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OPTIMAL APPROXIMATION BY CONTINUED FRACTIONS

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Abstract

Among all possible semiregular continued fraction expansions of an irrational number the one with the best approximation properties, in a well-defined and natural sense, is determined. Some properties of this so called optimal continued fraction expansion are described.

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1. Introduction

This paper is concerned with the approximation of irrational real numbers by convergents of semiregular continued fraction expansions. A semiregular continued fraction expansion of an irrational number x is an expansion

$$x = [b_0; \varepsilon_1 b_1, \varepsilon_2 b_2, \dots] = b_0 + \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2 + \cdots}} \tag{1.1}$$

where $\varepsilon_k = \pm 1$, and $b_k \in \mathbb{Z}_{\geq 1}$, for $k \geq 1$, with some constraints on b_k and ε_k (see [1, Section 1]). The finite truncations $p_n/q_n = [b_0; \varepsilon_1 b_1, \varepsilon_2 b_2, \dots, \varepsilon_n b_n]$ are called the convergents of this expansion and they form a sequence of rational approximations. The approximation constants, θ_n, are defined by $\theta_n = q_n |q_n x - p_n|$, for $n \geq 1$. Since

$$\left| x - \frac{p_n}{q_n} \right| = \frac{\theta_n}{q_n^2},$$

θ_n may be regarded as a measure of how well x is approximated by the nth convergent, taking the size of q_n into account.

Taking $\varepsilon_k = 1$ for every k in (1.1), we get the regular continued fraction expansion of x; for this we reserve the notation $\mathrm{RCF}(x) = [B_0; B_1, B_2, \dots]$. Its convergents are denoted by P_n/Q_n, and its approximation constants by Θ_n.

The main theorem proved in this paper can be formulated as follows.

(1.2) THEOREM. *For every x there is a uniquely defined semiregular continued fraction expansion, whose convergents and approximation constants we will denote by r_k/s_k and θ_k^*, with the following property.*

For every semiregular expansion of x and for every $N \geq 1$,

$$\frac{1}{M}\sum_{j=1}^{M}\theta_j \geq \frac{1}{N}\sum_{j=1}^{N}\theta_j^*,$$

where $M = \#\{j : q_j < s_{N+1}\}$.

It will turn out that this particular expansion is the so-called *optimal continued fraction* expansion, studied before in [1] and [3]. We will in the sequel denote this expansion for x by $\mathrm{OCF}(x) = [a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, \dots]$.

Theorem (1.2) thus states that among all possible semiregular expansions the OCF gives the best approximation, in terms of the mean of the approximation constants. We will try to clarify the particular formulation of the theorem and illustrate it with two consequences.

Firstly, because θ_n takes the size of the denominator q_n into account, it is fair not just to compare means of the *same number* of approximation constants for different expansions, but rather to compare these means for convergents with denominators up to the same bound, as is done in (1.2). From the viewpoint of diophantine approximation, the most interesting expansions yield convergents that are all regular convergents (cf. the next section); two sequences of convergents of such semiregular expansions of the same x contain infinitely many common rational approximations, unless x is equivalent to $g = (\sqrt{5}-1)/2 = [0; 1, 1, \dots]$. This shows the strength of the following corollary.

(1.3) COROLLARY. *Let x be some irrational number. Let r_k/s_k and θ_k^* denote the convergents and the approximation constants for the expansion of x defined in* (1.2).

For every semiregular expansion of x, with convergents and approximation *constants denoted by* p_n/q_n *and* θ_n *respectively, the following holds*:

$$\frac{p_{n+1}}{q_{n+1}} = \frac{r_{k+1}}{s_{k+1}} \quad \Rightarrow \quad \frac{1}{n}\sum_{j=1}^{n}\theta_j \geq \frac{1}{k}\sum_{j=1}^{k}\theta_j^*.$$

Secondly, we may assure that the expansions we compare have the same growth rate of their convergent denominators; this can be done for instance by restricting to the class of *fastest* expansions. For this notion, one compares semiregular expansions with the classical nearest integer continued fraction expansion, with convergents say R_k/S_k, whose denominators S_k are known to grow asymptotically as fast as possible. An expansion having the property that all the convergents are regular convergents is fastest if it has the property that

$$\frac{p_n}{q_n} = \frac{R_k}{S_k} \quad \Rightarrow \quad n = k.$$

(Notice that for such an expansion, as remarked above, $p_n/q_n = R_k/S_k$ infinitely often, for every x not equivalent to g.)

(1.4) Corollary. *Let x be some irrational number. Let r_k/s_k and θ_k^* denote the convergents and the approximation constants for the expansion of x defined in* (1.2).

For every fastest semiregular expansion of x, with convergents and approximation constants denoted by p_n/q_n and θ_n respectively, the following holds:

$$\frac{1}{n}\sum_{j=1}^{n}\theta_j \geq \frac{1}{n}\sum_{j=1}^{n}\theta_j^*.$$

Once we have established that the expansion of Theorem (1.2) coincides with the OCF-expansion, we can determine how well x can be approximated by semiregular continued fractions, by applying results from [3]. It is shown there that for every k the inequalities $\theta_k^* \leq \frac{1}{2}$ and $\theta_{k-1}^* + \theta_k^* \leq 2/\sqrt{5}$ hold; moreover, the distribution of θ_k^* is given for almost all x. In Section 4 below we will show that $\theta_{k-1}^* + \theta_k^* + \theta_{k+1}^* \leq 3/\sqrt{5}$. This leads to the following. (In fact (1.7) is the same as [3, Corollary (5.16)].)

(1.5) Theorem. *Let notations be as in* (1.2). *For every irrational x, and for every $n \geq 2$,*

$$\frac{1}{n}\sum_{j=1}^{n}\theta_j^* \leq \frac{1}{\sqrt{5}} = 0.44721\ldots. \tag{1.6}$$

For almost all irrational x,

$$(1.7)\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\theta_j^* = \frac{\arctan\frac{1}{2}}{4\log G} = 0.24087\ldots, \quad \textit{where } G = \frac{\sqrt{5}+1}{2}.$$

(1.8) Remarks. The mean of the approximation constants for some other semiregular expansions (for almost all x) is given by

$\frac{1}{4\log 2}0.36067\ldots$ for the RCF,

0.25 for Minkowski's diagonal continued fraction,

$\frac{\sqrt{5}-2}{2\log G} = 0.24528\ldots$ for the nearest integer continued fraction,

$0.24195\ldots$ for the α-expansion with $\alpha = 0.55821\cdots$, which yields the smallest value for any α-expansion.

For all this see [2] and [7].

Notice how close the value in (1.7) is to $\frac{1}{2}\log G = 0.24061\ldots$. This value gives an *a priori* lower bound for the mean of the θ's for almost all x as can be seen as follows. Any subexpansion of the RCF forms a subsequence of RCF-convergents with density at least $\log G/\log 2$ (a.a.); see [1, Section 3]. Therefore we get a lower bound for the mean by computing the mean of the fraction of size $\log G/\log 2$ of the regular convergents with smallest value of Θ. Since $\log G < \frac{1}{2}$ and since the values of Θ are equidistributed up to $\Theta = \frac{1}{2}$ with density $1/\log 2$, this is the mean of all Θ's smaller than $\log G$, which is $\frac{1}{2}\log G$ (for almost all x). However, the regular convergents thus selected (that is, those corresponding to $\Theta < \log G$) will not in general form together a semiregular continued fraction expansion (several consecutive regular convergents may be omitted); that explains why the value $\frac{1}{2}\log G$ is not attained.

The rest of this paper is organized as follows.

In Section 2 we study the distribution of Θ_n, belonging to the RCF expansion. In Section 3 it is shown that the OCF-convergents are obtained by throwing out the badly approximating regular convergents. In Section 4 we prove some interesting results about three consecutive approximation constants. In Section 5 we use the results of Sections 3 and 4 to show that the OCF-expansion has the property stated in Theorem (1.2) above.

Since so much is known about the OCF, we are also able to prove that this expansion with optimal approximation properties is periodic precisely

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for quadratic irrationals, and we can derive some statistics on its partial quotients, in Section 6.

Most results in this paper hold for rational x as well as for irrational numbers, but to avoid awkward notation they are only stated for irrational x, for which the semiregular expansions are infinite.

2. Selenius's theorem

In this section we fix an irrational number x and study the approximation properties of different expansions of x.

Every semiregular continued fraction convergent is either a regular convergent or a so-called intermediate convergent; more precisely, every semiregular expansion of x will have among its convergents at least one of the regular convergents P_{n-1}/Q_{n-1}, P_n/Q_n whenever $B_n = 1$, and it will contain at least one of P_{n-1}/Q_{n-1} and the intermediate convergents between P_{n-1}/Q_{n-1} and P_n/Q_n when $B_n > 1$. But if $B_n > 1$ then $Q_{n-1}|Q_{n-1}x - P_{n-1}| < 1/2$, while for intermediate convergents P/Q one knows $Q|Qx - P| > 1/2$. Thus, in looking for expansions with the best approximation properties, we may as well restrict ourselves to those expansion all of whose convergents are regular convergents. (For all this, see [1].)

Since such expansions necessarily have P_{n-1}/Q_{n-1} among their convergents if $B_n > 1$, the only remaining problem will be that of determining the distribution of the values of $\Theta_n = Q_n|Q_nx - P_n|$ whenever a block of m consecutive partial quotients equal to 1 occurs in the regular expansion.

First an obvious observation that will be used frequently.

(2.1) LEMMA. *Let* $\mathrm{RCF}(x) = [B_0; B_1, \ldots, B_{k-1}, B_k, \ldots]$ *and suppose that* $\mathrm{RCF}(x') = [B_0; B_1, \ldots, B_{k-1}, B'_k, \ldots]$ *with* $B_k \neq B'_k$. *Then*

$$B_k < B'_k \iff \begin{cases} k \text{ is even and } x < x', \text{ or} \\ k \text{ is odd and } x > x'. \end{cases}$$

We will sometimes abbreviate a block of m consecutive 1's as partial quotients in a continued fraction by 1^m.

(2.2) THEOREM. *Let* x *be an irrational number and*

$$\mathrm{RCF}(x) = [B_0; B_1, B_2, \ldots, 1^m, \ldots],$$

say $B_n \neq 1$, $B_{n+1} = \cdots = B_{n+m} = 1$, $B_{n+m+1} \neq 1$. *If* m *is odd then*

$$\Theta_n, \Theta_{n+2}, \ldots, \Theta_{n+m-1} > \frac{1}{\sqrt{5}} \quad \text{and} \quad \Theta_{n+1}, \Theta_{n+3}, \ldots, \Theta_{n+m-2} < \frac{1}{\sqrt{5}}.$$

If m is even then

$$\Theta_n > \Theta_{n+2} > \cdots > \Theta_{n+m-2} \quad \textit{and} \quad \Theta_{n+1} < \Theta_{n+3} < \cdots < \Theta_{n+m-1}.$$

Moreover, if $m \equiv 0 \bmod 4$, $m = 2k$ then

$$\Theta_{n+k-2} > \Theta_{n+k-1}, \qquad \Theta_{n+k} < \Theta_{n+k+1}$$

and

$$\Theta_n, \Theta_{n+2}, \ldots, \Theta_{n+k-2}, \Theta_{n+k+1}, \Theta_{n+k+3}, \ldots, \Theta_{n+m-1} > \frac{1}{\sqrt{5}};$$

if $m \equiv 2 \bmod 4$, $m = 2k$ then

$$\Theta_{n+k-2} < \Theta_{n+k-1}, \qquad \Theta_{n+k} > \Theta_{n+k+1}$$

and

$$\Theta_n, \Theta_{n+2}, \ldots, \Theta_{n+k-3}, \Theta_{j^*}, \Theta_{n+k+2}, \Theta_{n+k+4}, \ldots, \Theta_{n+m-1} > \frac{1}{\sqrt{5}},$$

where j^* is defined by

$$j^* = \begin{cases} n+k-1 & \iff \quad \Theta_{n+k-1} > \Theta_{n+k}; \\ \quad n+k & \iff \quad \Theta_{n+k-1} < \Theta_{n+k}. \end{cases} \tag{2.3}$$

PROOF. Recall (cf. for instance [6, page 29]) that

$$\Theta_i = \frac{1}{[0; B_i, \ldots, B_1] + B_{i+1} + [0; B_{i+2}, B_{i+3}, \ldots]}$$

so in particular, for $0 \le e \le m-1$

$$\Theta_{n+e} = \frac{1}{[0; 1^e, B_n, \ldots, B_1] + [0; 1^{m-1-e}, B_{n+m+1}, \ldots]}.$$

First suppose that m is odd. By Lemma (2.1),

$$[0; 1^e, B_n, \ldots, B_1] < [0; 1, 1, \ldots] = g \quad \text{if and only if } e \text{ is even}$$

and similarly

$$[0; 1^{m-1-e}, B_{n+m+1}, \ldots] < g \quad \text{if and only if } e \text{ is even.}$$

Hence

$$\Theta_{n+e} > (g+1+g)^{-1} = \frac{1}{\sqrt{5}} \quad \text{if and only if } e \text{ is even.}$$

Next suppose that $m = 2k$ is even; suppose that $m \ge 4$ (the case $m = 2$ is easily checked along the same lines). Let e be even, $0 \le e \le m-4$. Then by (2.1),

$$[0; 1^e, B_n, \ldots, B_1] < [0; 1^{e+2}, B_n, \ldots, B_1]$$

since e is even, and

$$[0; 1^{m-1-e}, B_{n+m+1}, \dots] < [0; 1^{m-3-e}, B_{n+m+1}, \dots]$$

since $m-1-e$ is odd, so

$$\begin{aligned}\Theta_{n+e} &= ([0; 1^e, B_n, \dots, B_1] + [0; 1^{m-1-e}, B_{n+m+1}, \dots] + 1)^{-1} \\ &> ([0; 1^{e+2}, B_n, \dots, B_1] + [0; 1^{m-3-e}, B_{n+m+1}, \dots] + 1)^{-1} \\ &= \Theta_{n+e+2}.\end{aligned}$$

Analogously one proves that $\Theta_{n+e} < \Theta_{n+e+2}$ for e odd, $1 \le e \le m-3$. If moreover k is even we get again by Lemma (2.1) that

$$\begin{aligned}\Theta_{n+k-2} &= ([0; 1^{k-2}, B_n, \dots, B_1] + [0; 1^{m-k+1}, B_{n+m+1}, \dots] + 1)^{-1} \\ &> ([0; 1^{m-k}, B_{n+m+1}, \dots] + [0; 1^{k-1}, B_n, \dots, B_1] + 1)^{-1} \\ &= \Theta_{n+k-1}\end{aligned}$$

and likewise that $\Theta_{n+k} < \Theta_{n+k+1}$.

Now we have

$$\begin{aligned}\Theta_{n+k-2} &= \max(\Theta_{n+k-2}, \Theta_{n+k-1}, \Theta_{n+k}), \\ \Theta_{n+k+1} &= \max(\Theta_{n+k-1}, \Theta_{n+k}, \Theta_{n+k+1}), \quad \text{and}\end{aligned}$$

$$\begin{aligned}&\min(\Theta_n, \Theta_{n+2}, \dots, \Theta_{n+k-2}, \Theta_{n+k+1}, \Theta_{n+k+3}, \dots, \Theta_{n+m-1}) \\ &\qquad = \min(\Theta_{n+k-2}, \Theta_{n+k+1}).\end{aligned}$$

But by a theorem of Tong [12], $\max(\Theta_i, \Theta_{i+1}, \Theta_{i+2}) > 1/\sqrt{5}$ and thus we see that

$$\min(\Theta_{n+k-2}, \Theta_{n+k+1}) > \frac{1}{\sqrt{5}}$$

implying

$$\Theta_n, \Theta_{n+2}, \dots, \Theta_{n+k-2}, \Theta_{n+k+1}, \Theta_{n+k+3}, \dots, \Theta_{n+m-1} > \frac{1}{\sqrt{5}}.$$

If k is odd we find, analogously to the above, that

$$\Theta_{n+k-2} < \Theta_{n+k-1} \quad \text{and} \quad \Theta_{n+k} > \Theta_{n+k+1}.$$

By the same argument as above then

$$\begin{aligned}\max(\Theta_{n+k-2}, \Theta_{n+k-1}, \Theta_{n+k}) &= \max(\Theta_{n+k-1}, \Theta_{n+k}, \Theta_{n+k+1}) \\ &= \max(\Theta_{n+k-1}, \Theta_{n+k}) > \frac{1}{\sqrt{5}}\end{aligned}$$

so with $\Theta_{j^*} = \max(\Theta_{n+k-1}, \Theta_{n+k})$ we find

$$\Theta_n, \Theta_{n+2}, \dots, \Theta_{n+k-3}, \Theta_{j^*}, \Theta_{n+k+2}, \Theta_{n+k+4}, \dots, \Theta_{n+m-1} > \frac{1}{\sqrt{5}}.$$

This completes the proof of (2.2).

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In [10] the method we employed to prove the above theorem was applied in such a way that $\Theta_n, \Theta_{n+1}, \ldots, \Theta_{n+m-1}$ could be ordered by size for every block of m consecutive 1's; the order depends only on m and on the relative sizes of $[0; B_{n+m+1}, \ldots]$ and $[0; B_n, \ldots, B_1]$. For our purposes however, the theorem and the following corollary suffice.

(2.4) Corollary. *Let* $\mathrm{RCF}(x) = [B_0; B_1, B_2, \ldots]$ *and suppose that* $B_n \neq 1$, $B_{n+1} = B_{n+2} = \cdots = B_{n+m} = 1$, $B_{n+m+1} \neq 1$, *for some* $n, m \geq 1$. *Let* Z *denote the set* $Z = \{n, n+1, \ldots, n+m-1\}$ *and put* $k = \lfloor m/2 \rfloor$. *Then a unique subset* $Y \subset Z$ *exists with the following properties*:

(i) $\#Y = k$;
(ii) *if* $n+d \notin Y$ *then* $n+d+1 \in Y$, *for every* d *with* $0 \leq d \leq m-2$;
(iii) $\Theta_i < \Theta_l$, *for every* $i \in Y$ *and* $l \in Z \backslash Y$;
(iv) $\Theta_l > 1/\sqrt{5}$ *for every* $l \in Z \backslash Y$.

Proof. Define Y by

$$Y = \begin{cases} \{n+1, n+3, \ldots, n+m-2\}, & \text{for } m \text{ odd;} \\ \{n+1, n+3, \ldots, n+k-1, n+k, & \\ \quad n+k+2, \ldots, n+m-2\}, & \text{for } m \equiv 0 \bmod 4; \\ \{n+1, n+3, \ldots, n+k-2, j^{**}, & \\ \quad n+k+1, n+k+3, \ldots, n+m-2\} & \text{for } m \equiv 2 \bmod 4; \end{cases}$$

with j^{**} such that $j^{**} + j^{*} = 2n + 2k - 1$, where j^{*} is as in (2.3). The result now immediately follows by Theorem (2.2) in each of the cases.

(2.5) Remarks. We mention some cases in which the above result is also valid, but that were left out of (2.4) for simplicity. If the expansion of x starts by $[B_0; 1, \ldots]$, that is if $n = 0$, the condition $B_n = B_0 \neq 1$ can be omitted. If m is infinite, then (i) should be replaced for instance by

$$\text{(i)}' \quad \forall k \quad \#(Y \cap Z_k) = \left\lfloor \frac{k}{2} \right\rfloor, \quad \text{where } Z_k = \{n, n+1, \ldots, n+k-1\} \cap Z.$$

Finally (2.4) holds for rational x too.

3. Connection with optimal continued fractions

In this section we show that the convergents of the optimal continued fraction expansion of x can be obtained from the regular expansion by skipping those convergents corresponding to index sets $Z \backslash Y$ as in (2.4). This is of importance to us, since it will be shown in the next section that the expansion with the best approximation properties, as in Theorem (1.2), actually is the same as the OCF.

We will not repeat the original definition of the OCF here as this is rather complicated (see [1], and [3]). That definition consists of an algorithm to compute successive partial quotients, without reference to the regular expansion. Here we prefer to define the OCF by describing the convergents in terms of a subsequence of the regular convergents.

(3.1) DEFINITION. Let $\{P_j/Q_j\}_{j=-1}^{\infty}$ be the RCF-convergents of $x = [B_0; B_1, B_2, \ldots]$. The sequence of *optimal continued fraction* convergents of x is obtained by applying the following rules.

(i) If $B_{j+1} > 1$ then P_j/Q_j is an OCF-convergent.

(ii) If $B_{j+1} = 1$ then

$$\frac{P_j}{Q_j} \text{ is an OCF-convergent}$$

$$\Longleftrightarrow \begin{cases} \text{either } \dfrac{P_{j-1}}{Q_{j-1}} \text{ is not an OCF-convergent,} \\ \text{or } \dfrac{P_{j-1}}{Q_{j-1}} \text{ is an OCF-convergent and } \Theta_j < \Theta_{j+1}. \end{cases}$$

(3.2) REMARKS. By definition, $P_{-1} = 1$ and $Q_{-1} = 0$.

Notice that as a consequence of (3.1), in case $B_{j+1} = 1$ and P_{j-1}/Q_{j-1} is not an OCF-convergent, then necessarily $\Theta_{j-1} > \Theta_j$.

Note that a semiregular expansion $x = [b_0; \varepsilon_1 b_1, \varepsilon_2 b_2, \ldots]$ is completely determined by the sequence of convergents p_k/q_k: the relation is given by the recurrence relations

$$\begin{array}{lll} p_{-1} = 1, & p_0 = b_0, & p_{k+1} = b_{k+1}p_k + \varepsilon_{k+1}p_{k-1} \\ q_{-1} = 0, & q_0 = 1, & q_{k+1} = b_{k+1}q_k + \varepsilon_{k+1}q_{k-1} \end{array} \qquad (k \geq 0).$$

That the above definition of the OCF describes the same semiregular expansion (for every x) as the definition given in [1] is proved in [1, Corollary (4.20)].

(3.3) PROPOSITION. *Let* $\{P_j/Q_j\}_{j=-1}^{\infty}$ *be the RCF-convergents of* $x = [B_0; B_1, B_2, \ldots]$. *Then*

$$\frac{P_j}{Q_j} \textit{ is not an OCF-convergent} \Longleftrightarrow B_{j+1} = 1,\ \Theta_{j-1} < \Theta_j \textit{ and } \Theta_j > \Theta_{j+1}.$$

PROOF. Notice that it is never the case that $\Theta_k = \Theta_{k+1}$ since x is irrational.

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The implication " $\Leftarrow$ " is immediate from part (ii) of Definition (3.1).

For " $\Rightarrow$ " we do the following. Suppose that P_j/Q_j is not an OCF-convergent; then $B_{j+1} = 1$ by (3.1)(i). Moreover, P_{j-1}/Q_{j-1} must be an OCF-convergent. But then (3.1)(ii) implies that $\Theta_j > \Theta_{j+1}$.

It remains to prove that $\Theta_{j-1} < \Theta_j$. In case $B_j > 1$ this is immediate from the inequality

$$V_j = [0; B_j, \dots, B_1] < [0; 1, B_{j+2}, B_{j+3}, \dots] = T_j$$

and the relations

$$\Theta_{j-1} = \frac{V_j}{1 + T_j V_j} \quad \text{and} \quad \Theta_j = \frac{T_j}{1 + T_j V_j}.$$

So we may assume that $B_j = 1$. Suppose now that $\Theta_{j-1} > \Theta_j$; this will lead to a contradiction in the following way. We have

$$B_j = 1, \quad \frac{P_{j-1}}{Q_{j-1}} \text{ is an OCF-convergent,} \quad \text{and} \quad \Theta_{j-1} > \Theta_j$$

so we get from (3.1)(ii) that

$$\Theta_{j-2} > \Theta_{j-1} \quad \text{and} \quad \frac{P_{j-2}}{Q_{j-2}} \text{ is not an OCF-convergent.}$$

The last assertion implies in particular that $B_{j-1} = 1$. Altogether we get

$$B_{j-1} = B_j = B_{j+1} = 1, \qquad \Theta_{j-2} > \Theta_{j-1} > \Theta_j > \Theta_{j+1}. \tag{3.4}$$

We use Theorem (2.2) to see that this is impossible: first of all, the length of the block of 1's of which B_{j-1}, B_j, B_{j+1} forms a part must be even, since otherwise the corresponding values of Θ would be alternately smaller and larger than $1/\sqrt{5}$ by (2.2), which is clearly not the case in (3.4). If $B_{j+2} > 1$ then $\Theta_{j-2} < \Theta_j$ by (2.2), again contradicting (3.4). Therefore also $B_{j+2} = 1$, but now (2.2) says that $\Theta_{j-2} > \Theta_j$ can only hold if $\Theta_{j-1} < \Theta_{j+1}$. This also contradicts (3.4) and it finishes the proof.

Thus (3.3) gives another description of the OCF. Below we give yet another one, this time establishing the connection with the considerations of the previous section.

(3.5) PROPOSITION. *Let $\{P_j/Q_j\}_{j=-1}^{\infty}$ be the RCF-convergents of $x = [B_0; B_1, B_2, \dots]$. Then*

$$\frac{P_j}{Q_j} \text{ is an OCF-convergent} \iff \begin{cases} B_{j+1} > 1 & \text{or} \\ B_{j+1} = 1 & \text{and } j \text{ satisfies the} \\ & \text{following condition.} \end{cases}$$

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Let $Z = \{n, n+1, \ldots, n+m-1\}$ *be such that* $j \in Z$, *with* n, m *as in* (2.4). *Then* $j \in Y$, *where* $Y \subset Z$ *is as in* (2.4).

PROOF. First we prove " $\Rightarrow$."

If $B_{j+1} \neq 1$ then P_j/Q_j is always an OCF-convergent by (3.3). So assume for the rest of this proof that $B_{j+1} = 1$. Suppose that P_j/Q_j is an OCF-convergent but $j \notin Y$; this leads to a contradiction in the following way.

From $P_j/Q_j \in \mathrm{OCF}$ and the fact that the OCF forms fastest expansions (which means by [1, Remark (3.11)] that $\lfloor (m+1)/2 \rfloor$ RCF-convergents out of m are skipped in passing to the OCF) we see that $m \geq 2$.

Let us first assume that $n < j < n+m-1$, which entails in particular $m \geq 3$. Then $B_j = B_{j+1} = B_{j+2} = 1$ and $j \notin Y$ implies by (2.4)(ii) that both $j-1$ and $j+1 \in Y$. From (2.4)(iii) we see that $\Theta_{j-1} < \Theta_j$ and $\Theta_j > \Theta_{j+1}$ but together with $P_j/Q_j \in \mathrm{OCF}$ these contradict (3.3).

Next let $n = j$; then $B_j > 1$ so $P_{j-1}/Q_{j-1} \in \mathrm{OCF}$ and moreover, $\Theta_{j-1} < \Theta_j$ (just as in the proof of (3.3)). By (2.4)(ii) and (2.4)(iii) again $j \notin Y$ implies that $j+1 \in Y$ and that $\Theta_j > \Theta_{j+1}$. But now we have $P_j/Q_j \in \mathrm{OCF}$, $\Theta_{j-1} < \Theta_j$ and $\Theta_j > \Theta_{j+1}$ while $B_{j+1} = 1$; these contradict (3.3).

Let finally $j = n+m-1$. As before $j \notin Y$ implies $j-1 \in Y$ and $\Theta_{j-1} < \Theta_j$. In this case (3.3) tells that $P_{j-1}/Q_{j-1} \in \mathrm{OCF}$, but then both P_{j-1}/Q_{j-1} and $P_j/Q_j \in \mathrm{OCF}$ which contradicts the fact that the OCF forms a fastest expansion; it is incompatible with the need to skip $\lfloor (m+1)/2 \rfloor$ out of $P_n/Q_n, \ldots, P_{j-1}/Q_{j-1}$, $P_j/Q_j = P_{n+m-1}/Q_{n+m-1}$ without skipping two consecutive elements.

This proves that $B_{j+1} = 1$ and $P_j/Q_j \in \mathrm{OCF}$ always imply that $j \in Y$.

The converse is just a matter of counting, using that the OCF forms fastest expansions.

That proves (3.5).

4. Three consecutive approximation constants

In this section we prove some results concerning triples $(\theta^*_{n-1}, \theta^*_n, \theta^*_{n+1})$. The reason to include them here is that we need one of them for the proof of our main theorem. Although only standard techniques are involved, our proofs of these results are very lengthy and we will only sketch them.

We need two auxiliary results. The first concerns general semiregular expansions.

(4.1) LEMMA. *Let* $x = [b_0; \varepsilon_1 b_1, \varepsilon_2 b_2, \dots]$ *be a semiregular continued fraction expansion of* x, *and let* θ_n *be defined as before. Then for* $n \geq 1$,

$$\theta_{n+1} = \varepsilon_{n+2}(\varepsilon_{n+1}\theta_{n-1} + b_{n+1}\sqrt{1 - 4\varepsilon_{n+1}\theta_{n-1}\theta_n} - b_{n+1}^2\theta_n). \tag{4.2}$$

PROOF. The main ingredient of the proof is that for every n,

$$\theta_{n-1} = \frac{v_n}{1 + t_n v_n} \quad \text{and} \quad \theta_n = \frac{\varepsilon_{n+1} t_n}{1 + t_n v_n} \tag{4.3}$$

where t_n and v_n are defined by

$$t_n = [0; \varepsilon_{n+1} b_{n+1}, \dots],$$

which is the "shift" of the expansion of x over n places, and

$$v_n = [0; b_n, \varepsilon_n b_{n-1}, \dots, \varepsilon_2 b_1] = \frac{q_{n-1}}{q_n}.$$

The proof of (4.3) is analogous to that of the special case of the regular expansion that we used before, and which can be found in [6, page 25].

Using that

$$t_n = \frac{\varepsilon_{n+1}}{b_{n+1} + t_{n+1}}, \quad \text{so} \quad t_{n+1} = \frac{\varepsilon_{n+1}}{t_n} - b_{n+1},$$

and that

$$v_{n+1} = \frac{1}{b_{n+1} + \varepsilon_{n+1} v_n},$$

we find from (4.3) that

$$\begin{aligned}
\theta_{n+1} = \frac{\varepsilon_{n+2} t_{n+1}}{1 + t_{n+1} v_{n+1}} &= \varepsilon_{n+2}\left(\frac{\varepsilon_{n+1} - b_{n+1} t_n}{t_n + (\varepsilon_{n+1} - b_{n+1} t_n)(1/(b_{n+1} + \varepsilon_{n+1} v_n))}\right) \\
&= \varepsilon_{n+2}\left(\frac{\varepsilon_{n+1}^2 v_n - b_{n+1}^2 t_n + \varepsilon_{n+1} b_{n+1}(1 - t_n v_n)}{\varepsilon_{n+1}(1 + t_n v_n)}\right) \\
&= \varepsilon_{n+2}\left(\frac{\varepsilon_{n+1} v_n}{1 + t_n v_n} - \frac{b_{n+1}^2 \varepsilon_{n+1} t_n}{1 + t_n v_n} + b_{n+1}\left(\frac{1 - t_n v_n}{1 + t_n v_n}\right)\right) \\
&= \varepsilon_{n+2}\left(\varepsilon_{n+1}\theta_{n-1} - b_{n+1}^2\theta_n + b_{n+1}\left(\frac{1 - t_n v_n}{1 + t_n v_n}\right)\right).
\end{aligned}$$

Combining this with

$$1 - 4\varepsilon_{n+1}\theta_{n-1}\theta_n = 1 - \frac{4\varepsilon_{n+1}^2 t_n v_n}{(1 + t_n v_n)^2} = \frac{(1 - t_n v_n)^2}{(1 + t_n v_n)^2}$$

completes the proof of (4.1).

(4.4) REMARKS. We will use (4.1) only for the OCF, but the result is interesting in itself. Special cases can be found in [4] also.

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(4.5) LEMMA. *For every* x *and every* $n \geq 1$;

$$(\theta^*_{n-1}, \theta^*_n) \in \Pi,$$

where $\Pi = \{(w, z) \in \mathbb{R} \times \mathbb{R} : w > 0, z > 0, 4w^2 + z^2 < 1, w^2 + 4z^2 < 1\}$.

Lemma (4.5) appears as [3, Theorem (5.1)(i)]; in the second part of that theorem in fact a distribution function for $(\theta^*_{n-1}, \theta^*_n)$ over Π is given, for almost all x. It is a consequence of the ergodic theory, developed in that paper.

(4.6) THEOREM. *Let* $\mathrm{OCF}(x) = [a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots]$ *and let* θ^*_n *for* $n \geq 1$ *be its approximation constants, as before. Let also for* $n \geq 1$,

$$\delta_n = \frac{\varepsilon_n + \varepsilon_{n+1}}{2}.$$

Then the following hold for $n \geq 2$:

$$\text{(4.7)} \qquad \frac{1}{a_{n+1}} \leq \theta^*_{n-1} + \theta^*_n + \theta^*_{n+1} \leq \frac{a_{n+1} + \delta_{n+1} + 1}{\sqrt{(a_{n+1} + \delta_{n+1})^2 + 1}};$$

$$\text{(4.8)} \qquad 0 < \theta^*_{n-1}\theta^*_n\theta^*_{n+1} \leq \frac{(a_{n+1} + \delta_{n+1})^2}{4\sqrt{((a_{n+1} + \delta_{n+1})^2 + 1)^3}};$$

and

$$\text{(4.9)} \qquad \frac{1}{\theta^*_{n-1}} + \frac{1}{\theta^*_n} + \frac{1}{\theta^*_{n+1}} \geq \frac{(a_{n+1} + \delta_{n+1} + 4)\sqrt{(a_{n+1} + \delta_{n+1})^2 + 1}}{a_{n+1} + \delta_{n+1}}.$$

Moreover, the bounds in (4.7)–(4.9) *are sharp.*

PROOF. The proof of (4.6) is laborious, but the idea is very simple. Use (4.2) in the special case of the OCF to express θ^*_{n+1} as a function of θ^*_{n-1}, θ^*_n, a_{n+1}, ε_{n+1} and ε_{n+2}. Next determine the subspaces $\Pi_{\varepsilon_{n+1}, a_{n+1}, \varepsilon_{n+2}}$ of Π on which ε_{n+1}, a_{n+1} and ε_{n+2} are constant. It turns out that the closure (under the ordinary euclidean topology) of each of $\Pi_{\varepsilon_{n+1}, a_{n+1}, \varepsilon_{n+2}}$ is a compact piece of Π. On these the sum, product and sum of the reciprocals of θ^*_{n-1}, θ^*_n and θ^*_{n+1} are now functions of the two variables θ^*_{n-1} and θ^*_n, which can easily be shown to take on extremes only on the boundaries. Calculating these extreme values gives (4.7)–(4.9).

The result stating that the bounds are sharp follows from the fact that the density function for pairs $(\theta^*_{n-1}, \theta^*_n)$ is nonzero on Π [3, Theorem (5.1)(ii)].

By way of example, we give the proof outlined above in some detail for one "generic" case. We consider $\Pi_{1,a,1}$, with $a > 2$, and we take its closure $\overline{\Pi}_{1,a,1}$ in order not to have to worry about the boundaries. To find these boundaries, we take a closer look at how Lemma (4.5) is proved in [3]. It is shown that $(t_n, v_n) \in \Upsilon$ for every $n \geq 1$, where $\Upsilon \subset [-1, 1] \times [0, 1]$ consisting of (t, v) such that

$$v \leq \min\left(\frac{2t+1}{t+1}, \frac{t+1}{t+2}\right) \quad \text{and} \quad v \geq \max\left(0, \frac{2t-1}{1-t}\right).$$

With $\varepsilon(t)$ denoting the sign of t,

$$\psi(t, v) = \left(\frac{v}{1+tv}, \frac{\varepsilon(t)t}{1+tv}\right)$$

defines a surjective, two-to-one mapping $\psi: \Upsilon \to \Pi$, sending (t_n, v_n) to $(\theta^*_{n-1}, \theta^*_n)$ by (4.3). On Υ an operator $\mathscr{W}$ is defined explicitly (as in [3, Definition (4.8)]) by

$$\mathscr{W}(t, v) = \left(\left|\frac{1}{t}\right| - \beta(t, v), \frac{1}{\beta(t, v) + \varepsilon(t)v}\right)$$

where

$$\beta(t, v) = \left\lfloor \left|\frac{1}{t}\right| + \frac{\lfloor|\frac{1}{t}|\rfloor + \varepsilon(t)v}{2(\lfloor|\frac{1}{t}|\rfloor + \varepsilon(t)v) + 1} \right\rfloor.$$

This $\mathscr{W}$ has the property that $\mathscr{W}^n(x, 0) = (t_n, v_n)$, for $x \in [-\frac{1}{2}, \frac{1}{2}]$. Now $\Pi_{1,a,1} = \psi\Upsilon_{1,a,1}$, if we define $\Upsilon_{1,a,1}$ as $\{(t, v) \in \Upsilon: \varepsilon(t) = \varepsilon(t') = 1$ and $\beta(t, v) = a\}$, where we have put $(t', v') = \mathscr{W}(t, v)$. An elementary calculation then shows that $\overline{\Pi}_{1,a,1}$ consists of $(w, z) \in \Pi$ for which

$$z \leq -\frac{1}{a^2}w + \frac{1}{a} \quad \text{and} \quad 4w^2 + 8a(a+1)wz + (2a^2+2a+1)^2z^2 \geq (2a+1)^2.$$

For the sum $\theta^*_{n-1} + \theta^*_n + \theta^*_{n+1}$ on $\Pi_{1,a,1}$, we need by (4.3) to look at the function $R(w, z) = w + z + (w + a\sqrt{1-4wz} - a^2z)$. Since the partial derivative of R with respect to z is negative on $\Pi_{1,a,1}$, extremes will be found on the boundaries; in fact $R(w, z)$ takes on the minimal value $1/a$ in the point $(0, 1/a)$ and the maximal value $(a+2)/\sqrt{(a+1)^2+1}$ in the point $((a+1)/2\sqrt{(a+1)^2+1}, 1/\sqrt{(a+1)^2+1}$. That proves (4.7) in this case.

The extremal values for the product and the sum of the reciprocals in this case are found by considering instead of R the functions

$$G(w, z) = wz(w + a\sqrt{1-4wz} - a^2z)$$

and

$$H(w\,,\,z)=\frac{1}{w}+\frac{1}{z}+\frac{1}{w+a\sqrt{1-4wz}-a^2z}.$$

The cases $\varepsilon_{n+1}=-1$ or $\varepsilon_{n+2}=-1$ can be dealt with similarly; the case that $a=2$ needs special attention, but follows in similar vein.

Checking all these cases completes the proof of (4.6).

(4.10) COROLLARY. *For every* x, *and for every* $n\geq 2$,

(4.11) $$\theta^*_{n-1}+\theta^*_n+\theta^*_{n+1}\leq\frac{3}{\sqrt{5}},$$

(4.12) $$\theta^*_{n-1}\theta^*_n\theta^*_{n+1}\leq\left(\frac{1}{\sqrt{5}}\right)^3,$$

(4.13) $$\frac{1}{\theta^*_{n-1}}+\frac{1}{\theta^*_n}+\frac{1}{\theta^*_{n+1}}\geq 3\sqrt{5}.$$

Moreover, the bounds in (4.11)–(4.13) *are sharp.*

PROOF. This is an immediate consequence of (4.6) and the fact that $a_n+\delta_n\geq 2$ for every n, since $a_n\geq 2$, and $a_n\geq 3$ if both $\varepsilon_{n+1}=-1$ and $\varepsilon_{n+2}=-1$.

In other words, the arithmetic, geometric and harmonic means of three consecutive optimal approximation coefficients are all bounded by $1/\sqrt{5}$. In fact we see easily that this holds for an arbitrary number of consecutive optimal approximation constants.

(4.14) COROLLARY. *For every* x, *for* $n\geq 1$ *and for every* $N\geq 2$,

(4.15) $$\frac{1}{N}\sum_{j=0}^{N-1}\theta^*_{n+j}\leq\frac{1}{\sqrt{5}},$$

(4.16) $$\sqrt[N]{\prod_{j=0}^{N-1}\theta^*_{n+j}}\leq\frac{1}{\sqrt{5}},$$

(4.17) $$\frac{N}{\sum_{j=0}^{N-1}\frac{1}{\theta^*_{n+j}}}\leq\frac{1}{\sqrt{5}}.$$

The constant $1/\sqrt{5}$ *is best possible in each case.*

PROOF. For the arithmetic mean as in (4.15), this is a consequence of the special cases $N=2$, proven in [3, Theorem (5.9)], and $N=3$, which is (4.11) above.

By a famous theorem of Cauchy the arithmetic mean R, the geometric mean G as in (4.16), and the harmonic mean H as in (4.17), always satisfy $H \le G \le R$.

The results are seen to be sharp by a density argument, as in the proof of (4.6).

That proves (4.14).

(4.18) REMARK. Of course it is also possible to derive analogues of (4.7)–(4.9) for two consecutive approximation constants. For the arithmetic mean we get for instance

$$\text{(4.19)} \qquad \frac{2a_{n+1}+1}{2a_{n+1}^2+2a_{n+1}+1} \le \theta_{n-1}^* + \theta_n^* \le \frac{a_{n+1}+\gamma_{n+1}+2}{2\sqrt{(a_{n+1}+\gamma_{n+1})^2+1}},$$

where $\gamma_{n+1} = \frac{\varepsilon_{n+1}-1}{2}$. These bounds are sharp, except for the upper bound in the case that $\varepsilon_{n+1} = -1$ and $a_{n+1} = 2$ (where it can be improved to $2/\sqrt{5}$). Since we do not need results of this type, we leave it at this.

5. Optimal approximation

Now we are ready to prove the main theorem. Throughout this section we use the following notation.

We fix an arbitrary irrational x, let again $(r_k/s_k)_{k\ge1}$ denote its sequence of OCF-convergents and $(p_k/q_k)_{k\ge1}$ its sequence of convergents for some other semiregular continued fraction expansion. Furthermore $\theta_k^* = s_k|s_kx - r_k|$ and $\theta_k = q_k|q_kx - p_k|$ for $k \ge 1$. We also introduce an auxiliary arithmetical function η; for every $N \ge 1$ we denote by $\eta(N)$ the number of convergents among $(p_k/q_k)_{k\ge1}$ with denominator up to s_{N+1}, so $\eta(N) = \#\{k : q_k < s_{N+1}\}$. Note that η depends on the semiregular expansion under consideration; but irrespective of this expansion we have the following.

(5.1) LEMMA. *For every* $N \ge 1$, $\eta(N) \ge N$.

PROOF. Suppose that for some x we have, for some semiregular expansion, that $\eta(N) < N$ for certain $N \ge 1$; fix that expansion and take the least N_0 with this property. Then $q_{N_0} \ge s_{N_0+1}$. We can now find a new semiregular expansion, for instance by taking $q_N = s_{N+1}$ for $N \ge N_0$, satisfying $\eta(N) < N$ for every $N > N_0$. This contradicts the fact that OCF(x) forms a fastest expansion: our newly defined continued fraction skips at least one more regular convergent [1, Section 3].

(5.2) THEOREM. *Let* $M \geq 1$ *be such that* p_M/q_M *is not an OCF-convergent. If* $\eta(N) \geq M$, *then*

$$\text{(5.3)} \qquad \frac{1}{\eta(N)} \sum_{j=1}^{\eta(N)} \theta_j > \frac{1}{N} \sum_{j=1}^{N} \theta_j^*.$$

PROOF. The idea of the proof is to supplement the sequence $\theta_1^*, \dots, \theta_N^*$ with elements $1/\sqrt{5}$ until it consists of $\eta(N)$ elements (note that $\eta(N) \geq N$ by (5.1)) and to show next that the mean of this new sequence is larger than that of $\theta_1^*, \dots, \theta_N^*$ and smaller than that of $\theta_1, \dots, \theta_{\eta(N)}$.

First note that M exists: because the OCF is a fastest expansion, no strict subsequence of OCF-convergents forms a continued fraction expansion; by assumption $(p_k/q_k)_{k\geq 1}$ differs from the OCF-expansion, so we must have $p_M/q_M \notin$ OCF for some M.

Let N be such that $\eta(N) \geq M$, so $q_M < s_{N+1}$. We define an injective map

$$\phi\colon \{k: 1 \leq k \leq N\} \to \{m: 1 \leq m \leq \eta(N)\}$$

as follows. Define $n(k)$ for $1 \leq k \leq N$ by the relation $s_k = Q_{n(k)}$ between OCF- and RCF-convergents. If $B_{n(k)+1} > 1$, there exists at least one m such that

$$\text{(5.4)} \qquad Q_{n(k)} \leq q_m < Q_{n(k)+1}, \quad \text{with } 1 \leq m \leq \eta(N)$$

by [1, Lemma (3.1)]; we let $\phi(k)$ be the smallest of these integers m.

Now suppose that $B_{n(k)+1} = 1$. If $r_k/s_k = p_m/q_m$ for some m with $1 \leq m \leq \eta(N)$, we put $\phi(k) = m$; otherwise let n be the largest integer for which $B_n > 1$ with $n \leq n(k)$. If $n(k) - n$ is even (or zero) we let $\phi(k)$ be such that $p_{\phi(k)}/q_{\phi(k)} = P_{n(k)+1}/Q_{n(k)+1}$, while for $n(k) - n$ odd we let $\phi(k)$ be such that $p_{\phi(k)}/q_{\phi(k)} = P_{n(k)-1}/Q_{n(k)-1}$. In any case such $\phi(k)$ exists again by [1, Lemma (3.1)]. Thus ϕ is well defined and injective and it has the property

$$\text{(5.5)} \qquad \text{if } \frac{r_k}{s_k} = \frac{p_m}{q_m} \text{ for some } m, \quad \text{then } \phi(k) = m.$$

Next we define $\lambda\colon \{m: 1 \leq m \leq \eta(N)\} \to \mathbb{R}$ by

$$\lambda(m) = \begin{cases} \theta^*_{\phi^{-1}(m)} & \text{if } m \in \operatorname{im}(\phi), \text{ the image of } \phi; \\ 1/\sqrt{5} & \text{otherwise.} \end{cases}$$

Since ϕ is injective, λ takes on each of the values $\theta_1^*, \theta_2^*, \dots, \theta_N^*$ exactly once, and the value $1/\sqrt{5}$ precisely $\eta(N) - N$ times. But we can show that in this way

$$\text{(5.6)} \qquad \lambda(m) \leq \theta_m$$

for $1 \le m \le \eta(N)$. For, suppose first that $m \in \operatorname{im}(\phi)$. If p_m/q_m is an OCF-convergent, then $\theta_m = \theta^*_{\phi^{-1}(m)} = \lambda(m)$; if p_m/q_m is not an OCF-convergent, it is either a secondary regular convergent (if $B_{n(k)+1} > 1$) and satisfies $\theta_m > \frac{1}{2}$ by Legendre's theorem, while $\frac{1}{2} > \theta^*_{\phi^{-1}(m)} = \lambda(m)$, or it is a regular convergent that is not an OCF-convergent, in which case $\theta_m > \theta^*_{\phi^{-1}(m)} = \lambda(m)$ by (2.4) and (3.5).

On the other hand, if $m \notin \operatorname{im}(\phi)$ we see from (5.5) that p_m/q_m is not an OCF-convergent, and (5.6) holds again by (2.4) and (3.5).

Since we have on the same grounds the strict inequality $\lambda(M) < \theta_M$, we get that

$$\sum_{j=1}^{\eta(N)} \theta_j > \sum_{j=1}^{\eta(N)} \lambda(j) = \sum_{j=1}^{N} \theta^*_j + \frac{\eta(N) - N}{\sqrt{5}}, \tag{5.7}$$

and thus

$$\begin{aligned}
\frac{1}{\eta(N)} \sum_{j=1}^{\eta(N)} \theta_j &> \frac{1}{\eta(N)} \left(\sum_{j=1}^{N} \theta^*_j + \frac{\eta(N) - N}{\sqrt{5}} \right) \quad \text{(by (5.7))} \\
&= \left(1 - \frac{N}{\eta(N)}\right)\left(\frac{1}{\sqrt{5}} - \frac{1}{N}\sum_{j=1}^{N} \theta^*_j\right) + \frac{1}{N}\sum_{j=1}^{N}\theta^*_j \\
&\ge \frac{1}{N}\sum_{j=1}^{N}\theta^*_j \quad \text{by (5.1) and (4.15).}
\end{aligned}$$

This proves (5.2).

This also proves Theorem (1.2), and its immediate consequences (1.3) and (1.4). Moreover, the uniquely determined expansion of Theorem (1.2) is now known to be the OCF-expansion.

6. Periodicity and partial quotients

The results of the previous sections yield an algorithm to compute the OCF expansion once the RCF is known: if $B_{n+1} > 1$ then P_n/Q_n is an OCF convergent; if $B_n \neq 1$, $B_{n+1} = B_{n+2} = \cdots = B_{n+m} = 1$, $B_{n+m+1} \neq 1$ then we can determine the set Y as in (2.4) which tells us by (3.5) precisely which convergents to skip. In other words, this gives us a singularization algorithm. An interesting consequence of this is the periodicity of OCF-expansions for quadratic irrationals, due to the fact that this singularization scheme almost only depends on m (the length of the block of 1's). There is only a slight complication in the case that $m \equiv 2 \bmod 4$ caused by the definition of j^* in (2.3). Still, the following holds.

(6.1) THEOREM. *The OCF of x is periodic if and only if x is a quadratic irrational number.*

PROOF. Since every periodic semiregular continued fraction converges to a quadratic irratinal number [8] we only need to prove one direction here. If the period of RCF(x) contains no 1's (as partial quotients), the period of the OCF is the same as the one for the RCF of x, for instance by the first part of (3.5). If precisely m consecutive 1's occur in the period of RCF(x), then for m odd or divisible by 4 the recipe for skipping RCF-convergents to find the OCF-convergents is periodic again because it depends only on m, not on x or n. If m is $2 \bmod 4$ however, there is dependence on x in finding the set Y of (2.4). Suppose that RCF(x) has preperiod of length q, period length p, and suppose that this period contains m consecutive 1's somewhere, with $m = 2k$, k odd; say

$$\mathrm{RCF}(x) = [B_0; B_1, B_2, \ldots, B_q, \overline{B_{q+1}, \ldots, B_n, 1^{2k}, B_{n+m+1}, \ldots, B_{q+p}}],$$

with $B_n \neq 1$, $B_{n+m+1} \neq 1$; here the bar denotes the period. We may as well write for RCF(x),

$$[B_0; B_1, B_2, \ldots, B_q, B_{q+1}, \ldots, B_n, 1^k, \overline{1^k, B_{n+m+1}, \ldots, B_{q+p}, B_{q+p+1}, \ldots, B_{n+p}, 1^k}],$$

with $B_{q+1} = B_{q+p+1}, \ldots, B_n = B_{n+p}$. Since k is odd, it is not hard to see that

$$(6.2) \qquad \Theta_{n+p+k-1} < \Theta_{n+p+k} \quad \Leftrightarrow \quad V_{n+p} > T_{n+p+m},$$

if we use as before that $\Theta_{n+p+k-1}$ equals

$$\frac{1}{[0; 1^{k-1}, B_{n+p}, \ldots, B_1] + [0; 1^k, B_{n+p+m+1}, B_{n+p+m+2}, \ldots] + 1}$$

and that

$$\Theta_{n+p+k} = \frac{1}{[0; 1^{k-1}, B_{n+p+m+1}, B_{n+p+m+2}, \ldots] + [0; 1^k, B_{n+p}, \ldots, B_1] + 1}.$$

Let $j \geq 0$ be minimal such that

$$B_{n+p-j} \neq B_{n+p+m+1+j} \quad \text{with } n+p-j \geq 1.$$

Now either

$$(6.3) \qquad n+p-j \geq n+m+1$$

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or

$$n + p - j \leq n. \tag{6.4}$$

In the first of these cases it is clear from periodicity that

$$V_{n+p} > T_{n+p+m} \Leftrightarrow V_{n+ip} > T_{n+ip+m} \quad \text{for every } i \geq 1,$$

and therefore

$$\Theta_{n+p+k-1} < \Theta_{n+p+k} \Leftrightarrow \Theta_{n+ip+k-1} < \Theta_{n+ip+k} \quad \text{for every } i \geq 1.$$

In the second case we see that the period 1^k, $B_{n+m+1}, \dots, B_{q+p}$, B_{q+p+1}, $\dots, B_{n+p}$, 1^k is symmetric, and two cases have to be distinguished again.

If the period length p is even we get

$$V_{n+p} > T_{n+p+m} \Leftrightarrow V_{n+ip} > T_{n+ip+m} \quad \text{for every } i \geq 1,$$

and thus

$$\Theta_{n+p+k-1} < \Theta_{n+p+k} \Leftrightarrow \Theta_{n+ip+k-1} < \Theta_{n+ip+k} \quad \text{for every } i \geq 1$$

as before; if the period length p is odd however we find

$$\Theta_{n+p+k-1} < \Theta_{n+p+k} \Leftrightarrow \begin{cases} \Theta_{n+ip+k-1} < \Theta_{n+ip+k} & \text{for } i \text{ odd}, \\ \Theta_{n+ip+k-1} > \Theta_{n+ip+k} & \text{for } i \text{ even}. \end{cases}$$

But then we have periodicity again, with period length p if (6.3) holds or (6.4) holds with p even, and with period length $2p$ if (6.4) holds with p odd. This proves (6.1).

(6.5) EXAMPLE. Let $\omega = (-39 + \sqrt{3029})/58$ (cf. [5]).

In the table we show the first 16 partial quotients and convergents of the RCF for ω; the fifth column shows the values of Θ. The sixth column shows the partial quotients for Minkowski's diagonal continued fraction (DCF) expansion, which can be defined as the expansion one gets by selecting precisely those regular convergents that satisfy $\Theta < 1/2$ (cf. [8, Section 45], [7]). The seventh column gives the first partial quotients for the nearest integer continued fraction (NICF) expansion of ω, the eighth those for the OCF. All of these expansions are periodic; the bold partial quotients in the table together constitute the period (the DCF and the OCF both have a preperiod). A blank in a column indicates that the regular convergent of that row is skipped. We see that the OCF skips the maximal number of regular convergents (3 in every period) just like the NICF, but unlike the DCF. At the same time the OCF selects only convergents with $\Theta < 1/2$, just as the DCF does; the NICF does not have this property.

TABLE

n	RCF	P	Q	Θ	DCF	NICF	OCF
1	3	1	3	0.5116...			
2	1	1	4	0.4238...	4	4	4
3	1	2	7	0.4520...	−2		−2
4	1	3	11	0.4551...	1	−3	
5	1	5	18	0.4176...	1		2
6	1	8	29	0.5271...		−3	
7	1	13	47	0.2362...	2	−2	−3
8	3	47	170	0.5269...			
9	1	60	217	0.4095...	−5	4	−5
10	1	107	387	0.45424578...	−2		
11	1	167	604	0.45424554...	1	−3	−3
12	1	274	991	0.4179...	1		−2
13	1	441	1595	0.5279...		−3	
14	1	715	2586	0.2363...	2	−2	2
15	3	2586	9353	0.5262...			
16	1	3301	11939	0.4192...	−5	4	−5

The example also illustrates how the first two regular periods together are used for one OCF-period; this is because the period length (7) of RCF(ω) is odd and it contains 6 ($\equiv 2 \bmod 4$) consecutive 1's (see Table above.)
In the rest of this section we have collected some facts about the partial quotients of optimal continued fractions. Let x be any irrational number and let SRCF(x) be some semiregular expansion of x, so SRCF(x) = $[b_0; \varepsilon_1 b_1, \varepsilon_2 b_2 \dots]$; then $M_{\mathrm{SRCF}}(x, B, k)$ will denote the number of partial quotients among the first k in this particular expansion that equal B. So

$$M_{\mathrm{SRCF}}(x, B, k) = \#\{j: 1 \le j \le k, b_j = B\}.$$

(6.6) THEOREM. *For every irrational number x and every semiregular expansion* SRCF(x) *that is fastest, we have*

$$(6.7) \qquad M_{\mathrm{SRCF}}(x, 2, k) \le M_{\mathrm{OCF}}(x, 2, k) \quad \textit{for infinitely many } k \ge 1;$$

moreover, for every $B \ge 1$ both

$$\lim_{k\to\infty} \frac{1}{k} M_{\mathrm{SRCF}}(x, B, k) \quad \textit{and} \quad \lim_{k\to\infty} \frac{1}{k} M_{\mathrm{OCF}}(x, B, k)$$

will exist for almost all x, and then

$$(6.8) \qquad \lim_{k\to\infty} \frac{1}{k} M_{\mathrm{SRCF}}(x, 2, k) \le \lim_{k\to\infty} \frac{1}{k} M_{\mathrm{OCF}}(x, 2, k).$$

PROOF. We sketch a proof.

All we have to do for (6.7) is to analyze what happens after every possible singularization according to fastest expansions; so we look again at a sequence of regular partial quotients $B_n \neq 1, 1, \dots, 1, B_{n+m+1} \neq 1$. For odd m, every fastest expansion singularizes in the same way since there is only one way to singularize $\lfloor (m+1)/2 \rfloor$ of the 1's.

If m is even, say $m = 2k$, there are several strategies possible: one may singularize the first, third, ..., $m-1$th 1, or one may singularize the second, fourth, ..., mth 1 or one may switch from the first to the second strategy somewhere in between. If one uses the first or the second strategy, $k-1$ new quotients 3 are introduced and 1 extra 2 (while also either B_n or B_{n+m+1} is increased by 1); if one switches strategy somewhere however, $k-2$ new 3's are created and 2 extra 2's (while also both B_n and B_{n+m+1} are increased by 1). Thus one sees without too much difficulty that the number of 2's is maximized only if one changes strategy (only the fact that at the endpoints B_n and B_{n+m+1} of the block 2's may disappear urges some caution, particularly if $m \equiv 2 \bmod 4$). From (3.5) and (2.4) it is clear that this always happens in the OCF. This finishes the sketch of the proof, using the usual ergodic properties for the existence of the limit in (6.8).

(6.9) EXAMPLE. Let us try to elucidate the argument above a little bit by giving an example. Let us see how a block of six consecutive 1's can be singularized for fastest expansions; underlined 1's are singularized below (and ε's are omitted).

(i) $B_n, \underline{1}, 1, \underline{1}, 1, \underline{1}, 1, B_{n+m+1} \mapsto B_n + 1, 3, 3, 2, B_{n+m+1}$;
(ii) $B_n, 1, \underline{1}, 1, \underline{1}, 1, \underline{1}, B_{n+m+1} \mapsto B_n, 2, 3, 3, B_{n+m+1} + 1$;
(iii) $B_n, \underline{1}, 1, 1, \underline{1}, 1, \underline{1}, B_{n+m+1} \mapsto B_n + 1, 2, 2, 3, B_{n+m+1} + 1$;
(iv) $B_n, \underline{1}, 1, \underline{1}, 1, 1, \underline{1}, B_{n+m+1} \mapsto B_n + 1, 3, 2, 2, B_{n+m+1} + 1$.

The cases (i) and (ii) correspond to the first and the second strategy in the proof (which are in general the singularization schemes of the NICF and Hurwitz's singular continued fraction). In the cases (iii) and (iv) strategies change; one of these (depending on the values of B_n and B_{n+m+1}) corresponds to the OCF.

One easily verifies that number of 2's is maximal and the number of 3's minimal in cases (iii) and (iv), irrespective of the values of B_n and B_{n+m+1}; but also note that these numbers are equal for all cases when $B_n = B_{n+m+1} = 2$.

Finally we provide some statistics about partial quotients; it is not hard (but very tedious) to find for almost all x the relative frequency of occurrence of the values of partial quotients for the OCF. In the table below we compare these to the same frequencies for some other fastest expansions, viz. those of the NICF and of the α-expansion with $\alpha = (-2+\sqrt{10})/2$; we also list the values for the (nonfastest) RCF. The α-expansion with $\alpha = (-2+\sqrt{10})/2 = 0.58113\ldots$ has the property that at the same time the frequency of 2's is maximal and the frequency of 3's is minimal for any fastest α-expansion. For the NICF the values can be found from results in [9].

B	RCF	OCF	NICF	α
1	0.41504...	–	–	–
2	0.16993...	0.22111...	0.19637...	0.22110...
3	0.09311...	0.20478...	0.22555...	0.19479...
4	0.05889...	0.12166...	0.12630...	0.12866...
5	0.04064...	0.07979...	0.08088...	0.08209...

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