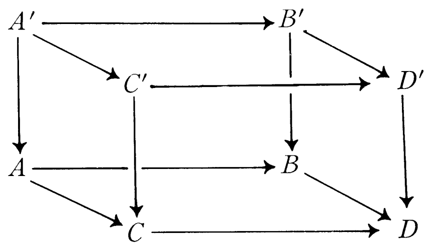
a) a whisker map $\alpha' : A' \to B'$;

b) a homotopy G' from $\gamma' \circ \alpha'$ to $\delta' \circ \gamma'$; and

c) a homotopy F_1 from $f_2 \circ \alpha'$ to $\alpha \circ f_1$ such that

$$f_4 \circ (-G') + F_3 \circ \alpha' + \gamma \circ F_1 \sim F_4 \circ \beta + \delta \circ F_2 + G \circ f_1.$$

But this is precisely the condition we need to make



homotopy commutative. That the rear face is a homotopy pull-back follows from Lemmas 12 and 14.

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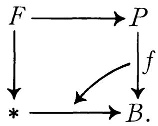
6. The case of *CW*-complexes. For convenience we work throughout this section in Top*.

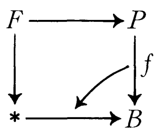
Let $f : P \to B$ be a map. Then we define the *fibre* of f to be the homotopy pull-back



Of course, this is defined only up to equivalence. We sometimes call F the fibre, rather than the whole square.

If f is a fibration we may take F to be the inverse image of the base point of B. Thus the following lemma does little more than assert the existence of the exact sequence of a fibration. However, we would like to mention the proof given.

LEMMA 32. *Let*

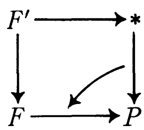


be a homotopy pull-back. Then there is a long exact sequence

$$\ldots \Pi_n F \to \Pi_n P \to \Pi_n B \to \Pi_{n-1} F \to \ldots \to \Pi_0 B.$$

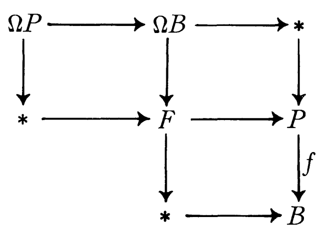
Proof. That $\Pi_n F \to \Pi_n P \to \Pi_n B$ is exact is immediate from the pull-back property.

Let



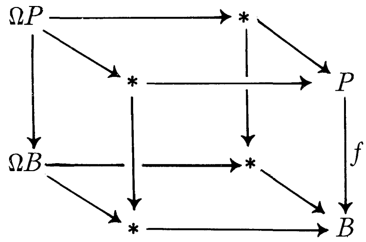
be the homotopy pull-back. Then, by Lemma 12, $F' \simeq \Omega B$. Thus we get a homotopy commutative diagram of homotopy pull-backs

(4)



where the map from ΩP to ΩB is Ωf, as follows.

The given diagram gives rise to a homotopy commutative diagram



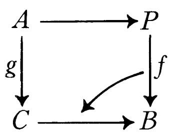
and, by the uniqueness part of the pull-back property for the lower square, the map $\Omega P \to \Omega B$ must be homotopic to Ωf.

The long exact sequence now follows by applying the first sentence of the proof to (4).

LEMMA 33. *If B is connected and F is contractible and P and B are CW-complexes, then f is a homotopy equivalence.*

Proof. It follows immediately from the exact sequence that $\Pi_n P \to \Pi_n B$ is an isomorphism for all n. Hence, by the J. H. C. Whitehead theorem, f is a homotopy equivalence.

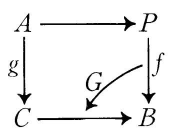
COROLLARY 34. *Suppose that*



is a homotopy pull-back, P and B are CW-complexes, and B is connected. If g is a homotopy equivalence then so is f.

Proof. The fibre of g is contractible and hence, by Lemma 12, so is the fibre of f.

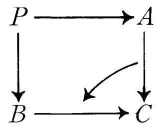
COROLLARY 35. *Suppose that*



is a homotopy pull-back, P and B are CW-complexes, and $\Pi_0 C \to \Pi_0 B$ is onto. If g is a homotopy equivalence then so is f.

Proof. The given square is obviously equivalent to a static square, so we may assume the given square is static. If we consider each component of B separately, together with its inverse images in P, C, and A, we can introduce base points and apply the previous corollary.

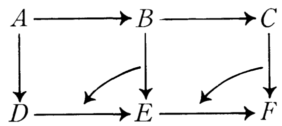
LEMMA 36. *Let*



be a homotopy pull-back, and suppose that A, B, C have the homotopy types of CW-complexes. Then P has the homotopy type of a CW-complex.

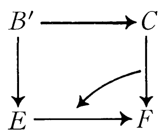
Proof. This is a corollary of Milnor [**7**, Theorem 3, p. 276].

LEMMA 37. *In the homotopy commutative diagram*

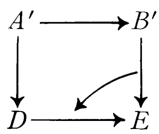


suppose that all the spaces are CW-complexes, that $\Pi_0 D \to \Pi_0 E$ is onto, and that the left and large squares are homotopy pull-backs. Then the right hand square is a homotopy pull-back.

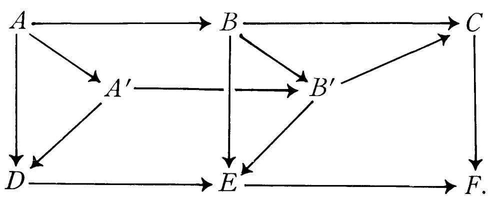
Proof. Let



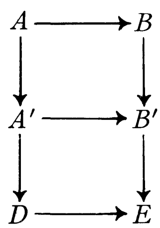
and



be homotopy pull-backs, and let $B \to B'$, $A \to A'$ be the corresponding whisker maps. Then we have a homotopy commutative diagram



By applying Lemma 8 to



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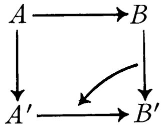
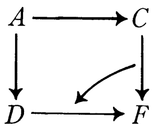
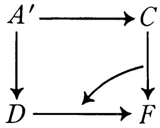
we see that



is a homotopy pull-back. However, by hypothesis and Lemma 12,

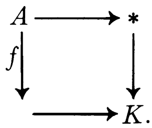
 and 

are both homotopy pull-backs, so $A \to A'$ is a homotopy equivalence.

Now $\Pi_0 D \to \Pi_0 E$ is onto, by hypothesis, so it follows easily that $\Pi_0 A' \to \Pi_0 B'$ is onto. But B and B' have the homotopy types of CW-complexes (the latter by Lemma 36). Hence, by Corollary 35, $B \to B'$ is a homotopy equivalence, which gives the desired result.

We now study the dual situation.

Let $f : A \to B$ be a map. Then we define the *cofibre* of f to be the homotopy push-out



Of course this is defined only up to equivalence. We sometimes call K the cofibre, instead of the whole square.

LEMMA 38. *For any Abelian coefficient group G there is a long exact sequence*

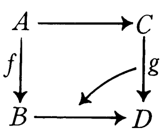
$$\tilde{H}^0(K; G) \to \tilde{H}^0(B; G) \to \tilde{H}^0(A; G) \to \tilde{H}^1(K; G) \to \dots$$

Proof. $\tilde{H}^n(X; G) = [X : K(G, n)]$, and the proof is dual to that of Lemma 32.

LEMMA 39. *If A and B are CW-complexes, A is simply connected and K is contractible then f is a homotopy equivalence.*

Proof. This is well known, so we omit the proof.

COROLLARY 40. *If*



is a homotopy push-out, A is a simply connected CW-complex, B is a CW-complex and g is a homotopy equivalence then f is a homotopy equivalence.

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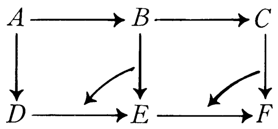
Proof. The cofibre of g is contractible, and hence so is the cofibre of f.

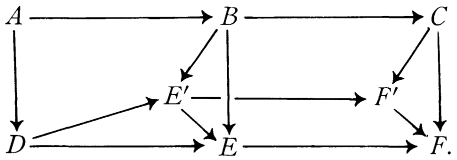
It is obvious that a homotopy push-out of CW-complexes has the homotopy type of a CW-complex.

LEMMA 41. *In the homotopy commutative diagram*



suppose that all the spaces are CW-complexes, that A is connected and B and D are simply connected, and that the right and large squares are homotopy push-outs. Then the left square is a homotopy push-out.

Proof. Dually to the proof of Lemma 37, we put in homotopy push-outs to form the diagram



Since A is connected and B and D are simply connected, it is clear that E' is simply connected. The rest of the proof is dual to Lemma 37, and is left to the reader.

Appendix 1. In this appendix we give the proofs which we omitted in Section 2. We give them in a different order.

LEMMA 2. *Let $f : A \to B$ be a homotopy equivalence, and suppose we are given two maps $g, h : X \to A$ and a homotopy H from $f \circ g$ to $f \circ h$. Then there is a homotopy G from g to h such that $f \circ G \sim H$.*

Proof. Let $f' : B \to A$ be the inverse homotopy equivalence, let F be a homotopy from $f' \circ f$ to 1_A and let F' be a homotopy from $f \circ f'$ to 1_B.

If H' is a homotopy from $f \circ g$ to $f \circ h$, possibly different from H, define a corresponding homotopy G from g to h by

$$G = (-F) \circ g + f' \circ H' + F \circ h.$$

This is defined at least up to equivalence. We will show how to choose H' so that $f \circ G \sim H$.

Now, $f \circ G \sim f \circ (-F) \circ g + f \circ f' \circ H' + f \circ F \circ h$. Thus $f \circ G \sim H$ if and only if $H \sim f \circ (-F) \circ g + f \circ f' \circ H' + f \circ F \circ h$
i.e., $f \circ F \circ g + H + f \circ (-F) \circ h \sim f \circ f' \circ H'$.

Now $f \circ f'$ is homotopic to 1_B by F' so that

$$f \circ f' \circ H' \sim F' \circ f \circ g + H' + (-F') \circ f \circ h.$$

Thus $f \circ G \sim H$ if and only if

$$H' \sim (-F') \circ f \circ g + f \circ F \circ g + H + f \circ (-F) \circ h + F' \circ f \circ h$$

and so we choose to define H' by this equation. This gives the required result.

COROLLARY 3. *If $f : A \to B$ is a homotopy equivalence, $g, h : X \to A$ are maps and G, H are homotopies from g to h such that $f \circ G \sim f \circ H$ then $G \sim H$.*

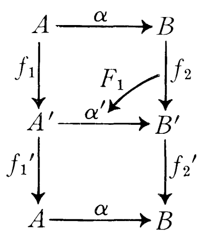
Proof. $G - H$ is a homotopy from g to g, and hence may be thought of as a map $k : X \times S^1 \to A$. Clearly $G \sim H$ if and only if k extends over $X \times D^2$, i.e., if and only if $k \sim g \circ p_1 : X \times S^1 \to A$.

But $f \circ G \sim f \circ H$, so $f \circ k \sim f \circ g \circ p_1 : X \times S^1 \to B$. Hence, by the lemma above, $k \sim g \circ p_1$ and $G \sim H$, as required.

Lemma 4 and Corollary 5 are the duals of the last two results and have the dual proofs.

Before we prove Lemma 1 we need an extra lemma.

LEMMA 42. *Let $f_1 : A \to A'$ be a homotopy equivalence, with inverse $f_1' : A' \to A$, and let H_A be a homotopy from $f_1' \circ f_1$ to 1_A. Similarly, let $f_2 : B \to B'$ be a homotopy equivalence, with inverse $f_2' : B' \to B$ and let H_B be a homotopy from $f_2' \circ f_2$ to 1_B. Consider the diagram*



in which there is one homotopy, as marked. Then there is a homotopy F_1' from $f_2' \circ \alpha'$ to $\alpha \circ f_1'$ such that

$$f_2' \circ F_1 + F_1' \circ f_1 \sim H_B \circ \alpha - \alpha \circ H_A.$$

Proof. The condition shows that we want

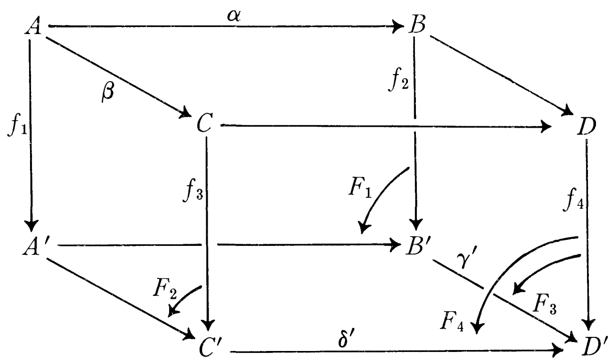
$$F_1' \circ f_1 \sim -f_2' \circ F_1 + H_B \circ \alpha - \alpha \circ H_A.$$

Such a homotopy F_1' exists by Lemma 2, and is unique by Corollary 3.

LEMMA 1. *On squares, equivalence is an equivalence relation.*

Proof. The reflexive and transitive properties are obvious. We show that equivalence is symmetric.

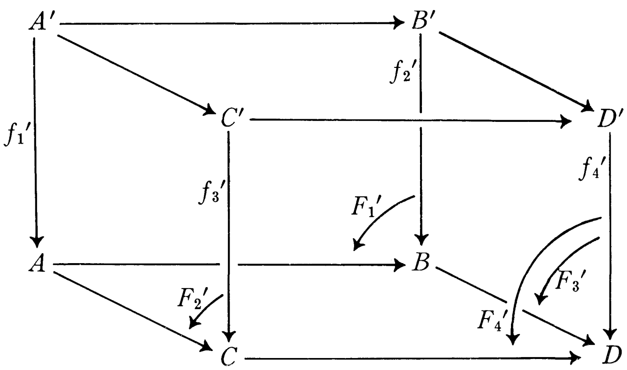
Thus, we suppose that we are given a homotopy commutative cube



in which the maps f_i are homotopy equivalences. The condition for homotopy commutativity may be written

$$f_4 \circ G + F_4 \circ \beta + \delta' \circ F_2 - G' \circ f_1 - \gamma' \circ F_1 - F_3 \circ \alpha \sim 0.$$

Let f_i' be a homotopy inverse of f_i for each i. Then we will define homotopies F_i' to make the following diagram homotopy commutative. This will complete the proof of the lemma.



Choose homotopies H_A, from $f_1' \circ f_1$ to 1_A, and H_B, H_C, H_D similarly. Then the homotopies F_i' are given by the previous lemma. This defines the cube, and it just remains to check that, if

$$K = f_4' \circ G' + F_4' \circ \beta' + \delta \circ F_2' - G \circ f_1' - \gamma \circ F_1' - F_3' \circ \alpha'$$

then $K \sim 0$.

We claim that $K \circ f_1 \sim 0$, as follows.

(i) $F_4' \circ \beta' \circ f_1 \sim -f_4' \circ \delta' \circ F_2 + F_4' \circ f_3 \circ \beta + \delta \circ f_3' \circ F_2$ since F_2 is a homotopy from $f_3 \circ \beta$ to $\beta' \circ f_1$

$$\sim -f_4' \circ \delta' \circ F_2 + [-f_4' \circ F_4 + H_D \circ \delta - \delta \circ H_c] \circ \beta + \delta \circ f_3' \circ F_2,$$

by the definition of F_4'.

(ii) $\delta \circ F_2' \circ f_1 \sim \delta \circ [-f_3' \circ F_2 + H_C \circ \beta - \beta \circ H_A]$

(iii) $G \circ f_1' \circ f_1 \sim \gamma \circ \alpha \circ H_A - H_D \circ \gamma \circ \alpha + f_4' \circ f_4 \circ G + H_D \circ \delta \circ \beta - \delta \circ \beta \circ H_A$

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(iv) $\gamma \circ F_1' \circ f_1 \sim \gamma \circ [-f_2' \circ F_1 + H_B \circ \alpha - \alpha \circ H_A]$.

(v) $F_3' \circ \alpha' \circ f_1 \sim -f_4' \circ \gamma' \circ F_1 + F_3' \circ f_2 \circ \alpha + \gamma \circ f_2' \circ F_1$
$\sim -f_4' \circ \gamma' \circ F_1 + [-f_4' \circ F_3 + H_D \circ \gamma - \gamma \circ H_B] \circ \alpha + \gamma \circ f_2' \circ F_1.$

Hence

$$\begin{aligned} K \circ f_1 \sim f_4' \circ G' \circ f_1 & \\ & - f_4' \circ \delta' \circ F_2 - f_4' \circ F_4 \circ \beta + H_D \circ \delta \circ \beta - \delta \circ H_C \circ \beta + \delta \circ f_3' \circ F_2 \\ & - \delta \circ f_3' \circ F_2 + \delta \circ H_C \circ \beta - \delta \circ \beta \circ H_A \\ & + \delta \circ \beta \circ H_A - H_D \circ \delta \circ \beta - f_4' \circ f_4 \circ G + H_D \circ \gamma \circ \alpha - \gamma \circ \alpha \circ H_A \\ & + \gamma \circ \alpha \circ H_A - \gamma \circ H_B \circ \alpha + \gamma \circ f_2' \circ F_1 \\ & - \gamma \circ f_2' \circ F_1 + \gamma \circ H_B \circ \alpha - H_D \circ \gamma \circ \alpha + f_4' \circ F_3 \circ \alpha + f_4' \circ \gamma' \circ F_1 \end{aligned}$$

and this expression cancels down to

$$f_4' \circ [G' \circ f_1 - \delta' \circ F_2 - F_4 \circ \beta - f_4 \circ G + F_3 \circ \alpha + \gamma' \circ F_1]$$

which is equivalent to the static homotopy since the original cube is homotopy commutative.

Now $K \circ f_1 \sim 0 \circ f_1$ implies $K \sim 0$, by Corollary 5. Hence the last cube is homotopy commutative and we have shown that equivalence of squares is symmetric and hence an equivalence relation, as required.

We now move to Lemma 6. This has been proved in much greater generality by Vogt [**13**]. However, for completeness, we include a proof here. We need three lemmas.

LEMMA 43. *Let $f : A \to A'$ be a homotopy equivalence, with inverse $f' : A' \to A$, and let H be a homotopy from $f' \circ f$ to 1_A. Then there is a homotopy H' from $f \circ f'$ to $1_{A'}$, unique up to equivalence, such that*

$$H \circ f' \sim f' \circ H'$$

and

$$f \circ H \sim H' \circ f.$$

Proof. $H \circ f'$ is a homotopy from $f' \circ f \circ f'$ to f'. Hence, by Lemma 2, there is a homotopy H' from $f \circ f'$ to $1_{A'}$ such that $H \circ f' \sim f' \circ H'$. By Corollary 3, such H' is unique up to equivalence.

Now we notice that, if K is a homotopy from k_0 to k_1 and L is a homotopy from l_0 to l_1 and $K \circ L$ is defined, then

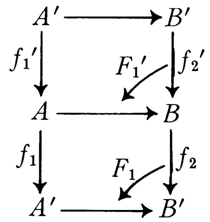
$$k_0 \circ L \sim K \circ l_0 + k_1 \circ L - K \circ l_1.$$

Thus

$$
\begin{aligned}
f \circ H \circ f' &\sim f \circ f' \circ H' \\
&\sim H' \circ f \circ f' + H' - H' \\
&\sim H' \circ f \circ f'
\end{aligned}
$$

so that $f \circ H \sim H' \circ f$ by Corollary 3. This completes the lemma.

LEMMA 44. *Under the circumstances of Lemma* 42, *let* H_A' *and* H_B' *be as constructed in Lemma* 43. *Then, in the following diagram, which has two homotopies,*



we have

$$f_2 \circ F_1' + F_1 \circ f_1' \sim H_B' \circ \alpha' - \alpha' \circ H_A'.$$

Proof. We calculate as follows:

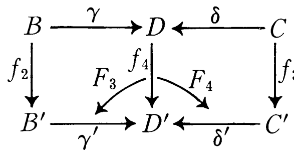
$$
\begin{aligned}
f_2' \circ (f_2 \circ F_1' + F_1 \circ f_1') + F_1' \circ f_1 \circ f_1' &\sim f_2' \circ f_2 \circ F_1' + \\
&\qquad (H_B \circ \alpha - \alpha \circ H_A) \circ f_1' \\
&\sim (H_B \circ f_2' \circ \alpha' + F_1' - H_B \circ \alpha \circ f_1') + (H_B \circ \alpha - \alpha \circ H_A) \circ f_1' \\
&\sim H_B \circ f_2' \circ \alpha' + F_1' - \alpha \circ H_A \circ f_1' \\
&\sim f_2' \circ H_B' \circ \alpha' + F_1' - \alpha \circ f_1' \circ H_A' \text{ by Lemma 43} \\
&\sim (f_2' \circ H_B' \circ \alpha' - f_2' \circ \alpha' \circ H_A') \\
&\qquad + (f_2' \circ \alpha' \circ H_A' + F_1' - \alpha \circ f_1' \circ H_A') \\
&\sim f_2' \circ (H_B' \circ \alpha' - \alpha' \circ H_A') + F_1' \circ f_1 \circ f_1'.
\end{aligned}
$$

Therefore $f_2' \circ (f_2 \circ F_1' + F_1 \circ f_1') \sim f_2' \circ (H_B' \circ \alpha' - \alpha' \circ H_A')$ and hence, by Corollary 3,

$$f_2 \circ F_1' + F_1 \circ f_1' \sim H_B' \circ \alpha' - \alpha' \circ H_A'$$

as required.

LEMMA 45. *Given*



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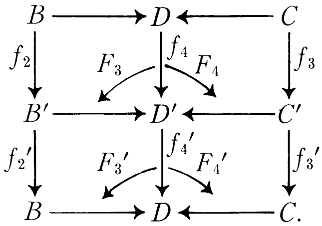
in which the maps f_i *are homotopy equivalences, the induced map* $f : E_{\gamma,\delta} \rightarrow E_{\gamma',\delta'}$ *is a homotopy equivalence.*

Proof. Clearly, if we change f_i by homotopies and alter F_3 and F_4 accordingly, we will change f by a homotopy.

Now use Lemma 42 above on each square to construct the homotopy commutative diagram



Then, clearly, the composition

$$E_{\gamma,\delta} \xrightarrow{f} E_{\gamma',\delta'} \rightarrow E_{\gamma,\delta}$$

is homotopic to the identity. By Lemma 44, we can construct a homotopy commutative diagram by putting the lower half of the diagram above on top of the upper half. Thus the composition

$$E_{\gamma',\delta'} \rightarrow E_{\gamma,\delta} \rightarrow E_{\gamma',\delta'}$$

is also homotopic to the identity. This completes the result.

LEMMA 6. *If a square is equivalent to a homotopy pull-back then it is a homotopy pull-back.*

Proof. The situation we are given is a homotopy commutative cube as shown on page 229, in which the lower face is a homotopy pull-back and all the vertical maps are homotopy equivalences. We wish to show that the upper face is also a homotopy pull-back.

Let $w : A \rightarrow E_{\gamma,\delta}$, and $w' : A' \rightarrow E_{\gamma',\delta'}$ be the whisker maps. Define $g : E_{\gamma,\delta} \rightarrow E_{\gamma',\delta'}$ by

$$g(b, \theta, c) = (f_2(b), \theta', f_3(c))$$

where

$$\begin{aligned} \theta'(t) &= F_3(b, 1 - 3t) \quad \text{if } t \leqq 1/3 \\ &= f_4\theta(3t - 1) \quad \text{if } 1/3 \leqq t \leqq 2/3 \\ &= F_4(c, 3t - 2) \quad \text{if } 2/3 \leqq t. \end{aligned}$$

Now g is a homotopy equivalence, by the previous lemma. Hence we need only show that $g \circ w \simeq w' \circ f_1$ to complete the proof of the lemma.

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Well, $g \circ w : A \to E_{\gamma',\delta'}$ is given by

$$g \circ w(a) = (f_2 \circ \alpha(a), (-F_3 \circ \alpha + f_4 \circ G + \beta \circ F_4)|a \times I, f_3 \circ \beta(a))$$

and $w' \circ f_1 : A \to E_{\gamma',\delta'}$ is given by

$$w' \circ f_1(a) = (\alpha' \circ f_1(a), G' \circ f_1|a \times I, \beta' \circ f_1(a)).$$

But $G' \circ f_1 \sim -\delta' \circ F_1 - F_3 \circ \alpha + f_4 \circ G + \beta \circ F_4 + F_2 \circ \gamma'$
so $w' \circ f_1$ is homotopic to the map from A to $E_{\gamma',\delta'}$ given by

$$a \mapsto (\alpha' \circ f_1(a), -\delta' \circ F_1 - F_3 \circ \alpha + f_4 \circ G + \beta \circ F_4 + F_2 \circ \gamma'|a \times I, \beta' \circ f_1(a))$$

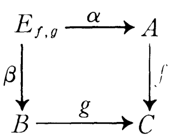
and this is obviously homotopic to $g \circ w$. This completes the proof of the lemma.

COROLLARY 7. *If, in the homotopy commutative cube on page 229 the upper and lower faces are homotopy pull-backs and the last three vertical maps are homotopy equivalences then the first map is a homotopy equivalence.*

Proof. Let $w' : A' \to E_{\gamma',\delta'}$ be the whisker map in the lower square. Then $w' \circ f_1$ is a homotopy equivalence by Lemma 6. But w' is a homotopy equivalence, and hence so is f_1.

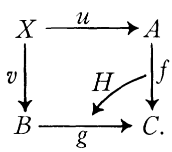
Lemma 8 and Corollary 9 are the duals of the last two results.

LEMMA 46. *The homotopy commutative square*



has the pull-back property.

Proof. Suppose we are given a homotopy commutative diagram



Define $\phi : X \to E_{f,g}$ by $\phi(x) = (u(x), H|x \times I, v(x))$. Then we obviously

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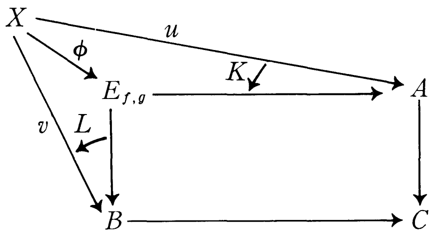
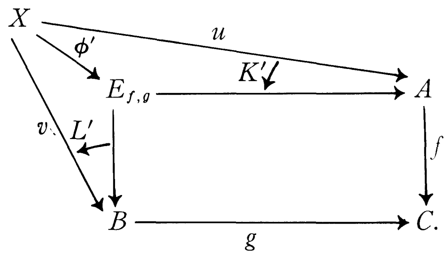
get a homotopy commutative diagram



in which the homotopies K and L are static.

Now suppose that we have another such homotopy commutative diagram



Let $\phi'(x) \in A \times C^I \times B$ be denoted by $(\phi_1'(x), \phi_2'(x), \phi_3'(x))$. Then homotopy commutativity means that

$$f \circ K' + \phi_2' + g \circ L' \sim H.$$

That is to say, there is a map $N : X \times I \times I \to C$ such that

$$\begin{aligned} N(x, 0, t) &= f \circ u(x) \\ N(x, 1, t) &= g \circ v(x) \\ N(x, s, 0) &= H(x, s) \\ N(x, s, 1) &= f \circ K'(x, 3s) \quad \text{for } s \leqq \tfrac{1}{3} \\ &= \phi_2'(x)(3s - 1) \quad \text{for } \tfrac{1}{3} \leqq s \leqq \tfrac{2}{3} \\ &= g \circ L'(x, 3s - 2) \quad \text{for } \tfrac{2}{3} \leqq s. \end{aligned}$$

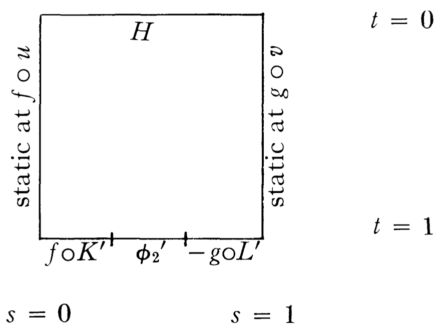


Diagram of N

We will use this to construct a homotopy M from ϕ to ϕ' satisfying axiom $PB4$.

We define maps $M_1 : X \times I \to A$, $M_2 : X \times I \times I \to C$, $M_3 : X \times I \to B$ so that we can set $M(x, s) = (M_1(x, s), M_2(x, s, \cdot), M_3(x, s))$, as follows.

$$M_1(x, s) = \begin{cases} u(x) & \text{for } s \leqq \frac{1}{2} \\ K'(x, 2s - 1) & \text{for } \frac{1}{2} \leqq s \end{cases}$$

$$M_2(x, s, t) = \begin{cases} N(x, t, 2s) & \text{for } s \leqq \frac{1}{2} \\ N(x, \frac{1}{3}(2s + 5t - 1 - 4st), 1) & \text{for } \frac{1}{2} \leqq s \end{cases}$$

$$M_3(x, s) = \begin{cases} v(x) & \text{for } s \leqq \frac{1}{2} \\ L'(x, 2 - 2s) & \text{for } \frac{1}{2} \leqq s. \end{cases}$$

Now it is simple to check that

(i) $M(x, s) \in E_{f,g}$;

(ii) M is continuous;

(iii) $M(x, 0) = \phi(x)$;

(iv) $M(x, 1) = \phi'(x)$;

(v) $\alpha \circ M \sim K'$;

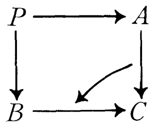
(vi) $\beta \circ M \sim -L'$.

This completes the lemma.

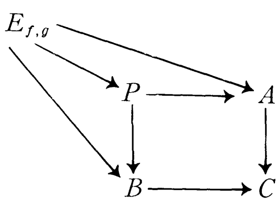
THEOREM 10. *A square has the pull-back property if and only if it is a homotopy pull-back.*

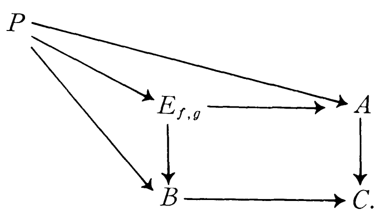
Proof. If a square is a homotopy pull-back then it follows readily from Lemma 46, using Lemma 2 and Corollary 3, that it has the pull-back property.

Conversely, suppose that



has the pull-back property. Then we get homotopy commutative diagrams:





In the usual way, the compositions

$$E_{f,g} \to P \to E_{f,g} \text{ and } P \to E_{f,g} \to P$$

must be homotopic to the respective identity maps. Hence

$$P \to E_{f,g}$$

is a homotopy equivalence, and the given square is a homotopy pull-back, as required.

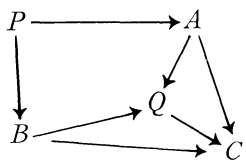
THEOREM 11. *A square has the push-out property if and only if it is a homotopy push-out.*

This is the dual of the previous theorem and is left to the reader.

Appendix 2. In this appendix we give two results which are needed in [**5**] and [**6**].

Versions of Theorem 47 have been given by Ganea [**4**], Nomura [**8**; **9**] and Svarc [**11**]. We work in Top*.

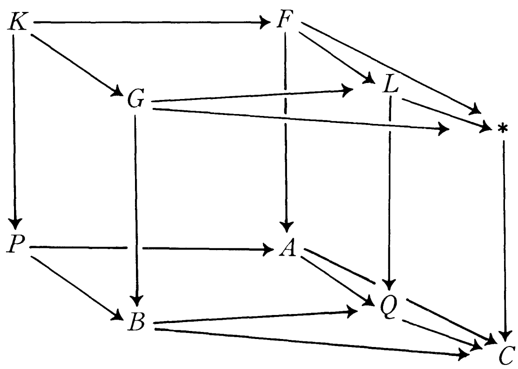
THEOREM 47. *In the homotopy commutative diagram*



suppose that the outside square is a homotopy pull-back, the inside square is a homotopy push-out, and A, B, C are connected. Let F and G be the fibres of $A \to C$, $B \to C$ respectively. Then the fibre of $Q \to C$ is $F * G$. It follows that, if $A \to C$ is r-connected and $B \to C$ is s-connected then $Q \to C$ is at least $(r + s + 1)$-connected.

Proof. By Lemma 31 we can construct, from the given diagram, a homotopy commutative diagram (with some spaces K and L):

(5)



in which the vertical squares are homotopy pull-backs.

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By applying Lemma 12 to

$$\begin{array}{ccccc} K & \longrightarrow & P & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & A & \longrightarrow & C \end{array}$$

and Lemma 14 to

$$\begin{array}{ccccc} K & \longrightarrow & G & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & * & \longrightarrow & C \end{array}$$

we see that the outside top square of (5) is a homotopy pull-back, so that $K \simeq F \times G$ in such a way that the maps $K \to F$ and $K \to G$ are homotopic to the projections.

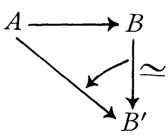
By Theorem 25, the square

$$\begin{array}{ccc} K & \longrightarrow & F \\ \downarrow & & \downarrow \\ G & \longrightarrow & L \end{array}$$

is a homotopy push-out, so that $L \simeq F * G$. This proves the theorem.

We now assume that we are working with CW-complexes.

LEMMA 48. *Let $A \to B$ be n-connected. Then there is a homotopy commutative diagram*



in which B' is obtained from A by attaching cells of dimension $\geqq n + 1$.

The proof is clear.

COROLLARY 49. *If*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a homotopy push-out and $A \to C$ is n-connected, then so is $B \to D$.

Proof. Replace $A \to C$ by the map $A \to C'$ given by the previous lemma. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C' & \longrightarrow & D' \end{array}$$

be the topological push-out. Since $A \to C'$ is a cofibration, this is also a homotopy push-out, and $B \to D'$ has the same homotopy type as $B \to D$. But D' is obtained from B by adding cells of dimension $\geqq n + 1$. Hence the result.

The following is a type of relative Hurewicz theorem.

THEOREM 50. *For any r-connected map $A \to C$ with C s-connected there is a map from the suspension of the fibre to the cofibre which is $(r + s + 1)$-connected.*

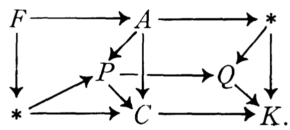
Proof. Take the homotopy pull-back

$$\begin{array}{ccc} F & \longrightarrow & A \\ \downarrow & & \downarrow \\ * & \longrightarrow & C, \end{array}$$

so that F is the fibre of $A \to C$, and construct successively three homotopy push-outs

$$\begin{array}{ccc} F & \longrightarrow & A \\ \downarrow & & \downarrow \\ * & \longrightarrow & P \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & * \\ \downarrow & & \downarrow \\ P & \longrightarrow & Q \end{array} \qquad \begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ C & \longrightarrow & K \end{array}$$

(thereby defining P, Q and K) to obtain a homotopy commutative diagram



By Lemma 13, K is the cofibre of $A \to C$. Also by Lemma 13, $Q \simeq \Sigma F$. Thus we have the desired map from ΣF to K.

By Theorem 47, $P \to C$ is $(r + s + 1)$-connected and hence, by Corollary 49, $\Sigma F \to K$ is $(r + s + 1)$-connected, as required.

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